# Ramanujan-inspired prime sums

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## Abstract

This paper develops a finite framework for studying sums of prime numbers through consecutive prime gaps. The motivation comes from Ramanujan's influence on arithmetic decompositions, partition methods, and summation ideas, while the argument remains within ordinary finite number theory. We study the partial sum of the first $n$ primes and prove an exact identity that writes this sum as a baseline term from the initial prime together with a weighted accumulation of consecutive prime gaps. Each gap receives a weight equal to the number of later primes affected by that gap. The same identity is then expressed geometrically by encoding each weighted gap as the slope of a right triangle. Exact numerical examples verify the formula, and the asymptotic discussion shows that the decomposition has the same leading scale as the classical growth predicted by the prime number theorem. This paper also separates finite identities from regularized infinite summations. This distinction is essential: the finite formula is exact, while divergent infinite prime sums require a separate summation theory and cannot be treated as ordinary convergent series.

## Keywords

prime sums, prime gaps, Ramanujan-inspired number theory, geometric encoding

## 1. Introduction

Prime numbers form one of the basic structures of arithmetic [1]. Their distribution has shaped number theory from Euler's product formula to the prime number theorem and modern work on prime gaps [2]. The usual objects in this area are the prime-counting function, Chebyshev functions, divisor functions, and asymptotic estimates for the sequence of primes [3]. A simpler finite object also carries useful information: the sum of the first $n$ primes. This sum records not only the size of the primes themselves, but also the way consecutive prime gaps accumulate through the sequence.

Let $p_n$ denote the $n$-th prime, and define the partial prime sum by

$$P_n = \sum_{k=1}^{n} p_k . \qquad (1)$$

The central observation of this paper is that $P_n$ has an exact decomposition in terms of consecutive prime gaps. If

$$g_i = p_{i+1} - p_i, \qquad (2)$$

then

$$P_n = 2n + \sum_{i=1}^{n-1} (n-i)\, g_i. \qquad (3)$$

The term $2n$ is the baseline contribution from the initial prime 2. The remaining term records the accumulated effect of the prime gaps. A gap $g_i$is not counted once. It is counted with weight $n - i$, because it affects every later prime from $p_{i+1}$through $p_n$. In this way, Formula (3) gives a finite history of how the prime sequence is built from its successive differences.

The motivation is partly Ramanujan-inspired. Ramanujan's work on highly composite numbers showed how arithmetic objects can be understood through structured decompositions, extremal behavior, and cumulative effects in multiplicative number theory [4]. Nicolas and Robin later studied Ramanujan's work in detail and clarified its connection with divisor functions, prime factors, and extremal arithmetic behavior [5]. Hardy and Ramanujan's paper on partitions remains another model for reorganizing arithmetic information into forms where cumulative structure becomes visible [6]. The present paper does not attribute Formula (3) to Ramanujan. It follows a related mathematical attitude: a finite arithmetic quantity is rewritten so that its internal structure becomes transparent.

The finite sum of the first $n$ primes belongs naturally to the classical theory of prime distribution. The prime number theorem gives the leading scale of prime growth, and Goldstein's historical account explains how this subject developed from numerical evidence into rigorous asymptotic theory [7]. More recent work studies $P_n$ directly. Axler obtained asymptotic formulas and explicit estimates for the sum of the first $n$ primes [8]. Sinha derived an asymptotic expansion for the same quantity and connected it with inequalities of Mandl and Robin type [9]. All these works address the analytic growth of $P_n$. The present paper has a different role in this case. It gives an exact finite decomposition that makes the contribution of each individual prime gap explicit.

Prime gaps provide the natural language for this decomposition. The differences $p_{i+1} - p_i$ contain fine information about the local behavior of the prime sequence. Soundararajan's survey on small gaps between primes shows how deep this subject becomes when one studies the distribution and size of these gaps [10]. Here the gaps are used in a more elementary but exact way. They are not estimated statistically. They are placed into a finite weighted identity for $P_n$. This gives a direct bridge between local increments of the prime sequence and the global partial sum.

