Recurrence Relations in the Tessellation of the Upper Half-Plane by the Full Modular Group.

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Abstract

The successive actions of the generators S and T of the full modular group $SL(2,\mathbb{Z})$ on the fundamental domain $\mathcal{F}$ lead to a tessellation of the upper half-plane $\mathbb{H}$. Each image can be obtained by acting on the fundamental domain with a word consisting of solely S's and T's. For instance $STTSTTT(\mathcal{F})$ is the image when the word $STTSTTT$ acts on $\mathcal{F}$. There is no shorter word that produces the image $STTSTTT(\mathcal{F})$. A recurrence relation will be derived for the number of images for a given minimal word length.

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1 Introduction

A member γ of the special linear group $SL(2,\mathbb{Z})$ is a 2×2 matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

with a, b, c and $d \in \mathbb{Z}$ and whose determinant is equal to one: $ad - bc = 1$. Each matrix γ is identified with the function $\gamma(z)$ which acts on a complex number z by a linear fractional (Möbius) transformation:

$$\gamma(z) = \frac{az+b}{cz+d}\,. \tag{2}$$

If $z \in \mathbb{H}$, then $\gamma(z) \in \mathbb{H}$.

The group $SL(2,\mathbb{Z})$ is generated by the two generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, see for instance [1, 2]. Applying (2) gives

$$S(z) = \frac{-1}{z} \tag{3}$$

and

$$T(z) = z + 1\,. \tag{4}$$

The tessellation starts with the fundamental domain $\mathcal{F} = \{z \in \mathbb{H} \,:\, |\mathrm{Re}(z)| \leq \frac{1}{2},\, |z| \geq 1\}$, see Figure 1.

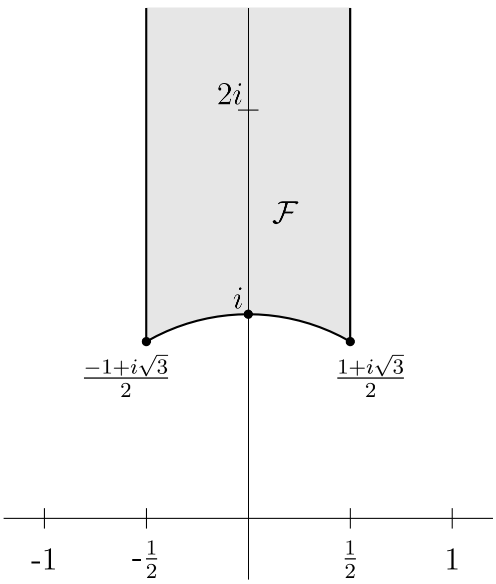

Figure 1: Standard fundamental domain $\mathcal{F}$.

For an image such as $TTSTT(\mathcal{F})$ the 'word' is $TTSTT$ and the word length is 5. A word length is minimal if there is no shorter word to arrive at the image. There are two images of $\mathcal{F}$ with word length 1: $T(\mathcal{F})$ and $S(\mathcal{F})$. There are four images with word length 2: $TT(\mathcal{F})$, $TS(\mathcal{F})$, $ST(\mathcal{F})$ and $SS(\mathcal{F})$. However, since

$$SS(z) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} (z) = \frac{-z+0}{0 \cdot z - 1} = z \,, \tag{5}$$

it follows that $SS(\mathcal{F}) = \mathcal{F}$. Therefore there are only three images with minimal word length 2. Any word that contains SS is not a word with minimal length. The word length can be reduced by cutting the string SS out of the word. The string SS is, so to speak, redundant. Among the eight words with length 3, the words TSS, SST and SSS do not have minimal length. Among the sixteen words with length 4, only 8 words have minimal length. For words of length 5 and longer another string becomes redundant. Since $TSTST(z) = S(z)$ it follows that $TSTST(\mathcal{F}) = S(\mathcal{F})$. Any word that contains $TSTST$ is never a word with minimal length, because the string $TSTST$ can be replaced by S. As a consequence of the redundancy of SS and $TSTST$ only 12 of the 32 words with length 5 have minimal length. In Figure 2 are shown some images of $\mathcal{F}$. Due to lack of space, the '$(\mathcal{F})$' is omitted in many image labels.

