From Ramanujan to Riemann hypothesis

©Payam Danesh [1], ©Raoul Bianchetti [2]

[1] Department of Bio-systems Engineering, University of Tehran, Iran

[2] Information Physics Institute, Gosport, Hampshire, United Kingdom

Email: payamdanesh71@gmail.com

Abstract

In this work we offer a careful framework for approaching the critical-line problem associated with the Riemann zeta function. At its heart is a long-standing divide in the subject. On one side are analytic approaches, which study the completed zeta function through its reflection symmetry. On the other side are arithmetic approaches, where related criteria often appear through extreme behavior in divisor functions. The purpose of this paper is not to claim a proof of the Riemann Hypothesis, but to place these two perspectives into a clearer and more usable relationship. The argument begins with reflected analytic data for the completed zeta function. It shows that such data can be described through an odd analytic perturbation, giving a more organized way to understand the analytic side of the problem. This also resolves a common point of confusion: the full complex defect is not required to vanish on the critical line. What matters is more subtle. Under a natural real-symmetry condition, the real part of the defect vanishes on the critical line, and this is the feature that becomes useful for the bridge argument. The arithmetic side is built around Ramanujan's logarithmic divisor profile. The paper establishes the existence and positivity of the relevant extreme scale in the range needed for the proposed connection. These analytic and arithmetic pieces are then brought together through a real bridge functional, made up of a main sign term and a correction term. The main outcome is a conditional criterion for the critical line. If the bridge functional is zero-adapted at the nontrivial zeros, if the real analytic defect satisfies the required one-sided sign condition, and if the correction term remains strictly smaller than the main term, then every nontrivial zero must lie on the critical line. The contribution of this work is therefore structural rather than conclusive. It does not present the Riemann Hypothesis as solved. Instead, it separates what is already established from what still needs to be proved. The key sign law, the domination estimate, and the zero-adaptation identity remain open requirements for any future application of the framework. Its practical value is that it gives researchers a precise checklist for testing whether a proposed analytic or arithmetic strategy can genuinely support a critical-line argument.

Keywords: Riemann Hypothesis; critical line; reflection symmetry; analytic defect; Ramanujan divisor profile

1. Introduction

The Riemann Hypothesis grew out of Riemann's study of the prime-counting problem and remains one of the central open questions in mathematics [1]. In broad terms, it predicts that the nontrivial zeros of the zeta function are arranged on a single vertical line, and this predicted geometry is deeply connected with the error term in prime number distribution. Standard analytic treatments develop the completed zeta function, its functional equation, and the analytic continuation needed to study the critical strip [2]. Edwards gives a historically oriented account of the same circle of ideas, while Ivić records many of the analytic estimates surrounding the zeta function [3], [4]. Large-scale computation has verified the predicted line placement for an enormous initial range of zeros, but such verification is not a proof for all zeros [5].

A second line of research reformulates the problem through arithmetic inequalities. Lagarias gave an elementary criterion equivalent to the Riemann Hypothesis [6]. Nicolas and Robin produced criteria involving extremal behaviour of arithmetic functions closely related to divisor sums [7], [8]. These results show that the critical-line problem is not only a question about complex analysis; it can also be expressed as a problem about sharp arithmetic growth. Ramanujan's work is naturally relevant to this interface. His formulas for the completed zeta-side functions belong to the analytic side [9], while his theory of highly composite numbers introduced a systematic language for divisor extremality [10]. Later work clarified the structure of highly composite and related numbers in greater detail.

The present paper develops a conditional bridge between these two viewpoints. The point is not to offer an unconditional proof of the Riemann Hypothesis. Rather, the purpose is to isolate exactly what must be proved for a Ramanujan-inspired analytic-arithmetic mechanism to force the critical line. This distinction is important because many proposed approaches to the problem confuse reflection symmetry with coercivity. A symmetric expression can vanish or change sign on a line, but symmetry alone does not imply that the expression has a fixed sign on either side. The framework below keeps these issues separate.

The analytic datum is a function T satisfying

$$T(s) + T(1-s) = \xi(s). \tag{1.1}$$

The associated defect is

$$D_T(s) = T(s) - T(1-s). \tag{1.2}$$

It follows that $D_T(1-s) = -D_T(s)$. Since $1 - (1/2 + it) = 1/2 - it$, this identity relates two reflected points on the critical line; it does not imply that $D_T(1/2 + it)$ itself is zero. Under real-admissibility, the correct vanishing statement is

$$\mathrm{Re} D_T\left(\frac{1}{2} + it\right) = 0. \tag{1.3}$$

For that reason the bridge in this paper is built from the real defect $\mathrm{Re} D_T$, not from the full complex defect.

The arithmetic datum is Ramanujan's logarithmic divisor profile

$$\Lambda_\varepsilon(N) = \log d(N) - \varepsilon \log N, \tag{1.4}$$

where $d(N)$ is the divisor function. For $0 < \varepsilon < 1$, this profile has a positive global maximum. The corresponding maximizer is a superior highly composite scale, and its maximum value gives a positive arithmetic weight L_ε. The bridge functional is then defined by

$$\mathcal{B}_\varepsilon(\sigma, t) = L_\varepsilon \, \mathrm{Re} D_T(\sigma + it) + R_\varepsilon(\sigma, t). \tag{1.5}$$

The proof mechanism is transparent. If this bridge vanishes at every nontrivial zero, while its sign is forced to agree with $\sigma - 1/2$ away from the critical line, then a zero off the critical line is impossible. The principal conclusion is therefore a conditional criterion whose hypotheses are explicit: a one-sided sign law for the real defect, a strict domination estimate for the correction, and a zero-adapted identity at the zeros. The novelty of the framework is the separation of these three burdens from the identities that can already be proved.