The same identity also has a geometric form. For fixed $n$, define

$$h_i = n - i. \qquad (4)$$

If an angle $\alpha_i$is chosen so that

$$\tan\, \alpha_i = \frac{g_i}{h_i}, \qquad (5)$$

then

$$g_i = h_i \tan\, \alpha_i, \qquad (6)$$

and the prime-sum identity becomes

$$P_n = 2n + \sum_{i=1}^{n-1} (n-i)^2 \tan\, \alpha_i. \qquad (7)$$

This trigonometric representation is exact. It should be read as a geometric encoding of the finite gap identity, not as an independent rule for predicting primes. The angle $\alpha_i$ is defined from the gap $g_i$, so the geometric form preserves the same arithmetic information in a visual form.

A separate issue concerns divergent infinite sums. Ramanujan's name is often connected with nonstandard summation ideas, and Ramanujan sums continue to appear in modern mathematical and signal-processing settings [11]. Such methods have value when they are used inside a precise regularization framework. They are not ordinary finite summation. For this reason, the present paper keeps the finite identity separate from any regularized infinite interpretation. The series of all primes diverges in the ordinary sense, and any finite value assigned to it would require separate definitions and separate proofs.

The paper proves the exact prime-gap decomposition for $P_n$, verifies it by finite examples, gives its geometric encoding, and explains its agreement with the known asymptotic scale of prime sums. The main conclusion is that partial prime sums have a simple weighted-gap structure. The result is finite, and independent of any regularized infinite summation.

## 2. Prime gaps and partial prime sums

Let

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \dots \qquad (8)$$

be the increasing sequence of prime numbers. For each integer $i \geq 1$, define the $i$-th prime gap by

$$g_i = p_{i+1} - p_i. \qquad (9)$$

Thus the first few gaps are

$$g_1 = 1, g_2 = 2, g_3 = 2, g_4 = 4, g_5 = 2. \qquad (10)$$

These gaps recover the prime sequence by telescoping. For every $k \geq 1$,

$$p_k = p_1 + \sum_{i=1}^{k-1} (p_{i+1} - p_i). \qquad (11)$$

Since $p_1 = 2$, this becomes

$$p_k = 2 + \sum_{i=1}^{k-1} g_i\,. \qquad (12)$$

Equation (12) is the basic finite relation used throughout the paper. It says that each prime is obtained from the first prime by adding all previous gaps.

We define the partial sum of the first $n$ primes by

$$P_n = \sum_{k=1}^{n} p_k\,. \qquad (13)$$

The notation $P_n$ is used deliberately. In standard number theory, $\pi(x)$ denotes the number of primes not exceeding $x$. Using $\pi(n)$ for a sum of primes would conflict with this standard notation. The symbol $P_n$ keeps the object clear.

The goal is to rewrite $P_n$ in terms of the gaps $g_i$. A gap that occurs early in the prime sequence appears in many later primes. A gap that occurs near the end appears only a few times. These repeated appearances

are the source of the weights in the main formula.
For example, take $n = 5$. The first five primes are

$$2,\ 3,\ 5,\ 7,\ 11. \tag{14}$$

They can be written through gaps as

$$\begin{aligned} 2 &= 2, \\ 3 &= 2 + g_1, \\ 5 &= 2 + g_1 + g_2, \\ 7 &= 2 + g_1 + g_2 + g_3, \\ 11 &= 2 + g_1 + g_2 + g_3 + g_4. \end{aligned} \tag{15}$$

Adding the five lines in (15), the baseline 2appears five times. The first gap $g_1$appears four times, $g_2$appears three times, $g_3$appears twice, and $g_4$appears once. Hence

$$P_5 = 2 \cdot 5 + 4g_1 + 3g_2 + 2g_3 + g_4. \tag{16}$$

This example shows the general pattern. In the sum $P_n$, the gap $g_i$appears exactly $n - i$ times. The main theorem in the next section formalizes this observation.