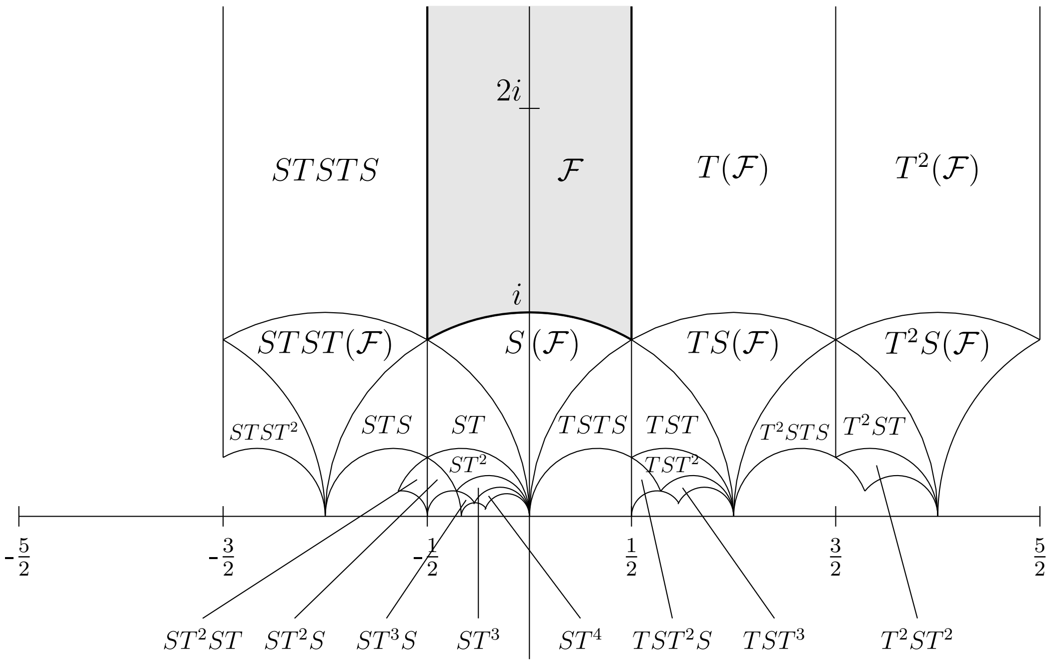

Figure 2: Images of $\mathcal{F}$.

The tree of images with increasing minimal word length is shown in the next graph.

