# Logical Foundations of Geometric Space and Natural Space-time Space

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**Abstract**
This short paper briefly discusses the fundamental concepts, intrinsic rules, and logical relationships of geometry from the most essential perspective, and its relationship with algebra. It also defines natural space-time space and its intrinsic logical relationships and significance. Most of the content in this paper is important achievements of the" Geometry of Space-time Structures " established by the author [1-5].

## 1. Concepts and Elements of Geometry [1-3]

Geometry is a precise quantitative logic concerning various morphological structures and their intrinsic relationships within space. Its basic elements include: a geometric space with defined dimensions, coordinate system, distance unit, and scale; points, lines, surfaces, and volumes; and fundamental morphological construction relationships.

With different scales, the morphological structure of the same object appears differently. "Shape" is relative. The "points" in geometric space have size and shape. A spatial point on a macroscopic scale may reveal rich internal structures when continuously magnified. Objectively, absolutely straight and uniform "lines, surfaces, and solids" do not exist. If so-called "manifolds" are used to represent various shapes and structures in geometric space, "smooth manifolds" are only a very small minority in natural reality.

The "shape" of geometric space is supported or presented by "intervals"; space and distance are real existences. The directional distance differential element dr or dS is a real number belonging to the real number set R, with $dr^2$ or $dS^2>0$. Coordinate points in geometric space are symbols marking spatial positions and have no size.

## 2. Mathematical Logic of Geometric Space — Coordinate, Dimension, Metric, Geometric Quantities , Characterization and Operation of Morphological Structures [1-4]

The position of any point P in any geometric space X is marked by numerical coordinates. Continuous coordinates and quantifiable coordinate functions are the logical basis for precise operations in geometric space. There can be an infinite number of geometric spaces, and precise quantitative comparisons and conversions between them require consistent scaling and a uniform, simple reference system (reference space).

Euclidean space serves as such a reference system. For the flat, uniform n-dimensional Euclidean space Rn, and its corresponding Cartesian coordinates $(x_1,x_2,...,x_n)$, the set form can be written as $Rn=\{p=(x_1,x_2,...,x_n) \mid \forall xi \in R\}$. The distance metric relation is $dS^2= \sum \delta_{ij} x_i\, x_j=\sum_{i=1}^{n} dx_i^2$. In this flat, uniform space, the distance S between any two points $P_1$ $(x_{11},x_{12},...,x_{1n})$ and $P_2$ $(x_{21},x_{22},...,x_{2n})$ is obtained by integration as $S=\sqrt{\sum_{i=1}^{n} (x_{2i}-x_{1i})^2}=\sqrt{\sum_{i=1}^{n} \Delta x_i^{\,2}}$. The coordinate variables $x_i$ are all of the same type of real numbers. The "points" of different dimensions in the entire space are of the same property and are connected. Euclidean space can be viewed as an ordered combination of uniform lattices: if 1 dimension has R points, then n dimensions have $R^n$ points. The dimension of a geometric space represents the overall symmetry of the space. The independent variables $x_i$ of the n dimensions in the Cartesian coordinate system are mutually independent, and the coordinate axes are mutually "perpendicular". After defining direction vectors, the operational relationships related to the position vector $\vec{r}(x_1,x_2,...,x_n)$ naturally incorporate the logic of operations such as functions, algebraic equations, vectors, and matrices. The key to precise calculation is determining geometric quantities and quantitative relations such as length, area, volume, neighbor and structural change relations, and morphological characterization,etc., based on position coordinates and distance differentials.

Any geometric space X can be naturally "embedded" into Euclidean space, using Euclidean space as a unified reference space. Using the same unit scale and position coordinate variables, the metric and distance differential of any geometric space (or manifold) can be determined, thereby revealing different spaces and their structures, and enabling precise calculation, analysis, and comparison. Clearly, the definition of the distance differential $dS^2$ determines the space structure properties. Compared to Euclidean space, general spaces and structures are "curved" and convergent.