## 3. Decomposition of prime sums

We now prove the finite identity stated in the Introduction. The proof is short, but it is the structural center of the paper. It shows that the sum of the first $n$primes is determined by the first prime and by a weighted record of the gaps that follow it.

**Theorem 3.1. Prime-gap decomposition**
For every integer $n \geq 1$,

$$P_n = 2n + \sum_{i=1}^{n-1} (n - i)\, g_i. \tag{17}$$

Equivalently,

$$\sum_{k=1}^{n} p_k = 2n + \sum_{i=1}^{n-1} (n - i)(\, p_{i+1} - p_i). \tag{18}$$

**Proof**
From (12), each prime $p_k$can be written as

$$p_k = 2 + \sum_{i=1}^{k-1} g_i\,. \tag{19}$$

Summing (19) over $k = 1, \dots, n$, we obtain

$$P_n = \sum_{k=1}^{n} \left(2 + \sum_{i=1}^{k-1} g_i\right). \tag{20}$$

Separating the constant term gives

$$P_n = 2n + \sum_{k=1}^{n} \sum_{i=1}^{k-1} g_i\,. \qquad (21)$$

We now count how often each fixed gap occurs. The gap $g_i$appears in the inner sum exactly when

$$k \geq i + 1. \qquad (22)$$

For a fixed $i$with $1 \leq i \leq n - 1$, this means

$$k = i + 1, i + 2, \dots, n. \qquad (23)$$

There are $n - i$such values of $k$. Therefore,

$$\sum_{k=1}^{n} \sum_{i=1}^{k-1} g_i = \sum_{i=1}^{n-1} (n - i)\, g_i. \qquad (24)$$

Substituting (24) into (21) gives

$$P_n = 2n + \sum_{i=1}^{n-1} (n - i)\, g_i. \qquad (25)$$

This proves the theorem.

**Corollary 3.2. Accumulated gap contribution**

For every integer $n \geq 1$,

$$P_n - 2n = \sum_{i=1}^{n-1} (n - i)\, g_i. \qquad (26)$$

**Proof**

Subtract $2n$from both sides of (17).

Formula (26) isolates the part of the prime sum produced by the gaps. The term $2n$is only the contribution of the initial prime repeated $n$times. Everything beyond that baseline is carried by the weighted gap sum.

The weight $n - i$has a direct meaning. The gap $g_i$shifts every later prime from $p_{i+1}$through $p_n$. It therefore contributes once to each of those later primes. That is why it appears $n - i$times in $P_n$. Early gaps have larger weights. Later gaps have smaller weights.

The notation in the decomposition is summarized in **Table 1**. The table is included to make clear that the index $k$ refers to primes, while the index $i$refers to gaps. This distinction prevents a common confusion in formulas involving prime sums.

**Table 1.** Notation and arithmetic role in the prime-gap decomposition

| Symbol | Definition | Role |
|---|---|---|
| $p_k$ | $k$-th prime | prime being summed |
| $g_i$ | $p_{i+1} - p_i$ | gap between consecutive primes |
| $P_n$ | $\sum_{k=1}^{n} p_k$ | sum of the first $n$ primes |
| $n - i$ | gap multiplicity | number of later primes affected by $g_i$ |
| $2n$ | $n$ copies of $p_1 = 2$ | baseline contribution |

## 4. Numerical verification and finite examples

The decomposition is finite, so it must agree exactly with ordinary prime sums. The examples below check the identity in small cases and show how the weights appear in practice.