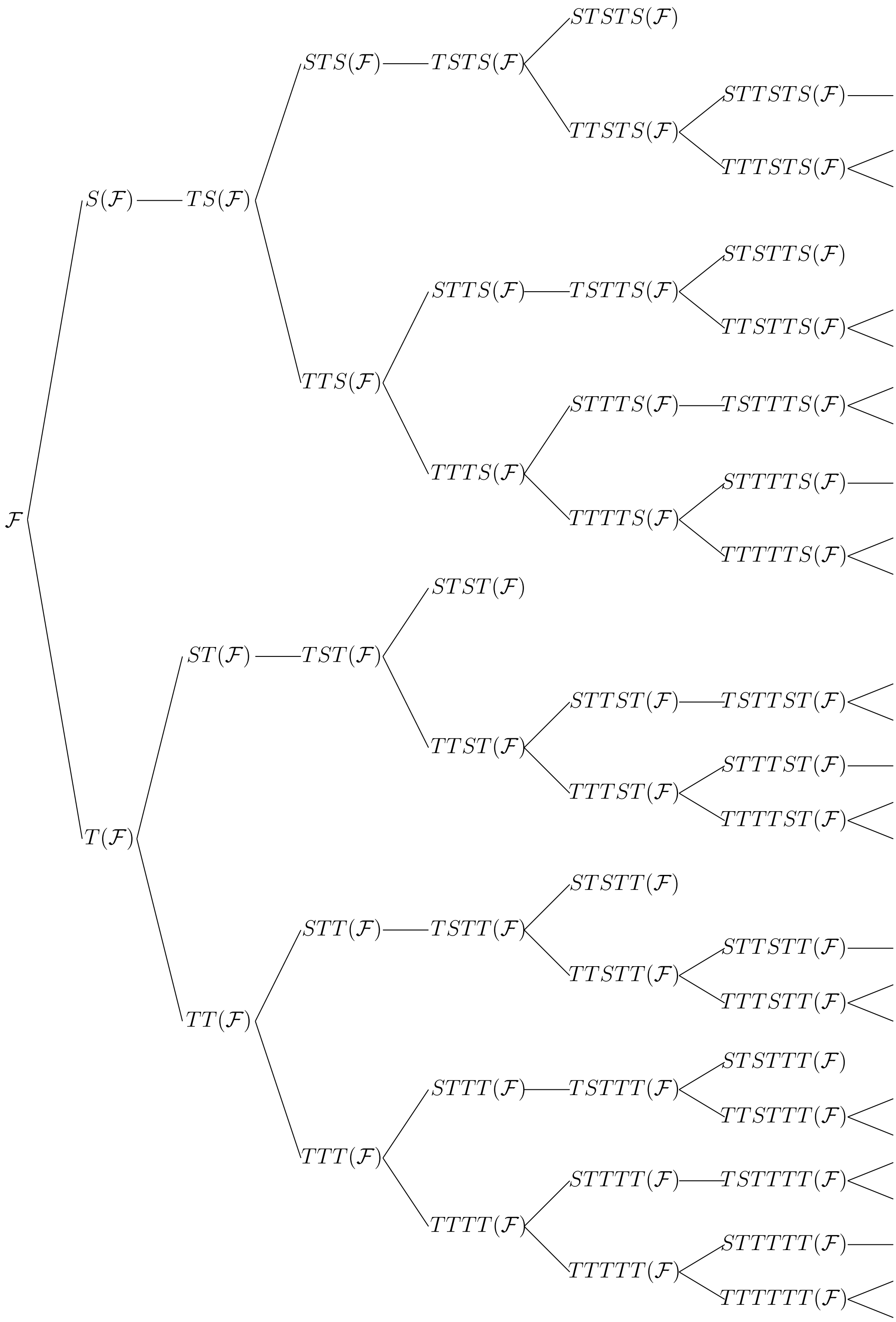

Figure 3: Tree of images.

The redundancy of SS and $TSTST$ is related to the fact that $S^2 = -I$ and $STSTST = -I$. Since SL(2,Z) is generated by S and ST any string equal to $-I$ must contain SS or $STSTST$. Therefore we only have to discard SS and $TSTST$ in the tree of images.

2 Recurrence relations

If we restrict ourselves to discarding words with SS, then an image with S on the left has one successor, while an image with T on the left has two successors. We show it is the arithmetic which leads to the Fibonacci numbers. Let σ_k be the number of images with no SS in the word, with word length k and with S on the left of the word. Let τ_k be the number of images with no SS in the word, with word length k and with T on the left of the word. Let ω_k be the number of images with no SS in the word and with word length k. An image with no SS in the word, with word length k and with S on the left of its word has only one successor: the same word with a T added to the left. An image with no SS in the word, with word length k and with T on the left has two successors: one with S added on the left and one with T added on the left. Therefore

$$\sigma_{k+1} = \tau_k \tag{6}$$

and

$$\tau_{k+1} = \sigma_k + \tau_k \, . \tag{7}$$

The initial values are $\sigma_1 = 1$ and $\tau_1 = 1$. The latter two equations imply the following second order system:

$$\tau_{k+1} = \tau_k + \tau_{k-1} \, , \tag{8}$$

with initial value $\tau_1 = 1$ and $\tau_2 = 2$. The latter is the recurrence relation that defines the Fibonacci numbers. If only the string SS is discarded, we would get the following table.

k	1	2	3	4	5	6	7	8	9	10
σ_k	1	1	2	3	5	8	13	21	34	55
τ_k	1	2	3	5	8	13	21	34	55	89
ω_k	2	3	5	8	13	21	34	55	89	144

Table 1: Fibonacci arithmetic.