B. Riemann argued [6] that the metric distance differential at any point P in any n-dimensional geometric space X is $dS^2 = \sum g_{ij} dx_i\, dx_j$, where $g_{ij}$ are the matrix elements of the metric tensor. However, this representation does not guarantee $dS^2 > 0$ for a real coordinate space. At sufficiently small scales, an orthogonal coordinate system is effective, and the origin of the Cartesian coordinate system can be translated to point P, so we can have,

$$dS^2=\sum_{i=1}^{n} g_{ii} dx_i^2=\sum_{i=1}^{n} g_i^2 dx_i^2 \quad , \quad g_{ii} = g_i^2 \geqslant 0 \tag{1}$$

Defining the second-order spatial distance differential, essentially determines the first and second derivative (rate of change) relationships of the spatial structure (or manifold), i.e., determines the intrinsic change relationships of the geometric manifold. This is precisely the content of differential geometry. Of course, for objective reality, this too is only an approximation or represents a very small minority of morphological structures.

Meaningful variables in geometric space are proportional to space capacities such as length, area, and volume, which have an exponential relationship with the coordinate variables. For any n-dimensional space structure or manifold, any n-dimensional capacity unit (volume element) is enclosed by at least n+1 (n-1)-dimensional manifolds. The n-dimensional spatial morphological relation function $f(x_1, x_2, \ldots, x_n)$ (i.e., an n-dimensional smooth manifold) can be decomposed into combination forms of various (n-1)-dimensional $f(x_1, x_2, \ldots, x_{n-1})$ and 1-dimensional $f(x_n)$. In other words, n-dimensional space structures contain nested combinations of various possible lower k-dimensional space structures ($0<k<n$), implying the existence of iterative expansion relations for various spaces or sets with the same kind properties, such as group relations. However, the more universal morphological structure relation in geometric space is the fractal structure of fractional dimension — self-similarity.

## 3. Spatial Morphology Structures, Fractals, and Fractional Dimensions [1-5]

In Euclidean space, the simplest geometric shape units include triangles (pyramids), squares (cubes), and circles (spheres). In any n-dimensional geometric space ($n \geq 2$), straight lines and completely smooth surfaces generally do not exist, triangles (pyramids), squares (cubes), circles (spheres), etc., all become closed units of arbitrary size surrounding a central point (topological equivalence), their uniquely determined macroscopic quantity is the rounded angle or solid angle $2^{n-1}\pi$ (with the center as the vertex).

Morphological structures precisely describable by continuous functions $f(x_1,x_2,...,x_n)$ constitute only a very small portion within a geometric space. More common are fractal structures with certain self-similarity, i.e., there exists similarity in morphology structure between large and small scales, between different levels, or between different regions within the same level, (this is also a symmetry of spatial structure). The logic is simple: any geometric space structure can be decomposed into numerous substructures at arbitrarily small scales. As the spatial extension combination set of these substructures, it must possess some regularity characteristic distinct from any other spatial morphology structure; otherwise, all structures would converge and become indistinguishable. The various spatial structures (material structures) existing in the natural world are all self-organized process states that "grow and expand" over time, and spatial structures are all fractal structures.

The analytical characteristic of fractal structures is having a fractional dimension, its intuitive definition and calculation method are: assuming a fractal structure is a bounded structure set in Euclidean space, when covering this structure set with circles or solid spheres of radius r, let N(r) represent the minimum number of circles or spheres corresponding to the minimum coverage, then the structure dimension D of the fractal is defined as $D=\lim_{r\to 0}\frac{\log N(r)}{\log(\frac{1}{r})}$ . Let $D_0$ represent the integer dimension n of Euclidean space, the space-filling degree $\theta$ of the fractal structure is defined as $\theta=\lim_{r\to 0}\frac{\log N(r)}{\log(\frac{1}{r})^{D_0}}=\frac{D}{D_0}$ . Clearly, $\theta$ is related to the proportion of certain capacities of the fractal structure in the corresponding Euclidean space, providing a simple and universal method for calculating the fractal dimension D.

The dimension of any geometric space is an inherent property of the space, a normal constant. From the perspective of coordinate independent variables, the dimension is the number of independent variables. From the perspective of a specific space structure, if the constraints forming the space structure remain stable, the number of coordinate independent variables presenting the space structure remains unchanged; each determined function equation reduces one coordinate independent variable (degree of freedom). The most fundamental geometric characteristic presented by any space structure is the stability of its structural dimension, which is also a topological invariance, corresponding to a large group.