For $\boldsymbol{n = 5}$, the first five primes are

$$\mathbf{2,\ 3,\ 5,\ 7,\ 11.} \tag{27}$$

The corresponding gaps are

$$\boldsymbol{g_1 = 1, g_2 = 2, g_3 = 2, g_4 = 4.} \tag{28}$$

Using Theorem 3.1, we get

$$\boldsymbol{P_5 = 2 \cdot 5 + 4g_1 + 3g_2 + 2g_3 + g_4.} \tag{29}$$

Substituting the gaps gives

$$\boldsymbol{P_5 = 10 + 4(1) + 3(2) + 2(2) + 1(4) = 28.} \tag{30}$$

The ordinary sum gives the same value:

$$\mathbf{2 + 3 + 5 + 7 + 11 = 28.} \tag{31}$$

For $\boldsymbol{n = 6}$, the first six primes are

$$\mathbf{2,\ 3,\ 5,\ 7,\ 11,\ 13.} \tag{32}$$

Their gaps are

$$\boldsymbol{g_1 = 1, g_2 = 2, g_3 = 2, g_4 = 4, g_5 = 2.} \tag{33}$$

The decomposition gives

$$\boldsymbol{P_6 = 2 \cdot 6 + 5g_1 + 4g_2 + 3g_3 + 2g_4 + g_5.} \tag{34}$$

Thus

$$P_6 = 12 + 5(1) + 4(2) + 3(2) + 2(4) + 1(2) = 41. \tag{35}$$

Again, this matches the direct computation

$$2 + 3 + 5 + 7 + 11 + 13 = 41. \tag{36}$$

The same verification for several small values of $n$ is given at **Table 2**. Each row compares the ordinary sum with the weighted-gap expression. The agreement is exact because both columns represent the same finite identity.

**Table 2.** Exact verification of the prime-gap decomposition for small prime sums.

| $n$ | first $n$ primes | weighted-gap expression | $P_n$ |
|---|---|---|---|
| $5$ | $2, 3, 5, 7, 11$ | $10 + 4(1) + 3(2) + 2(2) + 1(4)$ | $28$ |
| $6$ | $2, 3, 5, 7, 11, 13$ | $12 + 5(1) + 4(2) + 3(2) + 2(4) + 1(2)$ | $41$ |
| $7$ | $2, 3, 5, 7, 11, 13, 17$ | $14 + 6(1) + 5(2) + 4(2) + 3(4) + 2(2) + 1(4)$ | $58$ |
| $8$ | $2, 3, 5, 7, 11, 13, 17, 19$ | $16 + 7(1) + 6(2) + 5(2) + 4(4) + 3(2) + 2(4) + 1(2)$ | $77$ |

The triangular pattern of the weights is visible in the table. For $P_8$, the first gap has weight $7$, the second has weight $6$, and the weights decrease by one until the final gap has weight $1$. The arithmetic variation comes from the gaps themselves, while the weight pattern comes only from the finite summation structure.

The same structure is visible also in **Figure 1**. For $n = 12$, the figure displays both the decreasing multiplicity $n - i$and the actual weighted contribution $(n - i)g_i$.