For the counting of images with minimal word length both SS and $TSTST$ have to be discarded. Hereafter we let σ_k be the number of images with minimal word length k and with S on the left of the word. We let τ_k be the number of images with minimal word length k and with T on the left of the word. We let ω_k be the number of images with minimal word length k. For images with minimal word length the table is

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
σ_k	1	1	2	3	5	7	10	15	22	32	47	69	101	148	217	318
τ_k	1	2	3	5	7	10	15	22	32	47	69	101	148	217	318	466
ω_k	2	3	5	8	12	17	25	37	54	79	116	170	249	365	535	784

Table 2: Arithmetic for images with minimal word length.

For $k \geq 4$ we recognize in the latter table the following recurrence relations:

$$\sigma_{k+1} = \tau_k \tag{9}$$

and

$$\tau_{k+1} = \tau_k + \tau_{k-2}\,. \tag{10}$$

The equation (9) is self-evident. We will explain the recurrence relation (10) on the basis of the graph in Figure 3.

At the row of 4-step words we see the branch $STST(\mathcal{F})$ without a successor. Discarding the successor $TSTST(\mathcal{F})$ implies $\tau_5 = \tau_4 + \sigma_4 - 1$.

At the row of 5-step words we see both $STSTS(\mathcal{F})$ and $STSTT(\mathcal{F})$ without a successor. Discarding successors $TSTSTS(\mathcal{F})$ and $TSTSTT(\mathcal{F})$ implies $\tau_6 = \tau_5 + \sigma_5 - 2$.

At the row of 6-step words we see both $STSTTS(\mathcal{F})$ and $STSTTT(\mathcal{F})$ have no successor. Discarding the successors $TSTSTTS(\mathcal{F})$ and $TSTSTTT(\mathcal{F})$ implies $\tau_7 = \tau_6 + \sigma_6 - 2$.

A systematic way to arrive at the latter result is by adding from the row of 2-step words TS, ST and TT to the right of $STST$. For ST it leads to $STSTST$, which was discarded earlier because of the presence of $TSTST$. For TS and TT it results in the legitimate 6-step words $STSTTS$ and $STSTTT$ respectively. These two 6-step words have no successor, because the action of S on the words leads to SS on the left and the action of T on the words leads to $TSTST$ on the left. Since TS and TT are the 2-step words in Figure 3 starting with a T, the decrement by 2 is actually a decrement by τ_2. That is, $\tau_7 = \tau_6 + \sigma_6 - \tau_2$.

Applying the systematic approach to 7-step words, we add from the row of 3-step words STS, TTS, TST, STT and TTT to the right of $STST$. For STS and STT it leads to $STSTSTS$ and $STSTSTT$, which were discarded discarded earlier because of the presence of $TSTST$. For TTS, TST and TTT it results in the legitimate 7-step words $STSTTTS$,

$STSTTST$ and $STSTTTT$ respectively. These three 7-step words have no successor for the same reason as for 6-step words: the action of S on the words leads to SS on the left and the action of T on the words leads to $TSTST$ on the left. Since TTS, TST and TTT are the 3-step words in Figure 3 starting with a T, the decrement by 2 is actually a decrement by τ_2. So, here we have $\tau_8 = \tau_7 + \sigma_7 - \tau_3$.