For fractal structures with fractional dimensions, a fractional dimension implies that in the corresponding Euclidean space of integer dimension, the coordinate values of independent variables are discontinuous and discrete; independent variables on multiple different coordinate axes may all be discontinuous, forming discrete, random jump combinations in space (the independent variables are not independent, not continuous, and have intrinsic correlations). The spatial structure morphology naturally exhibits a certain discreteness and randomness, with intrinsic changes having inherent stochastic statistical correlation properties. Fractional dimensions, fractal structures, and statistical correlations are internally unified. The inherent constraint conditions governing the survival of general spatial fractal structures are stable, and their fractal dimensions are topological invariants. The invariance of the fractal dimension can also be proven by other methods [7].

The spatial structures of various objects in natural space, broadly speaking, are all fractal structures, with the geometric structure dimension D, $0 \leqslant D \leqslant 3$. Thus, Euclidean geometry, Riemannian geometry, Lobachevskian geometry, fractal geometry, etc., are naturally unified.

## 4. Geometric Characteristics of Infinitesimal Space and Infinite Distant Space [1-4]

The essential difference between geometric space and an algebraic set is that intervals (distances) completely separate neighboring point elements, forming ordered arrangements (relations) in true "space". "Interval, distance" have real existence. On any $x_i$ coordinate axis in an n-dimensional Euclidean space $(x_1, x_2, \ldots, x_n)$, for an arbitrarily infinitesimal quantity $\Delta x_i$(space, distance) near the independent variable $x_i$, as $\Delta x_i \rightarrow 0$ we always have $\Delta x_i \neq 0$, and $\Delta x_i$ can be subdivided into a second or arbitrarily m-th order infinitesimal divisions (partitions), similarly yielding $\Delta(\Delta x_i) \neq 0$ or $\Delta^m x_i \neq 0$; expressed in differential form: $dx_i \neq 0$ and $d^m x_i \neq 0$; the independent variable (space) is continuous and infinitely differentiable, (this differs completely from existing mathematical theory which posits $d(dx_i)=0$); $\Delta x_i$ or $dx_i$ itself is an indeterminate arbitrarily infinitesimal independent variable, its differential is also an arbitrary infinitesimal independent variable, i.e., itself (this does not contradict $(\Delta x_i)^m$ or $\prod_i^m \Delta x_i$ being arbitrarily high-order infinitesimals), and the rate of change (derivative) of each order of infinitesimal naturally takes the same value (i.e., it is an e-exponential map). Therefore, for the independent variable $x_i$ on any coordinate axis in uniform Euclidean space, we have,

$$\forall x_i,\ \Delta x_i=\Delta(\Delta x_i)=\Delta^m x_i,\ dx_i=d(dx_i)=d^m x_i\ ,\ \frac{dx_i}{dx_i}=\frac{d(dx_i)}{dx_i}=\frac{d^m x_i}{dx_i}=1=e^0 \qquad (2)$$

The essential logical meaning of Equation (2) is that within an infinitesimal region of the independent variable geometric space, uniform continuity and indivisible quantum discrete distribution coexist, mathematical logics such as continuity and discreteness, uniform distribution and quantum statistics, are unified. The "space" logic of infinitesimal $\Delta x_i$ of the independent variable is both the starting point where geometry differs from algebra and their connection point. Natural integers, rational numbers, real numbers, etc., can all be characterized and presented by the logical relations of the infinitesimal independent variable $\Delta x_i$. Since $\Delta x_i$ possesses indivisible quantum properties, it can represent a fundamental size unit "point" of real space(i.e.,the smallest space quantum unit), represented by 1; its space position uncertainty, or its geometric morphological equivalent quantity, is a function of itself — the natural exponential function, i.e., $e^{\Delta x_i}=e^1=e$. The irrational number e is precisely the embodiment of which the infinitesimal $\Delta x$ simultaneously possessing "continuity and discrete quantum uncertainty" in geometric space.

The above logic concerning $\Delta x_i$ and $dx_i$ can be extended to the entire n-dimensional Euclidean space Rn. Dimension is an inherent property of geometric space, representing the overall symmetry and the number of independent variables of the space (structure). The n-dimensional infinitesimal space volume (capacity) in Euclidean space is $\Delta V=\prod_i^n \Delta x_i$ or in differential form $dV=\prod_i^n dx_i$, (which can be simplified to $(\Delta x)^n$ or $(dx)^n$, respectively). $\Delta V$ or $dV$ also possesses the logical properties of continuous uniformity and discrete indivisible quanta, representing a fundamental space unit "point" with size and morphological structure in n-dimensional Euclidean space, its spatial morphological structure equivalent quantity(symmetric states) equals $e^n$.