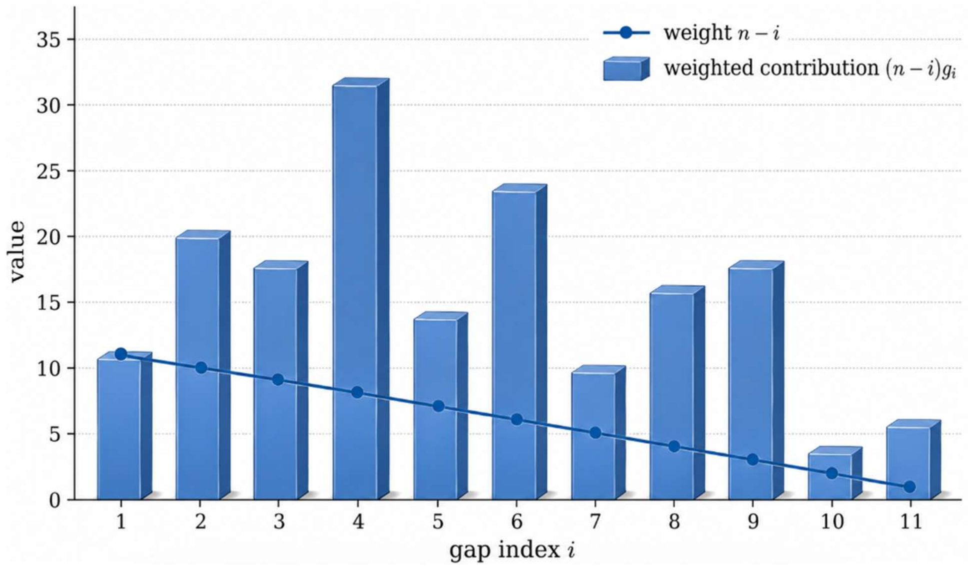

**Figure 1.** Weighted contribution of prime gaps to $P_n$.

The plot separates two effects: the deterministic decline of the weights and the arithmetic variation of the gaps. The $\boldsymbol{i}$-th gap $\boldsymbol{g_i}$contributes with weight $\boldsymbol{n-i}$, so early gaps have the largest structural influence on the finite prime sum.

## 5. Numerical verification and finite examples

The prime-gap decomposition also has a useful geometric form. This section does not introduce a new prime-generating rule. It gives an exact visual encoding of the same finite identity proved in Theorem 3.1.

Fix an integer $\boldsymbol{n \geq 2}$. For each gap $\boldsymbol{g_i}$, define

$$\boldsymbol{h_i = n - i.} \qquad (37)$$

The number $\boldsymbol{h_i}$is the multiplicity of the gap $\boldsymbol{g_i}$in the sum $\boldsymbol{P_n}$. Define an angle $\boldsymbol{\alpha_i}$by

$$\boldsymbol{\tan\ \alpha_i = \frac{g_i}{h_i}.} \qquad (38)$$

Equivalently,

$$\boldsymbol{g_i = h_i \tan\ \alpha_i.} \qquad (39)$$

Substituting (39) into the gap decomposition gives

$$\boldsymbol{P_n = 2n + \sum_{i=1}^{n-1} h_i\, g_i.} \qquad (40)$$

Since $\boldsymbol{g_i = h_i \tan\ \alpha_i}$, we obtain

$$\boldsymbol{P_n = 2n + \sum_{i=1}^{n-1} h_i^2 \tan\ \alpha_i.} \qquad (41)$$

Using $\boldsymbol{h_i = n - i}$, this becomes

$$\boldsymbol{P_n = 2n + \sum_{i=1}^{n-1} (n-i)^2 \tan\ \alpha_i.} \qquad (42)$$

**Proposition 5.1. Geometric form**

For every integer $\boldsymbol{n \geq 2}$,

$$\boldsymbol{P_n = 2n + \sum_{i=1}^{n-1} (n-i)^2 \tan\ \alpha_i}, \qquad (43)$$

where

$$\boldsymbol{\tan\ \alpha_i = \frac{p_{i+1} - p_i}{n-i}}. \qquad (44)$$

**Proof**

By definition,

$$\boldsymbol{p_{i+1} - p_i = g_i = (n-i) \tan\ \alpha_i}. \qquad (45)$$

Therefore,

$$\boldsymbol{(n-i) g_i = (n-i)^2 \tan\ \alpha_i}. \qquad (46)$$

Substitution into the prime-gap decomposition

$$\boldsymbol{P_n = 2n + \sum_{i=1}^{n-1} (n-i)\, g_i} \qquad (47)$$

gives (43).

This representation is exact since each angle is defined from the corresponding gap. It must be read in that direction. The angle does not predict the gap. The gap defines the angle, and the triangle records the weighted contribution of that gap to the finite prime sum.