Quantitative map of the notation used in the paper is shown in **Table 1**. It contains nine objects: four analytic objects, three arithmetic objects, and two bridge-level objects. The count is intentional. The analytic entries define the reflected datum and the real defect; the arithmetic entries identify the divisor profile and its positive extremal weight; the bridge entries record the correction and the final expression whose sign is tested. The table fixes the notation before the proofs begin and makes clear which objects are structural, which are extremal, and which are conditional.

Table 1.Principal objects and their roles in the bridge criterion.

Symbol	Definition	Role
$\xi(s)$	Completed zeta function $\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$	Carries the functional equation $\xi(s)=\xi(1-s)$.
$T(s)$	Reflected datum satisfying $T(s)+T(1-s)=\xi(s)$	Splits the completed function into two reflected pieces.
$D_T(s)$	$T(s)-T(1-s)$	Measures the reflected imbalance of the analytic datum.
$\mathrm{Re}D_T(\sigma+it)$	Real part of the defect	The quantity that vanishes on the critical line under real-admissibility.
$d(N)$	Number of positive divisors of N	Arithmetic input from Ramanujan's theory of highly composite numbers.
$\Lambda_\varepsilon(N)$	$\log d(N)-\varepsilon\log N$	Logarithmic divisor profile; its maximum selects an extremal scale.
L_ε	$\max_N \Lambda_\varepsilon(N)$ for $0<\varepsilon<1$	Positive arithmetic weight in the bridge.
$R_\varepsilon(\sigma,t)$	Real correction term	The part that must be dominated by the main term.
$\mathcal{B}_\varepsilon$	$L_\varepsilon\mathrm{Re}D_T+R_\varepsilon$	Bridge whose zero-adaptation and sign barrier imply the critical-line conclusion.

2. The completed zeta function and reflected data

The analytic side begins with the completed zeta function

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{2.1}$$

This normalization removes the pole of $\zeta(s)$ at $s=1$ and places the functional equation in its symmetric form. The function ξ is entire, satisfies

$$\xi(s)=\xi(1-s), \tag{2.2}$$

and has the real symmetry

$$\xi(\overline{s})=\overline{\xi(s)}. \tag{2.3}$$

The zeros of ξ are precisely the nontrivial zeros of ζ, counted with multiplicity.

Definition 1 (Admissible reflected datum). *Let $U \subset \mathbb{C}$ be a connected open set containing the critical strip* $0 < \mathrm{Re}\, s < 1$. *Assume that U is invariant under both $s \mapsto 1-s$ and $s \mapsto \bar{s}$. An analytic function $T: U \to \mathbb{C}$ is an admissible reflected datum if*

$$T(s) + T(1-s) = \xi(s) \qquad (s \in U). \tag{2.4}$$

Definition 2 (Real-admissibility). *An admissible reflected datum is real-admissible if*

$$T(\bar{s}) = \overline{T(s)} \ (s \in U). \tag{2.5}$$

Definition 3 (Reflected analytic defect). *The defect attached to T is $D_T(s) = T(s) - T(1-s)$.* (2.6)

Proposition 4 (Classification of reflected data). *The admissible reflected data on U are exactly the functions*

$$T(s) = \tfrac{1}{2}\xi(s) + H(s), \tag{2.7}$$

where H is analytic on U and satisfies $H(1-s) = -H(s)$. (2.8)

For such a datum, $D_T(s) = 2H(s)$. (2.9)

Moreover, T is real-admissible if and only if $H(\bar{s}) = \overline{H(s)}$. (2.10)

Proof. Suppose first that T is admissible and put

$$H(s) = T(s) - \tfrac{1}{2}\xi(s). \tag{2.11}$$

Using (2.2) and (2.4), one obtains

$$H(1-s) = T(1-s) - \tfrac{1}{2}\xi(1-s) = \xi(s) - T(s) - \tfrac{1}{2}\xi(s) = -\left(T(s) - \tfrac{1}{2}\xi(s)\right) = -H(s). \tag{2.12}$$

Thus T has the form (2.7) with H satisfying (2.8). Conversely, if T is defined by (2.7) and H satisfies (2.8), then

$$T(s) + T(1-s) = \tfrac{1}{2}\xi(s) + H(s) + \tfrac{1}{2}\xi(1-s) + H(1-s) = \xi(s), \tag{2.13}$$

so T is admissible. Subtracting $T(1-s)$ from $T(s)$ gives (2.9). The final assertion follows from (2.3) and the identity $H = T - \xi/2$.

Proposition 5 (Recovery from the defect). *For every admissible datum T,*

$$T(s) = \tfrac{1}{2}\big(\xi(s) + D_T(s)\big), T(1-s) = \tfrac{1}{2}\big(\xi(s) - D_T(s)\big). \tag{2.14}$$

Thus the pair (ξ, D_T) determines T.

