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AN INFINITE FAMILY OF SOLUTIONS TO THE GENERALIZED CANNONBALL PROBLEM

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ABSTRACT. In an earlier paper I considered the generalized cannonball problem for r -regular polygons and studied integer solutions to the associated Diophantine equation. In this note I prove that for every positive integer n , the triple

$$(r, a, b) = (3n + 2, 3n^2 - 2, 3n^3 - 3n + 1)$$

is a solution. Hence the generalized cannonball problem admits infinitely many positive integer solutions. I also compare this parametric family with the 858 tuples listed in the appendix of the earlier paper. Among those tuples, 802 are generated by the present family and the remaining 56 tuples are listed explicitly.

1. INTRODUCTION

The generalized cannonball problem asks for positive integer solutions to

$$\# := \frac{a(a+1)(a(r-2)+5-r)}{6} = \frac{b(b(r-2)+4-r)}{2}, \quad (1)$$

where the left-hand side is the a th r -gonal pyramidal number and the right-hand side is the b th r -gonal number. For background on the classical square case and on polygonal numbers, see Watson [2] and Weisstein [3]. In my earlier paper [1], I used a computer search and elliptic-curve methods to investigate Equation (1). In particular, I reported 858 solutions found for $3 \leq r \leq 10^5$, and I showed that there are no non-trivial solutions for $r = 5$, $r = 7$, and $r = 9$.

The aim of the present note is to record a simple infinite parametric family of solutions to Equation (1). The proof is elementary and proceeds by direct substitution. After proving the theorem, I compare the family with the tuples listed in the appendix of the earlier paper [1].

2. AN INFINITE PARAMETRIC FAMILY

Theorem 1. *Let n be a positive integer and define*

$$r := 3n + 2, \quad a := 3n^2 - 2, \quad b := 3n^3 - 3n + 1. \quad (2)$$

Then (r, a, b) is a positive integer solution to Equation (1). More precisely,

$$\# = \frac{(3n^2 - 2)(3n^2 - 1)(3n^3 - 3n + 1)}{2}. \quad (3)$$

Proof. From (2) I obtain

$$r - 2 = 3n, \quad 5 - r = 3 - 3n, \quad 4 - r = 2 - 3n.$$

Hence

$$\begin{aligned} a(r-2) + 5 - r &= (3n^2 - 2) \cdot 3n + 3 - 3n \\ &= 9n^3 - 9n + 3 \\ &= 3(3n^3 - 3n + 1) \\ &= 3b. \end{aligned}$$

Likewise,

$$\begin{aligned}
 b(r-2) + 4 - r &= (3n^3 - 3n + 1) \cdot 3n + 2 - 3n \\
 &= 9n^4 - 9n^2 + 2 \\
 &= (3n^2 - 2)(3n^2 - 1) \\
 &= a(a+1).
 \end{aligned}$$

Substituting these identities into Equation (1) gives

$$\begin{aligned}
 \frac{a(a+1)(a(r-2) + 5 - r)}{6} &= \frac{a(a+1) \cdot 3b}{6} \\
 &= \frac{a(a+1)b}{2},
 \end{aligned}$$

while

$$\begin{aligned}
 \frac{b(b(r-2) + 4 - r)}{2} &= \frac{b \cdot a(a+1)}{2} \\
 &= \frac{a(a+1)b}{2}.
 \end{aligned}$$

Therefore the two sides are equal, so (r, a, b) satisfies Equation (1). Since n is a positive integer, the parameters r , a , and b are positive integers as well. Finally,

$$\# = \frac{a(a+1)b}{2} = \frac{(3n^2 - 2)(3n^2 - 1)(3n^3 - 3n + 1)}{2},$$

which proves (3). □

Corollary 2. *The generalized cannonball problem has infinitely many positive integer solutions.*

Proof. Apply Theorem 1 for every positive integer n . □

3. COMPARISON WITH THE APPENDIX OF THE EARLIER PAPER

The appendix of the earlier paper [1] lists 858 tuples. A direct comparison with Theorem 1 shows that 802 of those 858 tuples are generated by the parametric family (2). The remaining 56 tuples are the following:

$r = 3$	$a = 3$	$b = 4$	$\# = 10$
$r = 3$	$a = 8$	$b = 15$	$\# = 120$
$r = 3$	$a = 20$	$b = 55$	$\# = 1540$
$r = 3$	$a = 34$	$b = 119$	$\# = 7140$
$r = 4$	$a = 24$	$b = 70$	$\# = 4900$
$r = 6$	$a = 11$	$b = 22$	$\# = 946$
$r = 8$	$a = 18$	$b = 45$	$\# = 5985$
$r = 8$	$a = 49785$	$b = 6413415$	$\# = 123395663059845$
$r = 8$	$a = 91839$	$b = 16068720$	$\# = 774611255177760$
$r = 10$	$a = 5$	$b = 7$	$\# = 175$
$r = 10$	$a = 6511$	$b = 303336$	$\# = 368050005576$
$r = 11$	$a = 10044$	$b = 581175$	$\# = 1519937678700$