For $\boldsymbol{n = 6}$, the gaps are

$$\boldsymbol{g_1 = 1, g_2 = 2, g_3 = 2, g_4 = 4, g_5 = 2}. \qquad (48)$$

The corresponding multiplicities are

$$\boldsymbol{h_1 = 5, h_2 = 4, h_3 = 3, h_4 = 2, h_5 = 1}. \qquad (49)$$

Geometric encoding is recorded in **Table 3**. The final column is the contribution $\boldsymbol{h_i^2 \tan\, \alpha_i}$, which is equal to $\boldsymbol{h_i g_i}$. The table shows that the trigonometric form reproduces the same exact arithmetic contribution as the gap formula.

**Table 3.** Geometric encoding for $P_6$.

| $\boldsymbol{i}$ | $\boldsymbol{g_i}$ | $\boldsymbol{h_i = 6 - i}$ | $\boldsymbol{\tan\, \alpha_i = g_i/h_i}$ | $\boldsymbol{h_i^2 \tan\, \alpha_i}$ |
|---|---|---|---|---|
| **1** | **1** | **5** | **1/5** | **5** |
| **2** | **2** | **4** | **1/2** | **8** |
| **3** | **2** | **3** | **2/3** | **6** |
| **4** | **4** | **2** | **2** | **8** |
| **5** | **2** | **1** | **2** | **2** |

The final column in Table 3 sums to

$$\mathbf{5 + 8 + 6 + 8 + 2 = 29.} \qquad (50)$$

Adding the baseline term $\boldsymbol{2n = 12}$, we obtain

$$\boldsymbol{P_6 = 12 + 29 = 41.} \qquad (51)$$

This agrees with the direct sum

$$\mathbf{2 + 3 + 5 + 7 + 11 + 13 = 41.} \qquad (52)$$

The geometric relation is most transparent in **Figure 2**. The vertical height is $\boldsymbol{h_i}$, the horizontal gap is $\boldsymbol{g_i}$, and the angle is chosen so that $\boldsymbol{\tan\, \alpha_i = g_i/h_i}$.

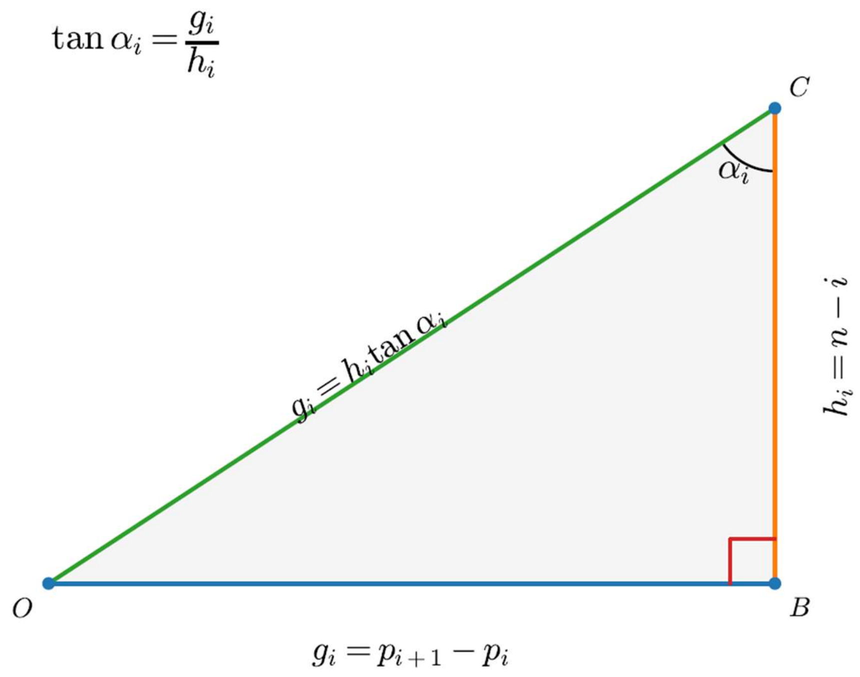

**Figure 2.** Right-triangle representation of one weighted prime gap

The weighted contribution is therefore $\boldsymbol{h_i g_i}$, equivalently $\boldsymbol{h_i^2 \tan \alpha_i}$. The gap $\boldsymbol{g_i}$is encoded as a slope relative to height $\boldsymbol{h_i = n - i}$, and its contribution to $\boldsymbol{P_n}$is $\boldsymbol{h_i g_i}$.