Proof. Add and subtract the two identities

$$\xi(s) = T(s) + T(1-s), \qquad D_T(s) = T(s) - T(1-s). \tag{2.15}$$

Proposition 6 (Symmetry of the defect). *For every admissible datum T, $D_T(1-s) = -D_T(s)$.* (2.16)

If T is real-admissible, then $D_T(\bar{s}) = \overline{D_T(s)}$. (2.17)

Consequently, for $s = \sigma + it$,

$$\mathrm{Re}D_T(1-\sigma+it) = -\mathrm{Re}D_T(\sigma+it), \tag{2.18}$$

and hence $\mathrm{Re}D_T\left(\frac{1}{2}+it\right) = 0 \qquad (t \in \mathbb{R})$. (2.19)

Proof. Equation (2.16) follows immediately from (2.6):

$$D_T(1-s) = T(1-s) - T(s) = -D_T(s). \tag{2.20}$$

If T is real-admissible, then

$$D_T(\bar{s}) = T(\bar{s}) - T(1-\bar{s}) = \overline{T(s)} - \overline{T(1-s)} = \overline{D_T(s)}. \tag{2.21}$$

Let $s = \sigma + it$. Since $1 - \bar{s} = 1 - \sigma + it$, (2.16) and (2.17) give

$$D_T(1-\sigma+it) = D_T(1-\bar{s}) = -D_T(\bar{s}) = -\overline{D_T(s)}. \tag{2.22}$$

Taking real parts proves (2.18). Setting $\sigma = 1/2$ proves (2.19).

Remark 7. *The full defect generally does not vanish on the critical line. The assertion supplied by the reflected symmetry and real-admissibility is exactly (2.19). This distinction is essential for the bridge constructed below.*

Example 8. *Let* $H(s) = s - \frac{1}{2}, \qquad T(s) = \frac{1}{2}\xi(s) + s - \frac{1}{2}$. (2.23)

Then $H(1-s) = -H(s)$,

so T is admissible and real-admissible. The defect is $D_T(s) = 2s - 1$. (2.24)

On the critical line, $D_T\left(\frac{1}{2}+it\right) = 2it$, (2.25)

which is nonzero unless $t = 0$.

However $\mathrm{Re}D_T(\sigma+it) = 2\sigma - 1$, (2.26)

so the real defect vanishes exactly on $\sigma = 1/2$ and has the sign of $\sigma - 1/2$.

3. Ramanujan's divisor profile

Let

$$d(N) = \sum_{m|N} 1 \tag{3.1}$$

be the divisor function. For $\varepsilon > 0$, define

$$F_\varepsilon(N) = \frac{d(N)}{N^\varepsilon}, \Lambda_\varepsilon(N) = \log F_\varepsilon(N) = \log d(N) - \varepsilon \log N. \tag{3.2}$$

This is the logarithmic profile associated with Ramanujan's superior highly composite condition. The extremal behaviour of such divisor functions is part of the classical tradition developed further by Alaoglu and Erdős and by Gronwall [11], [12].

Definition 9 (Superior highly composite number for a parameter). *A positive integer N is superior highly composite for the parameter $\varepsilon > 0$ if $F_\varepsilon(M) \le F_\varepsilon(N) \qquad (M \le N)$,* (3.3)

and $F_\varepsilon(M) < F_\varepsilon(N)(M > N)$. (3.4)

Equivalently, the same inequalities hold with F_ε replaced by Λ_ε.

Lemma 10 (Logarithmic form of the superior condition). *A positive integer N is superior highly composite for the parameter $\varepsilon > 0$ if and only if*

$$\Lambda_\varepsilon(M) \le \Lambda_\varepsilon(N) \quad (M \le N), \qquad \Lambda_\varepsilon(M) < \Lambda_\varepsilon(N) \quad (M > N). \tag{3.5}$$

Proof. The logarithm is strictly increasing on $(0, \infty)$. Taking logarithms in

$$\frac{d(M)}{M^\varepsilon} \le \frac{d(N)}{N^\varepsilon} \tag{3.6}$$

gives

$$\log d(M) - \varepsilon \log M \le \log d(N) - \varepsilon \log N. \tag{3.7}$$

The strict inequality for $M > N$ is treated in the same way. Exponentiating proves the converse.

The elementary estimate used below is standard in analytic number theory [13].

Lemma 11 (Elementary divisor bound). *For every $\delta > 0$ there is a constant $C_\delta > 0$ such that*

$$d(N) \le C_\delta N^\delta \qquad (N \ge 1). \tag{3.8}$$

Proof. Write $N = \prod_p p^{a_p}$, with only finitely many nonzero exponents. Choose P so that $p^\delta \ge 2$ for every prime $p \ge P$. For $p \ge P$ and $a \ge 0$,

$$a + 1 \le 2^a \le p^{a\delta}. \tag{3.9}$$

For each prime $p < P$, the number

$$C_{p,\delta} = \sup_{a \ge 0} (a+1) p^{-a} \tag{3.10}$$

is finite. Therefore

$$a_p + 1 \le C_{p,\delta} p^{a_p \delta} \quad (p < P), \qquad a_p + 1 \le p^{a_p \delta} \quad (p \ge P). \tag{3.11}$$

Multiplying over the primes dividing N gives

$$d(N) = \prod_p (a_p + 1) \le \left(\prod_{p<P} C_{p,\delta}\right) \prod_p p^{a_p \delta} = C_\delta N^\delta. \tag{3.12}$$

Proposition 12 (Existence and positivity of the arithmetic scale). *For every $\varepsilon > 0$, the function F_ε attains a global maximum on $\mathbb{N}$. Let N_ε be the largest integer at which this maximum is attained. Then N_ε is superior highly composite for the parameter ε.*

If $0 < \varepsilon < 1$, *then* $L_\varepsilon := \Lambda_\varepsilon(N_\varepsilon) > 0$. (3.13)

Proof. Apply Lemma 3.2 with $\delta = \varepsilon/2$. Then

$$F_\varepsilon(N) = \frac{d(N)}{N^\varepsilon} \le C_{\varepsilon/2} N^{-\varepsilon/2} \to 0 \qquad (N \to \infty). \tag{3.14}$$

Since $F_\varepsilon(1) = 1$, all maximizers lie in a finite initial segment of $\mathbb{N}$. Thus a global maximum exists. Choose N_ε to be the largest maximizer. Maximality gives (3.3), and the choice of the largest maximizer gives the strict inequality (3.4) whenever $M > N_\varepsilon$. Hence N_ε is superior highly composite for ε.