In general, for obtaining legitimate $k+1$-step words, we only have to add to the right of $STST$ the $k-4$-step words with a T at the left. The τ_{k-4} words with $k-4$ steps will have no successor. We therefore have

$$\tau_{k+1} = \tau_k + \sigma_k - \tau_{k-4} \,. \tag{11}$$

Substitution of $\sigma_k = \tau_{k-1}$ gives

$$\tau_{k+1} = \tau_k + \tau_{k-1} - \tau_{k-4} \,. \tag{12}$$

The latter is equivalent to the equation (10). To see the equivalence we rearrange the equation (10) to

$$\tau_{k-2} = \tau_{k+1} - \tau_k \,, \tag{13}$$

which is identical to

$$\tau_{k-4} = \tau_{k-1} - \tau_{k-2} \,. \tag{14}$$

The substitution of the latter into equation (12) results in equation (10).

Since $\sigma_{k+1} = \tau_k$ we have for $k \geq 5$

$$\sigma_{k+1} = \sigma_k + \sigma_{k-2} \,. \tag{15}$$

Since $\omega_{k+1} = \tau_{k+1} + \sigma_{k+1}$ there holds for $k \geq 5$

$$\omega_{k+1} = \omega_k + \omega_{k-2} \,. \tag{16}$$

The recurrence relation for σ_k, τ_k and ω_k is identical to the recurrence relation for Narayana's cow sequence [3]. The generated numbers differ only because of the different initial conditions. The initial conditions for σ_k are $\sigma_3 = 2$, $\sigma_4 = 3$ and $\sigma_5 = 5$. The initial conditions for τ_k are $\tau_2 = 2$, $\tau_3 = 3$ and $\tau_4 = 5$. The initial conditions for ω_k are $\omega_3 = 5$, $\omega_4 = 8$ and $\omega_5 = 12$.

3 Closed forms

In this section we derive a closed forms for σ_k, τ_k and ω_k.

For $k \geq 3$ a closed form for the sequence σ_k is given by

$$\sigma_k = A\alpha^k + B\beta^k + C\gamma^k \tag{17}$$

where A, B and C are constants, and where α, β and γ are the roots of the characteristic equation

$$\lambda^3 - \lambda^2 - 1 = 0\,. \tag{18}$$

The three roots of the characteristic equation are

$$\alpha = \frac{1 + \nu_- + \nu_+}{3}\,, \quad \beta = \frac{1 - \omega_+\nu_- - \omega_-\nu_+}{3}\,, \quad \gamma = \frac{1 - \omega_-\nu_- - \omega_+\nu_+}{3}\,. \tag{19}$$

where

$$\nu_\pm = \sqrt[3]{\frac{29 \pm 3\sqrt{93}}{2}}\,, \qquad \omega_\pm = \frac{1 \pm i\sqrt{3}}{2} \tag{20}$$

To obtain simple expressions for the constants A, B and C, we use the values σ_0, σ_1 and σ_2 would have had if the recurrence relation $\sigma_{k+1} = \sigma_k + \sigma_{k-2}$ would hold for $k \geq 0$. We denote these values as $\bar{\sigma}_0$, $\bar{\sigma}_1$ and $\bar{\sigma}_2$ respectively. With reversed engineering we then find $\bar{\sigma}_2 = \sigma_5 - \sigma_4 = 2$, $\bar{\sigma}_1 = \sigma_4 - \sigma_3 = 1$ and $\bar{\sigma}_0 = \sigma_3 - \bar{\sigma}_2 = 0$. By means of these three values we obtain for the constants A, B and C the equations

$$A + B + C = 0\,, \quad A\alpha + B\beta + C\gamma = 1\,, \quad A\alpha^2 + B\beta^2 + C\gamma^2 = 2\,. \tag{21}$$

The solution is

$$A = \frac{2 - (\beta + \gamma)}{\alpha^2 - \alpha(\beta + \gamma) + \beta\gamma}\,, \quad B = \frac{2 - (\gamma + \alpha)}{\beta^2 - \beta(\gamma + \alpha) + \gamma\alpha}\,, \quad C = \frac{2 - (\alpha + \beta)}{\gamma^2 - \gamma(\alpha + \beta) + \alpha\beta}\,. \tag{22}$$

To simplify the expressions for A, B and C the application of Vieta's formulas is useful [3]. For the characteristic equation (18) Vieta's formulas are

$$\alpha + \beta + \gamma = 1\,, \quad \alpha\beta + \alpha\gamma + \beta\gamma = 0\,, \quad \alpha\beta\gamma = 1\,. \tag{23}$$