AN INFINITE FAMILY OF SOLUTIONS TO THE GENERALIZED CANNONBALL PROBLEM

$r = 11$	$a = 16906$	$b = 1269127$	$\# = 7248070597636$
$r = 14$	$a = 6$	$b = 9$	$\# = 441$
$r = 17$	$a = 8583$	$b = 459096$	$\# = 1580765544996$
$r = 30$	$a = 17$	$b = 41$	$\# = 23001$
$r = 41$	$a = 204$	$b = 1683$	$\# = 55202400$
$r = 43$	$a = 33$	$b = 110$	$\# = 245905$
$r = 50$	$a = 34$	$b = 115$	$\# = 314755$
$r = 60$	$a = 5695$	$b = 248132$	$\# = 1785508245600$
$r = 88$	$a = 15$	$b = 34$	$\# = 48280$
$r = 145$	$a = 162$	$b = 1191$	$\# = 101337426$
$r = 276$	$a = 26$	$b = 77$	$\# = 801801$
$r = 290$	$a = 27646$	$b = 2653921$	$\# = 1014234341580401$
$r = 322$	$a = 28$	$b = 86$	$\# = 1169686$
$r = 374$	$a = 624$	$b = 9000$	$\# = 15064335000$
$r = 479$	$a = 75841$	$b = 12058561$	$\# = 34680418209778721$
$r = 506$	$a = 84670$	$b = 14224393$	$\# = 50988042196726505$
$r = 635$	$a = 133561$	$b = 28181161$	$\# = 251357275983804701$
$r = 740$	$a = 181546$	$b = 44660071$	$\# = 735978580461634001$
$r = 815$	$a = 220321$	$b = 59706721$	$\# = 1449128790279378041$
$r = 823$	$a = 113$	$b = 694$	$\# = 197427385$
$r = 953$	$a = 301465$	$b = 95564089$	$\# = 4342501377748042205$
$r = 980$	$a = 318826$	$b = 103936951$	$\# = 5282613053252028041$
$r = 1037$	$a = 357073$	$b = 123189841$	$\# = 7853443795373180441$
$r = 1152$	$a = 9215$	$b = 510720$	$\# = 149979784926720$
$r = 1616$	$a = 868330$	$b = 467161003$	$\# = 176119197621708804845$
$r = 1730$	$a = 995326$	$b = 573307201$	$\# = 283980510269980488041$
$r = 1832$	$a = 1116298$	$b = 680941171$	$\# = 424268043079628655221$
$r = 1940$	$a = 1251946$	$b = 808756471$	$\# = 633810330690581988041$
$r = 1976$	$a = 1298890$	$b = 854668963$	$\# = 720963068040690331685$
$r = 2006$	$a = 1338670$	$b = 894230893$	$\# = 801248186880443480405$
$r = 2033$	$a = 1374985$	$b = 930864169$	$\# = 879938975751183804245$
$r = 2117$	$a = 1491073$	$b = 1051205761$	$\# = 1168572980486666608961$
$r = 2378$	$a = 103$	$b = 604$	$\# = 432684460$
$r = 2386$	$a = 420$	$b = 4970$	$\# = 29437553530$
$r = 2453$	$a = 2002465$	$b = 1636013089$	$\# = 3280498330950060986705$
$r = 2510$	$a = 2096686$	$b = 1752828661$	$\# = 3852804024591814050701$
$r = 4980$	$a = 30810$	$b = 3122317$	$\# = 24264913354964425$
$r = 9325$	$a = 12691$	$b = 825436$	$\# = 3176083959788026$
$r = 9525$	$a = 2169$	$b = 58322$	$\# = 16195753597485$
$r = 16420$	$a = 6936$	$b = 333506$	$\# = 913053565546276$
$r = 19605$	$a = 1191$	$b = 23731$	$\# = 5519583702676$
$r = 31265$	$a = 259$	$b = 2407$	$\# = 90525801730$
$r = 31368$	$a = 14858$	$b = 1045635$	$\# = 17147031694579605$
$r = 83135$	$a = 1310$	$b = 27375$	$\# = 31148407558500$

4. CONCLUSION

I have proved that

$$(r, a, b) = (3n + 2, 3n^2 - 2, 3n^3 - 3n + 1)$$

gives a solution to the generalized cannonball problem for every positive integer n . This yields an infinite family of positive integer solutions. A comparison with the appendix of the earlier paper shows that 802 of the 858 listed tuples belong to the parametric family, while the remaining 56 are the exceptional tuples listed above.

REFERENCES

- [1] J.W.L. (Jan) Eerland, *Generalized Cannonball Problem*, viXra:2201.0048, 9 January 2022.
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- [3] Eric W. Weisstein, *Polygonal Number*, MathWorld—A Wolfram Resource.