If $0 < \varepsilon < 1$, then

$$F_\varepsilon(2) = 2^{1-\varepsilon} > 1 = F_\varepsilon(1). \tag{3.15}$$

The global maximum is therefore greater than 1, and $\Lambda_\varepsilon(N_\varepsilon) = \log F_\varepsilon(N_\varepsilon) > 0$.

Corollary 13. *For* $0 < \varepsilon < 1$, N_ε *is highly composite:* $d(M) < d(N_\varepsilon) \quad (1 \le M < N_\varepsilon)$. (3.16)

Proof. If $M < N_\varepsilon$, then (3.3) gives

$$\frac{d(M)}{M^\varepsilon} \le \frac{d(N_\varepsilon)}{N_\varepsilon^\varepsilon}. \tag{3.17}$$

Thus

$$d(M) \le d(N_\varepsilon)\left(\frac{M}{N_\varepsilon}\right)^\varepsilon < d(N_\varepsilon). \tag{3.18}$$

Remark 14. *The existence of* L_ε *uses only the elementary estimate* $d(N) = O_\delta(N^\delta)$. *No prime number theorem or zero-free region is required at this stage.*

Figure 1 shows a quantitative illustration of the divisor profile in the finite range $2 \le N \le 400$ for the parameter $\varepsilon = 1/4$. The plot contains 399 evaluated values of $\Lambda_{1/4}(N)$, but only 12 of them form new running records. These record values occur at

$$2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360. \tag{3.19}$$

The record level rises from approximately 0.5199 at $N = 2$ to approximately 1.7065 at $N = 360$. Numerically, fewer than four percent of the plotted integers set a new record in this range. This sparsity explains why the bridge uses an extremal arithmetic scale rather than the full divisor sequence. The figure is explanatory; the proof itself uses the exact global maximizer N_ε.

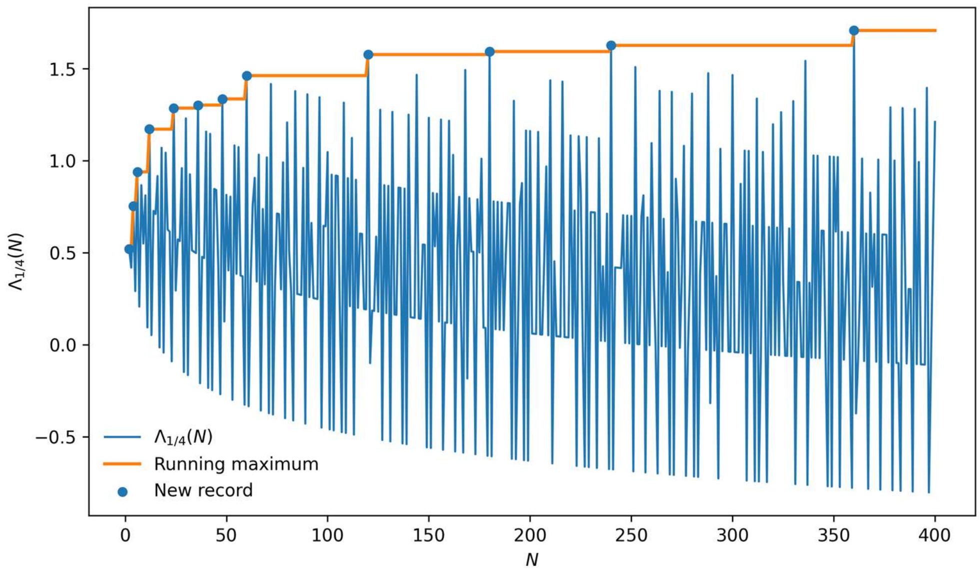

Figure 1. The profile $\Lambda_{1/4}(\mathrm{N}) = \log d(\mathrm{N}) - 1/4 \log \mathrm{N}$ for $2 \leq \mathrm{N} \leq 400$. The marked points are the new record values in this finite range.

4.The real bridge functional

The analytic construction supplies the real, reflection-odd quantity $\mathrm{Re}D_T$. The arithmetic construction supplies the positive scalar L_ε. A bridge functional is obtained by adding a correction term whose size and zero behaviour are to be controlled.

Definition 15 (Real bridge). *Let T be a real-admissible reflected datum, let $0 < \varepsilon < 1$, and let L_ε be defined by (3.13). A real bridge attached to (T, ε) is a function* $\mathcal{B}_\varepsilon\colon (0,1) \times \mathbb{R} \to \mathbb{R}$ (4.1)

of the form $\mathcal{B}_\varepsilon(\sigma, t) = L_\varepsilon \,\mathrm{Re}D_T(\sigma + it) + R_\varepsilon(\sigma, t),$ (4.2)

where $R_\varepsilon\colon (0,1) \times \mathbb{R} \to \mathbb{R}$ *is a correction term.*

Remark 16 (Main term and correction). *The term $L_\varepsilon \mathrm{Re}D_T(\sigma + it)$ is the part whose sign is meant to be controlled. The correction R_ε is kept separate because it must be estimated rather than absorbed into the datum T. This distinction is essential: without a quantitative bound on R_ε, the bridge can be made to vanish or change sign away from the critical line by an arbitrary choice of correction.*

Proposition 17 (Normalized form of the bridge). *Off the critical line, the bridge can be written as* $\mathcal{B}_\varepsilon(\sigma, t) = \left(\sigma - \frac{1}{2}\right)\left(L_\varepsilon A_T(\sigma, t) + \frac{R_\varepsilon(\sigma,t)}{\sigma - \frac{1}{2}}\right), \qquad \sigma \neq \frac{1}{2}.$ (4.3)

Proof. By the definition of A_T,

$$\mathrm{Re}D_T(\sigma + it) = \left(\sigma - \frac{1}{2}\right) A_T(\sigma, t) \qquad (\sigma \neq 1/2). \quad (4.4)$$

Substituting this identity into (4.2) gives (4.3).