By means of Vieta's formulas and the identity $\alpha^3 - \alpha^2 - 1 = 0$ the expression for A can be simplified to

$$A = \frac{2 - (1 - \alpha)}{\alpha^2 - \alpha(1 - \alpha) + \beta\gamma} = \frac{\alpha + 1}{2\alpha^2 - \alpha + \beta\gamma} = \frac{\alpha^2 + \alpha}{2\alpha^3 - \alpha^2 + 1} = \frac{\alpha^2 + \alpha}{\alpha^2 + 3}\,. \tag{24}$$

In a similar way we obtain

$$B = \frac{\beta^2 + \beta}{\beta^2 + 3}\,, \qquad C = \frac{\gamma^2 + \gamma}{\gamma^2 + 3}\,. \tag{25}$$

The constants A, B and C can be written as a second degree polynomial in α, β and γ respectively. Repeatedly applying $\alpha^3 = \alpha^2 + 1$ we obtain the identity

$$\begin{aligned} \left(7\alpha^2 + 8\alpha - 5\right)\left(\alpha^2 + 3\right) &= 7\alpha^4 + 8\alpha^3 + 16\alpha^2 + 24\alpha - 15 \\ &= 7\alpha(\alpha^2 + 1) + 8(\alpha^2 + 1) + 16\alpha^2 + 24\alpha - 15 \\ &= 7\alpha^3 + 7\alpha + 24\alpha^2 + 24\alpha - 7 \\ &= 7\alpha^2 + 7 + 24\alpha^2 + 31\alpha - 7 \\ &= 31\left(\alpha^2 + \alpha\right)\,. \end{aligned} \tag{26}$$

The same identity holds for β and γ. Quadratic equations for the constants therefore are

$$A = \frac{7\alpha^2 + 8\alpha - 5}{31} \quad , \qquad B = \frac{7\beta^2 + 8\beta - 5}{31} \quad , \qquad C = \frac{7\gamma^2 + 8\gamma - 5}{31} \, . \tag{27}$$

Substituting (19) and using $\nu_+\nu_- = 1$, $\omega_+\omega_- = 1$, $\omega_+^2 = -\omega_-$, $\omega_-^2 = -\omega_+$ we obtain

$$A = \frac{38}{279}\left(\nu_+ + \nu_-\right) + \frac{7}{279}\left(\nu_+^2 + \nu_-^2\right) \tag{28}$$

$$B = -\frac{38}{279}\left(\nu_+\omega_- + \nu_-\omega_+\right) - \frac{7}{279}\left(\nu_+^2\omega_+ + \nu_-^2\omega_-\right) \tag{29}$$

and

$$C = -\frac{38}{279}\left(\nu_+\omega_+ + \nu_-\omega_-\right) - \frac{7}{279}\left(\nu_+^2\omega_- + \nu_-^2\omega_+\right) \, . \tag{30}$$

For $k \geq 2$ a closed form for the sequence τ_k is given by

$$\tau_k = A\alpha^k + B\beta^k + C\gamma^k \tag{31}$$

where α, β and γ are given by (19). To obtain simple expressions for the constants A, B and C, we use the values τ_0, τ_1 and τ_2 would have if the recurrence relation $\tau_{k+1} = \tau_k + \tau_{k-2}$ would hold for $k \geq 0$. We denote these values as $\bar{\tau}_0$, $\bar{\tau}_1$ and $\bar{\tau}_2$ respectively. With reversed engineering we then find $\bar{\tau}_2 = \tau_2 = 2$, $\bar{\tau}_1 = \tau_4 - \tau_3 = 2$ and $\bar{\tau}_0 = \tau_3 - \tau_2 = 1$. By means of these three values we obtain for the constants A, B and C the equations

$$A + B + C = 1 \, , \quad A\alpha + B\beta + C\gamma = 2 \, , \quad A\alpha^2 + B\beta^2 + C\gamma^2 = 2 \, . \tag{32}$$