This logical relation also applies to geometric spaces of fractional dimension (fractal structures). The equivalent state quantity of the infinitesimal structure unit of the indivisible quantum "point" is $e^D$. This means that the morphological structure of an infinitesimal particle in the real world is the space representation (characterization and quantification) of its geometric dimension and degree of freedom. For example, single-particle quantum states, the degree of freedom of nucleon unit within atomic nuclei, etc., all follow this relation. Here, the dimension D can be an integer, i.e., $0 \leqslant D \leqslant 3$.

This infinitesimal quantum property of geometric space is precisely an objective reality of the natural world. Elementary particles have "size", are indivisible, and possess statistical quantum properties. The 3D natural space of the vacuum (which can contain light quanta) corresponds to 3D Euclidean space, natural space containing fundamental material quanta such as protons and electrons, as well as light quanta, corresponds to a general geometric space with $D \leqslant 3$.

For any geometric space structure (manifold), the mapping relation between the infinitesimal Δf of a function variable (or operator) and the infinitesimal independent variable Δx is also an exponential map, this is the Lie group-Lie algebra relationship. The logical foundation for why Lie algebras serve as an important algebraic tool in quantum field theory [8] lies precisely here.

The logic of geometric space at infinity ($r \to \infty$) is simpler. $\infty\infty \sim \infty$, $\Delta\infty \sim \infty$, continuous differentiability still exists, but the geometric morphological structure contracts and bends into a point. Although the point has enormous "capacity", it contains little clear or novel information. The continuously magnified image of the point at infinity is just "everything seen from the origin".

## 5. Essential Properties of Natural Space-time Space [1-3]

Natural space-time space is the real space of nature，it is a quasi-4-dimensional space composed of static 3D natural space plus quasi-1-dimensional time, with coordinates (x,y,z,ict) or abbreviated as (r,ict). Time t flies unidirectionally at the speed of light c; there is no independently existing time space connected to the 3D (x,y,z) space, the time dimension is naturally taken as "imaginary". Natural space-time space is a motion space, the imaginary coordinate ict precisely embodies the inherent motion and change of time and its nature as a primary variable acting independently in the natural world.

The distance differential dS of quasi-4-dimensional natural space-time space is defined as:

$$dS^2 = n_x^2 dx^2 + n_y^2 dy^2 + n_z^2 dz^2 - c^2 dt^2 \tag{3}$$

where n(x,y,z) represents the generalized refractive index tensor (orthogonal metric tensor) of space medium. When dt=0, equation (3) becomes the optical path differential in isochronous 3D natural space. In an infinitesimal space unit ($\Delta V \to 0$) or in the neighborhood of the coordinate origin, equation (3) is hyperbolic, approximating Minkowski space.

Points in natural space-time space, and any structures composed of these points, are open sets containing certain physical connotations. Lengths, surface areas, morphological structures and the space volumes they enclose are all measures and manifestations of matter and energy. The morphological structure in the space-time coordinate system (r,ict) can be precisely transformed with the energy state structure in the momentum-energy coordinate system (P,iE/c) via Fourier transform. The distance invariance of the (P,iE/c) coordinate system corresponds to the conservation of energy. $dS^2=0$, $c=dr/dt$, $E=cP$, the speed of light is maximal, no object can outrun time, the speed of light is the speed of time, the light quantum is the time quantum.

At the scale of the infinitesimal space-time quantum unit where $\Delta r \to 0$, $\Delta V \to 0$, $\Delta t \to 0$, space, time, and matter are unified. Δr, Δt, Δm, ΔE, etc., necessarily exhibit definite relationships, such as $\Delta E = \Delta m\, c^2$, $\Delta E = h\nu = h/\Delta t$, etc.; the nearest neighbor separation and combination between infinitesimal space units ΔV is a zero-range strong interaction accompanied by a material energy change of ±$\Delta mc^2$. The elementary particles – protons and electrons，in their free static state，their spatial sizes are influenced by electricity and uncertainty, but can be determined experimentally measuring their space-time coordinate hyperbolas.

As a reasonable generalization, in the era of optoelectronic information, various dynamic virtual scenes created by humans can be characterized and described using imaginary spaces such as (ix,iy,iz,ct) or (ix,iy,ct). Time used corresponds to the real world, here as a real coordinate.

The same logical relationships can have different representations, the relationship between geometry and algebra is relatively unified. However, nature itself is geometric space, which determines the dominant position of geometry in mathematics.

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