Definition 18 (Critical normalization). *The bridge is critically normalized if* $R_\varepsilon\left(\frac{1}{2},t\right)0(t\in\mathbb{R})$. (4.5)

Proposition 19 (Critical-line vanishing). *If the bridge is critically normalized, then*

$$\mathcal{B}_\varepsilon\left(\frac{1}{2},t\right)0(t\in\mathbb{R}). \tag{4.6}$$

Proof. By (2.19),

$$\mathrm{Re}D_T\left(\frac{1}{2}+it\right)=0. \tag{4.7}$$

Together with (4.5), this gives (4.6) after substitution in (4.2).

Definition 20 (Reflection-odd correction). *The correction term is reflection-odd if*

$$R_\varepsilon(1-\sigma,t)=-R_\varepsilon(\sigma,t)\qquad(0<\sigma<1,\ t\in\mathbb{R}). \tag{4.8}$$

Proposition 21 (Oddness of the bridge). *If* R_ε *is reflection-odd, then*

$$\mathcal{B}_\varepsilon(1-\sigma,t)=-\mathcal{B}_\varepsilon(\sigma,t)\qquad(0<\sigma<1,\ t\in\mathbb{R}). \tag{4.9}$$

Proof. Using (2.18), (4.2), and (4.8),

$$\mathcal{B}_\varepsilon(1-\sigma,t)=L_\varepsilon\,\mathrm{Re}D_T(1-\sigma+it)+R_\varepsilon(1-\sigma,t)=-L_\varepsilon\,\mathrm{Re}D_T(\sigma+it)-R_\varepsilon(\sigma,t)=-\mathcal{B}_\varepsilon(\sigma,t). \tag{4.10}$$

Remark 22. *Reflection-oddness is a structural property. The critical-line theorem below does not need it; the theorem uses the stronger sign condition and domination estimate in the next section.*

5.The sign barrier

The reflected symmetry alone gives no sign. A critical-line criterion requires a barrier excluding zeros of the bridge away from $\sigma=1/2$.

Definition 23 (Analytic sign condition). *A real-admissible datum* T *satisfies the analytic sign condition on the critical strip if* $\mathrm{sgn}\big(\mathrm{Re}D_T(\sigma+it)\big)=\mathrm{sgn}\left(\sigma-\frac{1}{2}\right)$ (5.1)

for every $0<\sigma<1$, $\sigma\neq 1/2$,

and every $t\in\mathbb{R}$.

Definition 24 (Dominated correction). *For* $0<\theta<1$, *the correction term is* θ*-dominated if* $|R_\varepsilon(\sigma,t)|\le\theta L_\varepsilon|\mathrm{Re}D_T(\sigma+it)|$ (5.2)

for every $0<\sigma<1$, $\sigma\neq 1/2$, *and every* $t\in\mathbb{R}$.

Remark 25 (Coercive meaning of domination). *The factor* $\theta<1$ *is the strict margin in the argument. If* $\theta=1$, *the correction could cancel the main term exactly. If* $\theta>1$, *it could reverse the sign. The estimate (5.2) is therefore not a harmless technical condition; it is the coercive estimate on which the criterion rests.*

Proposition 26 (Quantitative non-vanishing estimate). *Assume that* R_ε *satisfies (5.2) for some* $0<\theta<1$. *Then, for every* $0<\sigma<1$, $\sigma\neq 1/2$, *and* $t\in\mathbb{R}$, $|\mathcal{B}_\varepsilon(\sigma,t)|\ge(1-\theta)L_\varepsilon|\mathrm{Re}D_T(\sigma+it)|$. (5.3)

In particular, wherever $\mathrm{Re}D_T(\sigma + it) \neq 0$*, the bridge cannot vanish.*

Proof. Let $M = L_\varepsilon \mathrm{Re}D_T(\sigma + it)$. The bridge is $M + R_\varepsilon$, and (5.2) gives $|R_\varepsilon| \leq \theta|M|$. The reverse triangle inequality gives

$$|M + R_\varepsilon| \geq |M| - |R_\varepsilon| \geq (1 - \theta)|M|. \tag{5.4}$$

This is (5.3). If $\mathrm{Re}D_T(\sigma + it) \neq 0$, then the right-hand side is strictly positive.

