Explicit U(2) matrices and the emergence of electroweak charges

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Abstract

We make the $U(2)$ structure completely explicit in terms of 2×2 matrices and show how Woit's [1] construction on $\Lambda^*(\mathbb{C}^2)$ reproduces the chiral electroweak charges of the first lepton and quark generations.

1 Explicit form of U(2) matrices

An element $U \in U(2)$ is a 2×2 complex matrix satisfying $U^\dagger U = \mathbf{1}_2$. A convenient parametrization is

$$U = e^{i\alpha} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1, \qquad \alpha \in \mathbb{R}. \tag{1}$$

The overall phase $e^{i\alpha}$ generates the $U(1)$ factor, while the 2×2 matrix

$$S = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in SU(2), \qquad \det S = 1, \tag{2}$$

generates the $SU(2)$ factor. The Lie algebra generators of $SU(2)$ in this representation are the Pauli matrices

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

and the $U(1)$ generator is proportional to the identity,

$$Y_0 = \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

Exponentiating, a general $U(2)$ element can be written as

$$U = e^{i\alpha Y_0}\, e^{i\vec{\theta}\cdot\vec{T}}, \qquad \vec{\theta} \in \mathbb{R}^3. \tag{5}$$

2 Action of U(2) on $\Lambda^*(\mathbb{C}^2)$

Let $\mathbb{C}^2$ have basis $\{e_1, e_2\}$. The exterior algebra

$$\Lambda^*(\mathbb{C}^2) = \Lambda^0(\mathbb{C}^2) \oplus \Lambda^1(\mathbb{C}^2) \oplus \Lambda^2(\mathbb{C}^2) \cong \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \tag{6}$$

has the explicit basis

$$\Lambda^0 : \{1\}, \qquad \Lambda^1 : \{e_1, e_2\}, \qquad \Lambda^2 : \{e_1 \wedge e_2\}. \tag{7}$$

An element $U \in U(2)$ acts on $\mathbb{C}^2$ by

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \mapsto \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = U \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{8}$$

so explicitly

$$e_1' = u_{11}e_1 + u_{12}e_2, \qquad e_2' = u_{21}e_1 + u_{22}e_2. \tag{9}$$

The induced action on $\Lambda^*(\mathbb{C}^2)$ is:

- On Λ^0: $1 \mapsto 1$ (trivial representation).
- On Λ^1: the fundamental doublet,

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \mapsto U \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{10}$$

- On Λ^2:

$$e_1' \wedge e_2' = (u_{11}u_{22} - u_{12}u_{21})\, e_1 \wedge e_2 = \det(U)\, e_1 \wedge e_2. \tag{11}$$

Using the explicit form (1), we have

$$\det(U) = e^{2i\alpha}, \tag{12}$$

so under the $U(1)$ phase $e^{i\alpha}$ the three components transform with weights

$$\Lambda^0 : \; e^{0\cdot i\alpha}, \qquad \Lambda^1 : \; e^{1\cdot i\alpha}, \qquad \Lambda^2 : \; e^{2\cdot i\alpha}. \tag{13}$$

The $SU(2)$ part acts nontrivially only on Λ^1, making it a doublet, while Λ^0 and Λ^2 are $SU(2)$ singlets.

3 Lepton hypercharges from U(2) weights

Woit's identification for a lepton generation is

$$\Lambda^0(\mathbb{C}^2) \sim \nu_R, \qquad \Lambda^1(\mathbb{C}^2) \sim (\nu_L, e_L), \qquad \Lambda^2(\mathbb{C}^2) \sim e_R. \tag{14}$$

From the $U(1)$ weights above, we assign a hypercharge Y proportional to the $U(1)$ weight k on Λ^k. Choosing

$$Y = -\,k, \tag{15}$$

we obtain

$$Y(\Lambda^0) = 0, \qquad Y(\Lambda^1) = -1, \qquad Y(\Lambda^2) = -2. \tag{16}$$

Thus:

- ν_R (from Λ^0) is an $SU(2)$ singlet with $Y = 0$;
- (ν_L, e_L) (from Λ^1) form an $SU(2)$ doublet with $Y = -1$;
- e_R (from Λ^2) is an $SU(2)$ singlet with $Y = -2$.

These are precisely the Standard Model electroweak quantum numbers for a lepton generation, with electric charge given by $Q = T_3 + Y/2$.

4 Quark hypercharges via a vacuum shift

For quarks, Woit uses the same $U(2)$ action but shifts the "vacuum" $U(1)$ charge so that the average hypercharge of a full generation (leptons plus quarks) vanishes. Concretely, starting from the lepton hypercharges $Y_{\text{lep}} = \{0, -1, -2\}$ on $\Lambda^0, \Lambda^1, \Lambda^2$, one introduces a constant shift

$$Y_{\text{quark}} = Y_{\text{lep}} + Y_0, \tag{17}$$

and chooses Y_0 so that the resulting assignments match

$$(u_L, d_L): \; Y = +\frac{1}{3}, \qquad u_R: \; Y = +\frac{4}{3}, \qquad d_R: \; Y = -\frac{2}{3}. \tag{18}$$

In this way, the same explicit $U(2)$ matrices (1) acting on $\Lambda^*(\mathbb{C}^2)$, together with an appropriate choice of overall $U(1)$ normalization and shift, reproduce the chiral electroweak charges of the first generation of leptons and quarks.

Reference

[1] https://ai.vixra.org/abs/2503.0012