## 6. Asymptotic scale and finite versus infinite summation

The identity

$$\boldsymbol{P_n = 2n + \sum_{i=1}^{n-1} (n - i)\, g_i} \qquad (53)$$

is exact for every positive integer $\boldsymbol{n}$. It is also consistent with the classical asymptotic scale of prime sums.

The prime number theorem gives

$$\boldsymbol{p_n \sim n \log\, n.} \qquad (54)$$

From this, one obtains the standard leading behavior

$$\boldsymbol{P_n = \sum_{k=1}^{n} p_k \sim \frac{1}{2} n^2 \log\, n.} \qquad (55)$$

The gap decomposition agrees with this scale. Prime gaps near $\boldsymbol{p_i}$ have average size comparable to $\mathbf{log}\,\boldsymbol{i}$. Replacing $\boldsymbol{g_i}$ heuristically by $\mathbf{log}\,\boldsymbol{i}$ in (53) gives

$$\sum_{i=1}^{n-1}(n-i)\log\, i. \qquad (56)$$

This weighted sum has leading order

$$\frac{1}{2}n^2\log\, n. \qquad (57)$$

Thus the finite decomposition has the same dominant growth as the usual asymptotic estimate for $\boldsymbol{P_n}$.

The comparison of exact values of $\boldsymbol{P_n}$ with the leading asymptotic scale $\frac{1}{2}n^2\log\, n$ is plotted in **Figure 3**. The plotted ratio is not meant as a new theorem. It gives a numerical view of the known asymptotic relation and shows that the exact finite decomposition has the expected growth scale. The curve shows $P_n/(\frac{1}{2}n^2\log\, n)$ for finite $\boldsymbol{n}$, with the horizontal line marking the asymptotic level $\mathbf{1}$.