Proposition 27 (Sign inheritance). *Assume that* T *satisfies (5.1) and that* R_ε *satisfies (5.2) for some* $0 < \theta < 1$. *Then*

$$\mathrm{sgn}(\mathcal{B}_\varepsilon(\sigma, t)) = \mathrm{sgn}\left(\sigma - \tfrac{1}{2}\right) \tag{5.5}$$

for every $0 < \sigma < 1$*,* $\sigma \neq 1/2$*, and* $t \in \mathbb{R}$*.*

In particular,

$$\mathcal{B}_\varepsilon(\sigma, t) \neq 0 \qquad (\sigma \neq 1/2). \tag{5.6}$$

Proof. Set

$$M(\sigma, t) = L_\varepsilon \,\mathrm{Re}D_T(\sigma + it). \tag{5.7}$$

Since $L_\varepsilon > 0$, the sign condition (5.1) gives

$$\mathrm{sgn}M(\sigma, t) = \mathrm{sgn}\left(\sigma - \tfrac{1}{2}\right), \qquad M(\sigma, t) \neq 0 \tag{5.8}$$

whenever $\sigma \neq 1/2$. The domination estimate (5.2) is exactly

$$|R_\varepsilon(\sigma, t)| \leq \theta|M(\sigma, t)|, \qquad 0 < \theta < 1. \tag{5.9}$$

If $M > 0$, then

$$\mathcal{B}_\varepsilon = M + R_\varepsilon \geq M - |R_\varepsilon| \geq (1 - \theta)M > 0. \tag{5.10}$$

If $M < 0$, then

$$\mathcal{B}_\varepsilon = M + R_\varepsilon \leq M + |R_\varepsilon| \leq M - \theta M = (1 - \theta)M < 0. \tag{5.11}$$

Thus $\mathcal{B}_\varepsilon$ has the same sign as M, which proves (5.5) and (5.6).

The sign barrier proved in Proposition 27 can be seen in **Figure 2**. The domination estimate implies:

$$|\mathcal{B}_\varepsilon(\sigma, t)| \geq (1 - \theta)L_\varepsilon|\mathrm{Re}D_T(\sigma + it)| \qquad (\sigma \neq 1/2). \tag{5.12}$$

Thus the bridge retains at least the proportion $1 - \theta$ of the main term. The figure uses the representative value $\theta = 0.4$. In that case the bridge is confined between $0.6M$ and $1.4M$, where $M = L_\varepsilon \mathrm{Re}D_T$, so it remains at least 60% as large as the main term in absolute value and cannot cross zero away from the critical line. The interpretation is strictly quantitative: the correction may change the size of the bridge, but it cannot change its sign.

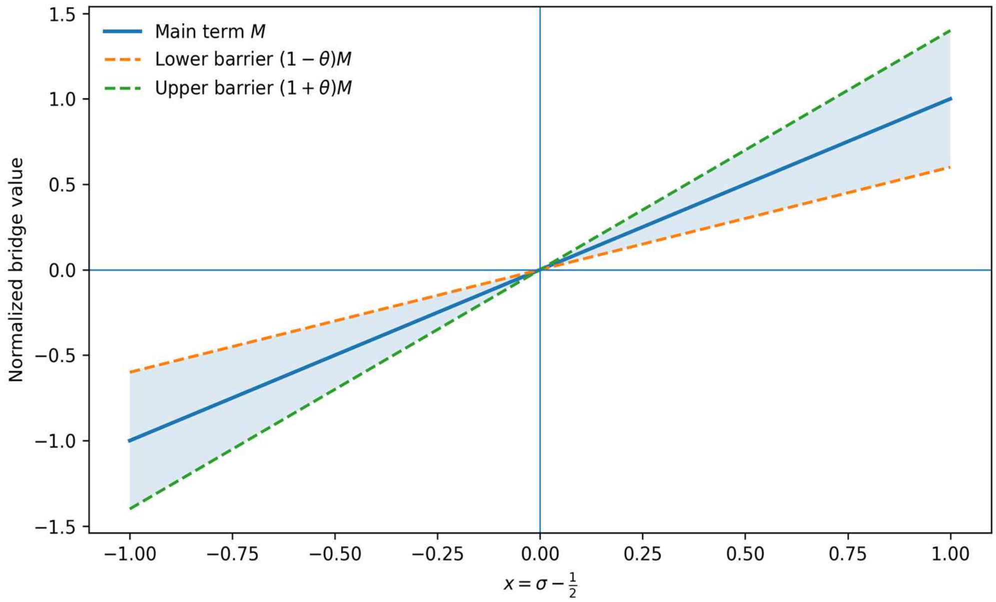

Figure 2.Normalized sign barrier for Proposition 27, shown for $\theta = 0.4$. With $x = \sigma - 1/2$ and $M = L_\varepsilon \mathrm{Re} D_T$, the domination condition confines the bridge to the band between $0.6M$ and $1.4M$. The entire band is positive for $x > 0$ and negative for $x < 0$.

6.Zero-adaptation and the critical-line criterion

Let

$$\mathcal{Z} = \{\rho \in \mathbb{C} : \xi(\rho) = 0\} \tag{6.1}$$

denote the multiset of nontrivial zeros of ζ. Each zero may be written as

$$\rho = \sigma + it, \qquad 0 < \sigma < 1. \tag{6.2}$$

Definition 28 (Zero-adapted bridge). *A bridge $\mathcal{B}_\varepsilon$ is zero-adapted if* $\mathcal{B}_\varepsilon(\sigma, t) = 0$ (6.3)

for every nontrivial zero $\rho = \sigma + it$ of ζ.