The solution is

$$A = \frac{2 - 2(\beta + \gamma) + \beta\gamma}{\alpha^2 - \alpha(\beta + \gamma) + \beta\gamma} \, , \quad B = \frac{2 - 2(\gamma + \alpha) + \gamma\alpha}{\beta^2 - \beta(\gamma + \alpha) + \gamma\alpha} \, , \quad C = \frac{2 - 2(\alpha + \beta) + \alpha\beta}{\gamma^2 - \gamma(\alpha + \beta) + \alpha\beta} \, . \tag{33}$$

By means of Vieta's formulas (23) the expression for A, B and C can be simplified to

$$A = \frac{2\alpha^2 + 1}{\alpha^2 + 3} \quad , \qquad B = \frac{2\beta^2 + 1}{\beta^2 + 3} \quad , \qquad C = \frac{2\gamma^2 + 1}{\gamma^2 + 3} \, . \tag{34}$$

Also here the constants for A, B and C can be written as a second degree polynomial in α, β and γ respectively. Applying $\alpha^3 = \alpha^2 + 1$ we obtain identity

$$\left(15\alpha^2 - 5\alpha + 7\right)\left(\alpha^2 + 3\right) = 31\left(2\alpha^2 + 1\right) \, . \tag{35}$$

The same identity holds for β and γ. Quadratic equations for the constants therefore are

$$A = \frac{15\alpha^2 - 5\alpha + 7}{31} \quad , \qquad B = \frac{15\beta^2 - 5\beta + 7}{31} \quad , \qquad C = \frac{15\gamma^2 - 5\gamma + 7}{31} \, . \tag{36}$$

Substituting (19) and using $\nu_+\nu_- = 1$, $\omega_+\omega_- = 1$, $\omega_+^2 = -\omega_-$, $\omega_-^2 = -\omega_+$ we obtain

$$A = \frac{1}{3} + \frac{5}{93}\left(\nu_+ + \nu_+^2 + \nu_- + \nu_-^2\right) \tag{37}$$

$$B = \frac{1}{3} - \frac{5}{93}\left(\nu_+\omega_- + \nu_+^2\omega_+ + \nu_-\omega_+ + \nu_-^2\omega_-\right) \tag{38}$$

and

$$C = \frac{1}{3} - \frac{5}{93}\left(\nu_+\omega_+ + \nu_+^2\omega_- + \nu_-\omega_- + \nu_-^2\omega_+\right) . \tag{39}$$

For $k \geq 3$ a closed form for the sequence ω_k is given by

$$\omega_k = A\alpha^k + B\beta^k + C\gamma^k \tag{40}$$

where α, β and γ are given by (19). To obtain simple expressions for the constants A, B and C, we use the values ω_0, ω_1 and ω_2 would have if the recurrence relation $\omega_{k+1} = \omega_k + \omega_{k-2}$ would hold for $k \geq 0$. We denote these values as $\bar{\omega}_0$, $\bar{\omega}_1$ and $\bar{\omega}_2$ respectively. With reversed engineering we then find $\bar{\omega}_2 = \omega_5 - \omega_4 = 4$, $\bar{\omega}_1 = \omega_4 - \omega_3 = 3$ and $\bar{\omega}_0 = \omega_3 - \bar{\omega}_2 = 1$. By means of these three values we obtain for the constants A, B and C the equations

$$A + B + C = 1\,, \quad A\alpha + B\beta + C\gamma = 3\,, \quad A\alpha^2 + B\beta^2 + C\gamma^2 = 4\,. \tag{41}$$

The solution is

$$A = \frac{4 - 3(\beta+\gamma) + \beta\gamma}{\alpha^2 - \alpha(\beta+\gamma) + \beta\gamma}\,, \quad B = \frac{4 - 3(\gamma+\alpha) + \gamma\alpha}{\beta^2 - \beta(\gamma+\alpha) + \gamma\alpha}\,, \quad C = \frac{4 - 3(\alpha+\beta) + \alpha\beta}{\gamma^2 - \gamma(\alpha+\beta) + \alpha\beta}\,. \tag{42}$$

