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AN INFINITE FAMILY OF SOLUTIONS TO THE GENERALIZED CANNONBALL PROBLEM

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Abstract. In an earlier paper I considered the generalized cannonball problem for r-regular polygons and studied integer solutions to the associated Diophantine equation. In this note I prove that for every positive integer n, the triple

$$(r, a, b) = \left(3n + 2,\, 3n^2 - 2,\, 3n^3 - 3n + 1\right)$$

is a solution. Hence the generalized cannonball problem admits infinitely many positive integer solutions. I also compare this parametric family with the 858 tuples listed in the appendix of the earlier paper. Among those tuples, 821 are generated by the present family and the remaining 37 tuples are listed explicitly.

1. Introduction

The generalized cannonball problem asks for positive integer solutions to

$$\# := \frac{a\,(a+1)\,(a\,(r-2)+5-r)}{6} = \frac{b\,(b\,(r-2)+4-r)}{2}, \tag{1}$$

where the left-hand side is the ath r-gonal pyramidal number and the right-hand side is the bth r-gonal number. For background on the classical square case and on polygonal numbers, see Watson [2] and Weisstein [3]. In my earlier paper [1], I used a computer search and elliptic-curve methods to investigate Equation (1). In particular, I reported 858 solutions found for $3 \le r \le 10^5$, and I showed that there are no non-trivial solutions for $r = 5$, $r = 7$, and $r = 9$.

The aim of the present note is to record a simple infinite parametric family of solutions to Equation (1). The proof is elementary and proceeds by direct substitution. After proving the theorem, I compare the family with the tuples listed in the appendix of the earlier paper [1].

2. An infinite parametric family

Theorem 1. *Let n be a positive integer and define*

$$r := 3n + 2, \qquad a := 3n^2 - 2, \qquad b := 3n^3 - 3n + 1. \tag{2}$$

Then (r, a, b) is a positive integer solution to Equation (1). More precisely,

$$\# = \frac{(3n^2-2)(3n^2-1)(3n^3-3n+1)}{2}. \tag{3}$$

Proof. From (2) I obtain

$$r - 2 = 3n, \qquad 5 - r = 3 - 3n, \qquad 4 - r = 2 - 3n.$$

Hence

$$\begin{aligned} a\,(r-2) + 5 - r &= \left(3n^2 - 2\right) \cdot 3n + 3 - 3n \\ &= 9n^3 - 9n + 3 \\ &= 3\left(3n^3 - 3n + 1\right) \\ &= 3b. \end{aligned}$$

Likewise,

$$\begin{aligned} b\,(r-2)+4-r &= \left(3n^3-3n+1\right)\cdot 3n+2-3n \\ &= 9n^4-9n^2+2 \\ &= \left(3n^2-2\right)\left(3n^2-1\right) \\ &= a\,(a+1)\,. \end{aligned}$$

Substituting these identities into Equation (1) gives

$$\begin{aligned} \frac{a\,(a+1)\,(a\,(r-2)+5-r)}{6} &= \frac{a\,(a+1)\cdot 3b}{6} \\ &= \frac{a\,(a+1)\,b}{2}, \end{aligned}$$

while

$$\begin{aligned} \frac{b\,(b\,(r-2)+4-r)}{2} &= \frac{b\cdot a\,(a+1)}{2} \\ &= \frac{a\,(a+1)\,b}{2}. \end{aligned}$$

Therefore the two sides are equal, so (r, a, b) satisfies Equation (1). Since n is a positive integer, the parameters r, a, and b are positive integers as well. Finally,

$$\# = \frac{a\,(a+1)\,b}{2} = \frac{(3n^2-2)(3n^2-1)(3n^3-3n+1)}{2},$$

which proves (3). □

Corollary 2. *The generalized cannonball problem has infinitely many positive integer solutions.*

Proof. Apply Theorem 1 for every positive integer n. □

3. Comparison with the appendix of the earlier paper

The appendix of the earlier paper [1] lists 858 tuples. Testing membership in Theorem 1 by the three parameters (r, a, b) shows that 821 of those 858 tuples are generated by the parametric family (2). For every such tuple, the value of # is then obtained from Equation (1). The remaining 37 tuples are the following:

$r = 3$	$a = 3$	$b = 4$	$\# = 10$
$r = 3$	$a = 8$	$b = 15$	$\# = 120$
$r = 3$	$a = 20$	$b = 55$	$\# = 1540$
$r = 3$	$a = 34$	$b = 119$	$\# = 7140$
$r = 4$	$a = 24$	$b = 70$	$\# = 4900$
$r = 6$	$a = 11$	$b = 22$	$\# = 946$
$r = 8$	$a = 18$	$b = 45$	$\# = 5985$
$r = 8$	$a = 49785$	$b = 6413415$	$\# = 123395663059845$
$r = 8$	$a = 91839$	$b = 16068720$	$\# = 774611255177760$
$r = 10$	$a = 5$	$b = 7$	$\# = 175$
$r = 10$	$a = 6511$	$b = 303336$	$\# = 368050005576$
$r = 11$	$a = 10044$	$b = 581175$	$\# = 1519937678700$
$r = 11$	$a = 16906$	$b = 1269127$	$\# = 7248070597636$
$r = 14$	$a = 6$	$b = 9$	$\# = 441$
$r = 17$	$a = 8583$	$b = 459096$	$\# = 1580765544996$
$r = 30$	$a = 17$	$b = 41$	$\# = 23001$
$r = 41$	$a = 204$	$b = 1683$	$\# = 55202400$
$r = 43$	$a = 33$	$b = 110$	$\# = 245905$
$r = 50$	$a = 34$	$b = 115$	$\# = 314755$
$r = 60$	$a = 5695$	$b = 248132$	$\# = 1785508245600$
$r = 88$	$a = 15$	$b = 34$	$\# = 48280$
$r = 145$	$a = 162$	$b = 1191$	$\# = 101337426$
$r = 276$	$a = 26$	$b = 77$	$\# = 801801$
$r = 322$	$a = 28$	$b = 86$	$\# = 1169686$
$r = 374$	$a = 624$	$b = 9000$	$\# = 15064335000$
$r = 823$	$a = 113$	$b = 694$	$\# = 197427385$
$r = 1152$	$a = 9215$	$b = 510720$	$\# = 149979784926720$
$r = 2378$	$a = 103$	$b = 604$	$\# = 432684460$
$r = 2386$	$a = 420$	$b = 4970$	$\# = 29437553530$
$r = 4980$	$a = 30810$	$b = 3122317$	$\# = 24264913354964425$
$r = 9325$	$a = 12691$	$b = 825436$	$\# = 3176083959788026$
$r = 9525$	$a = 2169$	$b = 58322$	$\# = 16195753597485$
$r = 16420$	$a = 6936$	$b = 333506$	$\# = 913053565546276$
$r = 19605$	$a = 1191$	$b = 23731$	$\# = 5519583702676$
$r = 31265$	$a = 259$	$b = 2407$	$\# = 90525801730$
$r = 31368$	$a = 14858$	$b = 1045635$	$\# = 17147031694579605$
$r = 83135$	$a = 1310$	$b = 27375$	$\# = 31148407558500$

4. Conclusion

I have proved that

$$(r, a, b) = \big(3n + 2,\, 3n^2 - 2,\, 3n^3 - 3n + 1\big)$$

gives a solution to the generalized cannonball problem for every positive integer n. This yields an infinite family of positive integer solutions. A comparison with the appendix of the earlier paper shows that 821 of the 858 listed tuples belong to the parametric family, while the remaining 37 are the exceptional tuples listed above.

References

[1] J.W.L. (Jan) Eerland, *Generalized Cannonball Problem*, viXra:2201.0048, 9 January 2022.
[2] G.N. Watson, *The Problem of the Square Pyramid*, Messenger of Mathematics **48** (1918), 1–22.
[3] Eric W. Weisstein, *Polygonal Number*, MathWorld—A Wolfram Resource.

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