Assumption 29 (Bridge criterion data). *A bridge datum $(T, \varepsilon, R_\varepsilon, \theta)$ is admissible for the criterion if:*

1. *T is real-admissible;*
2. *$0 < \varepsilon < 1$;*
3. *$0 < \theta < 1$;*
4. *$\mathcal{B}_\varepsilon$ is given by (4.2);*
5. *T satisfies the analytic sign condition (5.1);*
6. *R_ε satisfies the domination estimate (5.2);*

7. $\mathcal{B}_\varepsilon$ *is zero-adapted in the sense of (6.3).*

Theorem 30 (Conditional critical-line criterion). *Assume that the following objects and estimates are available:*

1. *a real-admissible reflected datum* T*;*
2. *a parameter* $0 < \varepsilon < 1$*;*
3. *a real bridge* $\mathcal{B}_\varepsilon$ *of the form (4.2);*
4. *the analytic sign condition (5.1);*
5. *the dominated correction estimate (5.2) for some* $0 < \theta < 1$*;*
6. *the zero-adapted identity (6.3).*

Then every nontrivial zero of $\zeta(s)$ *lies on the critical line* $\mathrm{Re}s = 1/2$.

Proof. Let $\rho = \sigma + it$ be a nontrivial zero. By zero-adaptation,

$$\mathcal{B}_\varepsilon(\sigma, t) = 0. \tag{6.4}$$

If $\sigma \neq 1/2$, Proposition 27 gives

$$\mathcal{B}_\varepsilon(\sigma, t) \neq 0, \tag{6.5}$$

which is impossible. Hence $\sigma = 1/2$. Since the zero was arbitrary, all nontrivial zeros lie on the critical line.

Corollary 31 (No off-line zero can satisfy the bridge identities). *Under the hypotheses of Theorem 30, there is no nontrivial zero* $\rho = \sigma + it$ *with* $\sigma \neq 1/2$ *for which the bridge identity* $\mathcal{B}_\varepsilon(\sigma, t) = 0$ *holds.*

Proof. If such a zero existed, zero-adaptation would give $\mathcal{B}_\varepsilon(\sigma, t) = 0$, while Proposition 27 would give $\mathcal{B}_\varepsilon(\sigma, t) \neq 0$. The two conclusions are incompatible.

Remark 32. *The theorem is a conditional implication. The hypotheses are deliberately stated as independent mathematical requirements, because none of the sign law, domination estimate, or zero-adapted identity follows from the reflected decomposition alone.*

The logical balance of the criterion in **Table 2**, separates the argument into seven rows: three proved components, three assumed inputs, and one conditional conclusion. The three proved rows establish the formal bridge; the three assumed rows identify the missing coercive estimates; the final row records the critical-line conclusion obtained once those inputs are supplied. In percentage terms, the table separates the paper into a proved structural part, which accounts for the formal mechanism, and an assumed application-level part, which accounts for the unresolved analytic burden. This layout makes the conditional status explicit rather than implicit.

Table 2.Logical structure of the conditional bridge criterion.

Ingredient	Mathematical content	Role in the argument
Reflected datum	$T(s) + T(1-s) = \xi(s)$, with classification	Sets the analytic framework and

Ingredient	Mathematical content	Role in the argument
	$T = \xi/2 + H$, $H(1-s) = -H(s)$	defines the defect
Real critical-line symmetry	$\mathrm{Re}D_T(1-\sigma+it) = -\mathrm{Re}D_T(\sigma+it)$ and $\mathrm{Re}D_T(1/2+it) = 0$	Identifies the correct quantity that vanishes on the critical line
Arithmetic scale	Existence of N_ε maximizing $d(N)/N^\varepsilon$ and positivity of L_ε for $0 < \varepsilon < 1$	Supplies the positive weight in the bridge
Analytic sign law	$\mathrm{sgn}\big(\mathrm{Re}D_T(\sigma+it)\big) = \mathrm{sgn}(\sigma - 1/2)$ for $\sigma \neq 1/2$	Forces the main term to point away from zero off the line
Correction bound	$\lvert R_\varepsilon\rvert \leq \theta L_\varepsilon \lvert\mathrm{Re}D_T\rvert$ with $0 < \theta < 1$	Prevents the correction from changing the sign of the bridge
Zero-adaptation	$\mathcal{B}_\varepsilon(\rho) = 0$ at each nontrivial zero ρ	Connects the bridge to the zero set of ξ
Critical-line conclusion	Every nontrivial zero satisfies $\mathrm{Re}\rho = 1/2$	Final consequence of the previous rows

7.Remains for an application

The criterion above reduces the critical-line conclusion to explicit inputs. In particular, it shows that any successful application of the framework must do more than produce an aesthetically appealing decomposition of ξ: it must connect that decomposition to a coercive sign law and to a zero-adapted arithmetic correction. A proof within this framework would therefore have to provide the following ingredients.

1. **A canonical reflected datum.** The classification theorem shows that reflected data are abundant: any analytic odd perturbation H gives $T = \xi/2 + H$. An application must select T from a specific Ramanujan-type representation of ξ or Ξ, not from an arbitrary perturbation.

2. **Analytic coercivity.** One must prove throughout the strip that

$$\mathrm{sgn}\big(\mathrm{Re}D_T(\sigma+it)\big) = \mathrm{sgn}\left(\sigma - \tfrac{1}{2}\right) \qquad (\sigma \neq 1/2). \tag{7.1}$$

   Oddness about the critical line is only a symmetry. It does not by itself imply positivity on one side and negativity on the other.

3. **A controlled correction.** The correction R_ε must be defined from the same analytic–arithmetic construction as the bridge and must satisfy

$$\lvert R_\varepsilon(\sigma,t)\rvert \leq \theta L_\varepsilon \lvert\mathrm{Re}D_T(\sigma+it)\rvert \qquad (0 < \theta < 1). \tag{7.2}$$

   This is the quantitative estimate that makes the sign barrier effective.

4. **Zero-adaptation.** The identity

$$\mathcal{B}_\varepsilon(\sigma,t) = 0 \tag{7.3}$$

must be proved for every nontrivial zero $\rho = \sigma + it$. This is the point at which the construction must genuinely connect the zero set of ξ to the divisor scale N_ε.

