Arbitrary approximation of the shifted-LRC by the LRC

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Abstract: - Consider an instance of the Shifted Lonely Runner Conjecture (S-LRC) where all n runners (except the stationary runner 0) have integer speeds and start from real values in [0,1[at time t=0. We show that one can derive an alternative vector of starting points that can be made to be arbitrarily close to the initial vector of starting points. The alternative starting point of each runner i is a rational in [0,1[and is expressible as (q_i / P) where P is a large prime and q_i is an integer in [0, P-1]. The S-LRC instance with the alternative starting points, allows a minimal loneliness gap of f, if and only if, the corresponding LRC allows a minimal loneliness gap of f, where f is a desired fraction in]0,1[. This finding is important in the light of recent counter-examples to the shifted-LRC.

1. Introduction

The reader is referred to papers [1][2] for an introduction to the S-LRC and LRC (where the n-1 moving runners have the same starting point as the stationary runner 0 at time t=0). Recently, counter-examples [1] have been found to the shifted-LRC (with as low as 5 runners), in which certain vectors of rational starting points of the runners prevent the minimal 1/n gap. The aim of this paper is to show that the S-LRC can be approximated by the LRC with the same vector of speeds, but with an alternative vector of starting points, to any desired level of accuracy. Thus, it aims to motivate the research community to narrow its focus to the LRC. So, if a result is obtained that the LRC's minimal loneliness gap tends to 1/n, it automatically implies that the S-LRC's minimal loneliness gap also tends to 1/n (not necessarily equal to 1/n).

2. The approach

We refer the reader to the list of notations we used in our previous paper [2]. We further make notations for the initial and alternative vectors of starting points for the S-LRC in Theorem 1.

<u>Theorem 1</u>: Denote the initial given vector of n-1 real starting points in the S-LRC instance as $A = <a_1, a_2, ... , a_{n-1}>$. There exists an alternative vector of n-1 rational starting points denoted as $B = <(q_1 / P), (q_2 / P), ... , (q_{n-1} / P)>$, where P is an arbitrarily large prime number, and where each q_i is an integer in [0, P-1] such that MAX(absolute((q_i / P) - a_i), over integers i in [1,n-1]) is arbitrarily small.

<u>Proof</u>: We know from well-known theories on prime numbers that if $\{P_1, P_2, ... P_M\}$ is the set of prime numbers below (P_M + 1), then the number P = (($P_1 P_2 ... P_M$) + 1) is also prime. We are thus able to choose a larger prime number P, for which we can keep obtaining an optimum value of integer q_i in [0,P-1] such that absolute((q_i / P) - a_i) will tend to decrease from the earlier value of absolute((q_i / P) - a_i) calculated from the earlier and smaller prime P. There will be exceptions to the statement that absolute((q_i / P) - a_i) will keep decreasing if we choose larger values of P, for example if a_i is a rational whose denominator is a prime constant in which case absolute((q_i / P) - a_i) will be 0 when P is that prime constant and will increase momentarily for a higher value of P. But it is true that there exists an optimum value of integer q_i in [0,P-1] such that absolute((q_i / P) - a_i) will tend to 0 as P tends to infinity.

<u>Hence Proved</u>

We now state the main result of this paper in Theorem 2, where the symbol ↔ denotes "if and only if".

<u>Theorem 2</u>: Let Δ be a positive real tending to 0 and f be a fraction. (Minimal loneliness gap of the S-LRC ≥ (f - Δ)) ↔ (Minimal loneliness gap of the LRC ≥ f)

<u>Proof</u>: This result comes from the joint application of Theorem 1 of this paper, and the Lemmas 5.1 and 5.2 of our previous paper [2]. We remark that since the non-zero speeds of the runners are integers, the positions of the runners in the S-LRC (and separately in the LRC) are periodic, irrespective of the real values in A. This remark is crucial since it establishes a finite amount of time in which the minimal loneliness gap needs to be checked. Consider P sectors on the circle numbered from 0 to P-1, where the arc of sector j is in [(j-0.5)/P, (j+0.5)/P]. We observe the following situations at different times.

Situation at time t=0 for both the LRC and the S-LRC: From Theorem 1, it is clear that each alternative starting point in B is actually the center of the arc of some sector, so the position of each runner i at time t=0 for the S-LRC is given by I_i/P where I_i is some integer. The equivalent situation for the LRC is that the position of each runner i is 0, which is also the center of Sector 0, at time t=0.
Situation at the target time t=T for the LRC: If time t=T exists for the LRC such that each runner i is separated by atleast f from the stationary runner R_0, it would mean that its position is given by J_i+y_i , where -f > y_i > f, where absolute(y_i) < 1, where J_i is some integer.
Situation at time t=T/P for the LRC: This means that at the time t=T/P, the position of runner i would be given by (J_i/P + y_i/P), where J_i/P is the center of some sector and -f/P > y_i/P > f/P, and where absolute(y_i/P) < 1/P.
Situation at t=T/P for the S-LRC: At time t=T/P, the position of runner i would therefore be ((I_i+J_i)/P + y_i/P), where (I_i+J_i)/P is the center of some sector and -f/P > y_i/P > f/P, and where absolute(y_i/P) < 1/P.
Situation at the target time t=T for the S-LRC: At time t=T, the position of runner i would therefore be $I_i+J_i+y_i$, where -f > y_i > f, which would mean that the runners are again separated from R_0 by f.

We observe from Theorem 1, that the elements of alternative vector of starting points B is separated from the corresponding elements of initial vector of starting points A, by a distance $\pm\Delta$, where Δ is positive and that tends to 0 as P becomes a larger prime. The Theorem follows.

Hence Proved

References

[1] Monica Blanco, Francisco Criado, Francisco Santos, *Coloopless zonotopes and counterexamples to the Shifted Lonely Runner Conjecture*, Mar 2026, https://arxiv.org/abs/2603.24784.
[2] Deepak Ponvel Chermakani, *An Average Loneliness Gap of 1/n Can Allow a Minimum Loneliness Gap of 1/(2n)*, May 2026, https://www.vixra.org/abs/2605.0011.

About the author

I, Deepak Ponvel Chermakani, am a citizen of India and I wrote this paper, which is original to the best of my knowledge, out of my own interest and initiative during my spare time. I completed a fulltime two-year Master of Science Degree in Electrical Engineering from University of Hawaii at Manoa USA (www.hawaii.edu) in Aug 2015, a fulltime one-year Master of Science Degree in Operations Research with Computational Optimization from University of Edinburgh UK (www.ed.ac.uk) in Sep 2010, a fulltime four-year Bachelor of Engineering Degree in Electrical and Electronic Engineering, from Nanyang Technological University Singapore (www.ntu.edu.sg) in Jul 2003, fulltime high schooling from National Public School Indiranagar in Bangalore India in Jul 1999, and fulltime middle schooling from Bishop Cotton Boys School in Bangalore India in Jul 1997. I am most grateful to my parents (my mother Mrs. Kanaga Rathinam Chermakani and my father Mr. T. Chermakani) for their sacrifices in educating me and bringing me up.