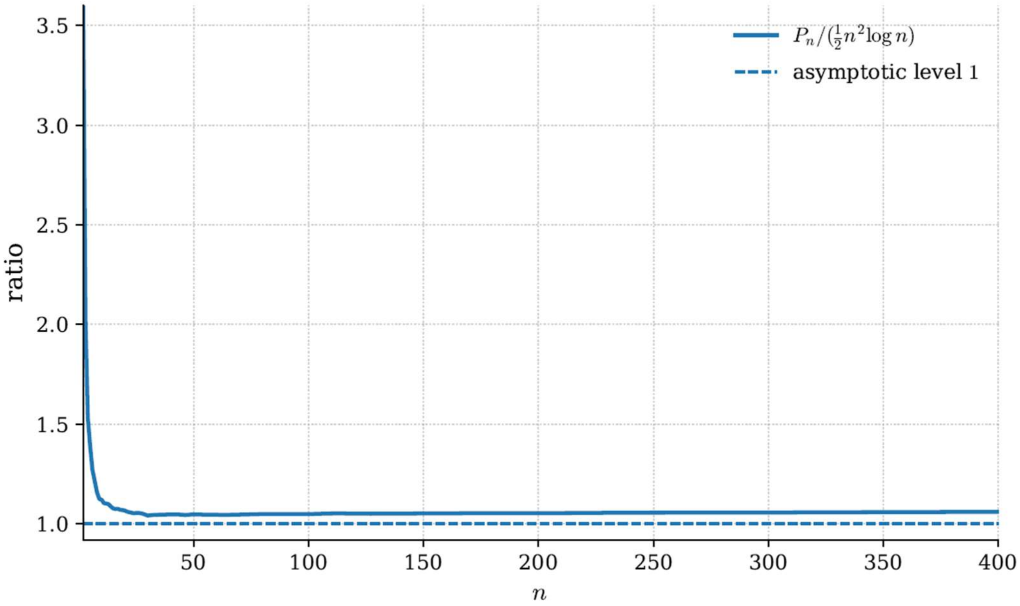

**Figure 3.** Partial prime sums relative to the leading asymptotic scale

We now separate the finite theorem from infinite summation. The series

$$\sum_{k=1}^{\infty} p_k \qquad (58)$$

diverges in the ordinary sense because

$$\boldsymbol{p_k} \to \infty. \qquad (59)$$

Therefore,

$$\mathbf{2 + 3 + 5 + 7 + \cdots} \qquad (60)$$

has no finite value as an ordinary convergent series. Similarly,

$$\sum_{\boldsymbol{k=1}}^{\infty} \mathbf{1} \qquad (61)$$

diverges in the ordinary sense.

Assigning a finite value to a divergent series requires a specified regularization method. Such methods have legitimate uses in analytic continuation, spectral theory, and mathematical physics. They do not turn a divergent series into an ordinary convergent sum.

For this reason, the finite identity (53) is kept separate from any regularized interpretation of infinite prime sums. Any regularized treatment of

$$\sum_{\boldsymbol{k=1}}^{\infty} \boldsymbol{p_k} \qquad (62)$$

would require its own definitions, convergence principles, and proofs. It is not part of the finite result proved in this paper.

## 7. Conclusion

This work gives an exact finite decomposition for the sum of the first $\boldsymbol{n}$ primes. If $\boldsymbol{p_n}$denotes the $\boldsymbol{n}$-th prime and

$$\boldsymbol{g_i = p_{i+1} - p_i}, \qquad (63)$$

then the partial prime sum satisfies

$$\boldsymbol{P_n = \sum_{k=1}^{n} p_k = 2n + \sum_{i=1}^{n-1} (n - i)\, g_i}. \qquad (64)$$

The proof is a direct telescoping argument followed by a change in the order of summation. The formula shows that each prime gap contributes according to the number of later primes it affects. Early gaps receive larger weights, while later gaps receive smaller weights. This gives a finite structural description of the prime sum rather than only its numerical value.

The same identity also has the geometric form

$$\boldsymbol{P_n = 2n + \sum_{i=1}^{n-1} (n-i)^2 \tan\ \alpha_i,} \qquad (65)$$

where

$$\boldsymbol{\tan\ \alpha_i = \frac{p_{i+1} - p_i}{n-i}.} \qquad (66)$$

This form encodes each weighted gap as the slope of a right triangle. It does not introduce a separate rule for predicting primes. It preserves the same arithmetic information in geometric notation.

The examples confirm the identity exactly for finite values of $\boldsymbol{n}$. The asymptotic discussion shows that the decomposition is consistent with the classical scale

$$\boldsymbol{P_n \sim \frac{1}{2} n^2 \log\ n.} \qquad (67)$$

The finite theorem is also independent of any regularized infinite summation. The series of all primes diverges in the ordinary sense, and any finite assignment to such a series requires a separate summation theory. Thus the result proved here belongs fully to finite number theory. The Ramanujan-inspired aspect of the paper lies in the organization of arithmetic information. A partial prime sum is rewritten as a weighted history of consecutive prime gaps.

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We express our sincere respect for Srinivasa Ramanujan, whose work continues to shape the way mathematicians think about arithmetic structure, partitions, summation, and hidden order in finite and infinite processes. We also thank the authors whose work on highly composite numbers, prime sums, prime gaps, and Ramanujan-related methods helped place this study in its proper mathematical context. Their contributions provided the background needed to state the present finite decomposition clearly and without overstatement.

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