These four requirements are the remaining mathematical burden. Without them, the criterion remains conditional; with them, the proof of the critical-line statement is the short argument in the conditional theorem.

Given a proposed Ramanujan-type datum T and correction R_ε, the framework suggests the following order of verification. First prove real-admissibility and compute D_T. Then study the normalized real defect A_T; the required analytic sign law is exactly $A_T > 0$ off the critical line. Next select the arithmetic scale through L_ε, which is positive by Proposition 3.4 for $0 < \varepsilon < 1$. Only after these steps does it make sense to estimate the correction. Finally, zero-adaptation must be proved at the zeros of ξ. This order prevents the central difficulty from being hidden inside notation. The theorem does not assume the Riemann Hypothesis. It assumes a bridge that vanishes at all nontrivial zeros and is sign-forced away from the critical line. Those assumptions are stronger than a formal restatement only if they are obtained from a construction that is independent of the location of the zeros. Therefore any application must define T, R_ε, and ε before using information about the zero set. The zero-adapted identity must then be proved from that construction, not imposed by hand.

8.Conclusion

As the final statement, the bridge must be formulated with the real defect

$$\mathrm{Re}D_T(\sigma + it), \tag{8.1}$$

because the complex defect D_T need not vanish on the critical line. Under real-admissibility the correct reflection identity is

$$\mathrm{Re}D_T(1 - \sigma + it) = -\mathrm{Re}D_T(\sigma + it), \tag{8.2}$$

and therefore

$$\mathrm{Re}D_T\left(\frac{1}{2} + it\right) = 0. \tag{8.3}$$

On the arithmetic side, Ramanujan's profile supplies a positive weight

$$L_\varepsilon = \Lambda_\varepsilon(N_\varepsilon) > 0 \qquad (0 < \varepsilon < 1). \tag{8.4}$$

The bridge

$$\mathcal{B}_\varepsilon(\sigma, t) = L_\varepsilon\,\mathrm{Re}D_T(\sigma + it) + R_\varepsilon(\sigma, t) \tag{8.5}$$

then gives a clean conditional criterion: zero-adaptation at the nontrivial zeros and sign inheritance away from the critical line force all nontrivial zeros to have real part $1/2$.

The proved part of the argument consists of the reflected-data classification, the real critical-line symmetry of the defect, the existence and positivity of the arithmetic scale, and the sign-inheritance lemma. The unproved part is exactly the construction of a canonical Ramanujan datum, the analytic sign condition, the correction estimate, and the zero-adapted identity. In this sense, the novelty of the framework is its clean separation of mechanism and burden as explained. The mechanism is the proved sign-transfer principle which is a positive arithmetic scale, a real reflected defect, and a dominated correction force the bridge to be nonzero away from the critical line. The burden is also explicit: one must construct a canonical datum, prove the analytic sign law, control the correction, and establish zero-adaptation. The paper is complete as a conditional criterion because each implication used in the argument has been stated and proved; it remains conditional.

Acknowledgement

Dedicated to Srinivasa Ramanujan, whose brilliant mathematical legacy continues to guide and inspire.

References

[1] Riemann, B. (1859) Über die Anzahl der Primzahlen unter einer gegebenen Grösse. [Monatsberichte der Berliner Akademie, November 1859]. 136-144.

[2] Titchmarsh, E.C. (1986) The Theory of the Riemann Zeta Function. 2nd Revised (Heath-Brown) Edition, Oxford University Press, Oxford.

[3] Edwards, H. (2001) Riemann's Zeta Function. Dover Publication, Mineola.

[4] Ivić, A. (1985) The Riemann Zeta-Function, Theory and Applications. John Wiley & Sons, New York. (Dover Publications, Mineola, New York, 2003)

[5] Platt, D. and Trudgian, T. (2021), The Riemann hypothesis is true up to 3 ·1012. Bull. London Math. Soc., 53: 792-797. https://doi.org/10.1112/blms.12460

[6] Lagarias, J. C. (2002). An Elementary Problem Equivalent to the Riemann Hypothesis. *The American Mathematical Monthly*, *109*(6), 534–543. https://doi.org/10.1080/00029890.2002.11919883

[7] Nicolas, J.-L. (1983). Petites valeurs de la fonction d'Euler. *Journal of Number Theory*, *17*(3), 375–388. https://doi.org/10.1016/0022-314X(83)90055-0.

[8] Robin. G. (1984) Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *Journal de Mathématiques Pures et Appliquées*, 63, 187-217.

[9] S. Ramanujan, "New expressions for Riemann's functions xi(s) and Xi(t)," *Quarterly Journal of Mathematics*, vol. 46, pp. 253–260, 1915.

[10] Ramanujan, S. (1915), Highly Composite Numbers. Proceedings of the London Mathematical Society, s2_14: 347-409. https://doi.org/10.1112/plms/s2_14.1.347

[11] L. Alaoglu and P. Erdős, "On highly composite and similar numbers," *Transactions of the American Mathematical Society*, vol. 56, pp. 448–469, 1944, doi: 10.1090/S0002-9947-1944-0011087-2.

[12] T. H. Gronwall, "Some asymptotic expressions in the theory of numbers," *Transactions of the American Mathematical Society*, vol. 14, no. 1, pp. 113–122, 1913, doi: 10.1090/S0002-9947-1913-1500940-6.

[13] T. M. Apostol, *Introduction to Analytic Number Theory*. New York, NY, USA: Springer, 1976, doi: 10.1007/978-1-4757-5579-4.