By means of Vieta's formulas (23) the expression for A, B and C can be simplified to

$$A = \frac{3\alpha^2 + \alpha + 1}{\alpha^2 + 3} \quad , \quad B = \frac{3\beta^2 + \beta + 1}{\beta^2 + 3} \quad , \quad C = \frac{3\gamma^2 + \gamma + 1}{\gamma^2 + 3}\,. \tag{43}$$

Also here the constants for A, B and C can be written as a second degree polynomial in α, β and γ respectively. Applying $\alpha^3 = \alpha^2 + 1$ we obtain the identity

$$\left(22\alpha^2 + 3\alpha + 2\right)\left(\alpha^2 + 3\right) = 31\left(3\alpha^2 + \alpha + 1\right) . \tag{44}$$

Similar identities are obtained for β and γ. Quadratic equations for the constants therefore are

$$A = \frac{22\alpha^2 + 3\alpha + 2}{31} \quad , \quad B = \frac{22\beta^2 + 3\beta + 2}{31} \quad , \quad C = \frac{22\gamma^2 + 3\gamma + 2}{31}\,. \tag{45}$$

Substituting (19) and using $\nu_+\nu_- = 1$, $\omega_+\omega_- = 1$, $\omega_+^2 = -\omega_-$, $\omega_-^2 = -\omega_+$ we obtain

$$A = \frac{1}{3} + \frac{53}{279}\left(\nu_+ + \nu_-\right) + \frac{22}{279}\left(\nu_+^2 + \nu_-^2\right) \tag{46}$$

$$B = \frac{1}{3} - \frac{53}{279}\left(\nu_+\omega_- + \nu_-\omega_+\right) - \frac{22}{279}\left(\nu_+^2\omega_+ + \nu_-^2\omega_-\right) \tag{47}$$

and

$$C = \frac{1}{3} - \frac{53}{279}\left(\nu_+\omega_+ + \nu_-\omega_-\right) - \frac{22}{279}\left(\nu_+^2\omega_- + \nu_-^2\omega_+\right) . \tag{48}$$

4 Conclusions

The numbers ω_k of fundamental domain images with minimal word length k initially obey the Fibonacci recurrence: $\omega_{k+1} = \omega_k + \omega_{k-1}$ for $k \leq 4$. For $k \geq 5$ the numbers ω_k obey the Narayana equation (16). The sequence of ω_k differs from Narayana's cow sequence because of the different initial conditions.

A closed form can be constructed for the sequence of ω_k, albeit an inconvenient one. From sequence A097333 of the OEIS [4] it can be inferred that

$$\tau_k = \sum_{m=0}^{k} \binom{k-m}{\lfloor \frac{m}{2} \rfloor} \quad , \quad k \geq 2 \, , \tag{49}$$

$$\sigma_k = \sum_{m=0}^{k-1} \binom{k-1-m}{\lfloor \frac{m}{2} \rfloor} \quad , \quad k \geq 3 \, . \tag{50}$$

and ω_k is the sum of the latter two summations for $k \geq 3$. The latter also is not very practical. The sequence ω_k is most conveniently computed using the recurrence relation (16) itself.

For $k = 1$ through 50 the sequence for ω_k is
$1, 2, 3, 5, 8, 12, 17, 25, 37, 54, 79, 116, 170, 249, 365, 535, 784, 1149, 1684, 2468, 3617, 5301,$
$7769, 11386, 16687, 24456, 35842, 52529, 76985, 112827, 165356, 242341, 355168,$
$520524, 762865, 1118033, 1638557, 2401422, 3519455, 5158012, 7559434, 11078889,$
$16236901, 23796335, 34875224, 51112125, 74908460, 109783684, 160895809, 235804269.$

For $\omega_k \geq 5$ the sequence for ω_k also appears in the sequence A179070 and the sequence A372760 of the OEIS [4].

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