Towards a Floer theory for Mars
I - Twisted Zeeman systems

Urs Frauenfelder
Universität Augsburg

Joa Weber*
UNICAMP

May 29, 2026

Abstract

In this article we study periodic orbits of an electron attracted by a proton subject to Lorentz, electric, and Euler forces where each of them is allowed to depend periodically on time. This setup is motivated by the elliptic restricted three-body-problem where the Lorentz force corresponds to Coriolis force, the Coulomb force is replaced by the gravitational force, and the electric force of an external source is a combination of centrifugal forces and gravitational forces of other bodies. This is a singular version of a Euler-Hamilton system as discussed in [FW26b]. The singularity is due to collisions of the electron with the proton, respectively of two masses. Due to the possibility of collisions this problem has to be regularized.

We show how periodic collisional solutions of this problem can be detected variationally in a non-local Lagrangian setup as well as in a non-local Hamiltonian setup.

Contents

*Email: urs.frauenfelder@math.uni-augsburg.de joa@unicamp.br

1 Introduction

In 1609 the emperor of the Holy Roman Empire of the German Nation Rudolf II received a dedication of a rather special book. In it the author claimed that he had taken the God of War captive. This author was Johannes Kepler and the book was the *Astronomia Nova* [Kep09][1] in which Kepler showed that the orbit of Mars is an ellipse. In fact, Mars was the Roman god of war and in the *Astronomia Nova* Kepler overcame the Aristotelian Worldview that on heaven everything is moving on perfect circles.

Mars and Floer theory. Studying periodic orbits in the vicinity of Mars also leads to new structures in Floer theory which we describe in this paper. In view of the high eccentricity of Mars the system consisting of sun, Mars and a satellite cannot well be described by the circular restricted[2] three body problem, but has to be attacked by the elliptic restricted three body problem. Because of the eccentric movement of Mars, the Coriolis force in the elliptic restricted three body problem is not constant, but *depends periodically on time*.
A basic system where the Coriolis force is time-dependent is a merry-go-round which accelerates and decelerates. How such systems can be described using time-dependent symplectic forms was the subject of the recent work [FW26b]. Like the Lorentz force of a magnetic field, the Coriolis force depends linearly on the velocity and therefore can be modeled by twisting the standard symplectic form on the cotangent bundle. In the case where the magnetic field is time-dependent, a new force shows up, the Euler force, which in contrast to the Lorentz force depends on the choice of time-dependent primitive of the time-dependent symplectic form.
In the case of a satellite around Mars a new issue shows up and these are collisions with Mars. Although one does not want to put a satellite on a collisional orbit with Mars, to obtain the global picture of periodic orbits one has to take into account these collisional orbits as well. In fact, considering a homotopy periodic orbits appear in families and such families can go through collisions.

Variational approach to collisional orbits. For global theories like Floer theory the crucial ingredient is a variational approach to periodic orbits. In order to obtain a variational approach as well to collisional orbits, we apply in this paper the new regularization technique of Barutello, Ortega, and Verzini [BOV21] of blowing up the loop space, see also [Fra25], to the case of time-dependent magnetic fields.

To treat these kind of problems we develop a general setup which we refer to as twisted Zeeman systems.[3] A **Stark-Zeeman system**, as introduced in [CFvK17], describes the motion of an electron attracted by a proton and sub-

[1] Deutsche Übersetzung [Kep37]. English translation [Kep92].
[2] "Restricted" means that the satellite is considered massless and does not attract the sun and mars, but on the other hand is attracted by sun and Mars.
[3] A **planar twisted Zeeman system** describes the motion of an electron in the plane attracted by a proton in a time-dependent magnetic field which admits a primitive depending twisted periodically on time. Such a magnetic field is necessarily time periodic, see Remark 2.2.

ject to a magnetic and electric field. In a **twisted Zeeman system** the magnetic field is allowed to depend on time and, in contrast to a time-independent case, the dynamics also depends on the choice of the time-dependent primitive of the time-dependent symplectic form. While the magnetic field is assumed to depend periodically on time, the time-dependence of the primitive is not necessarily periodic, but is allowed to be twisted-periodic. The additional freedom of allowing this twist makes it possible to incorporate the electric field in the twist as well, as observed in [FW26b, §3.3]. Therefore the contribution of Stark, i.e. the electric field, is not needed any more, so that every Stark-Zeeman system can be re-interpreted as a twisted Zeeman system.

Blowing up the loop space requires a reparametrization of the loop which depends on the loop. Therefore in the regularized system the symplectic form on the loop space becomes non-local. In the present article we introduce a Lagrangian and a Hamiltonian action functional which are related by a non-local Legendre transformation, see also [FW21, CFV23], and show that the critical points correspond to periodic solutions of twisted Zeeman systems allowing collisions of the electron with the proton. In part II [FW26d] we then show that the linearized L^2-gradient flow equation of the non-local Hamiltonian action functional for this non-local symplectic form is a Fredholm operator by showing that its Hessian field almost extends by applying our general Fredholm result from the recent article [FW26c].

1.1 Main results

We twist the Lagrangian of the Kepler problem with a 1-form on the plane $\mathbb{C}$ depending periodically on time $\theta_{t+1} = \theta_t$. The physical interpretation of this 1-form is the following. The exterior derivative with respect to space gives rise to a time-dependent magnetic field $d\theta_t$. The time derivative $\dot{\theta}_t$ of the 1-form gives rise to a force known as Euler force which can for example be felt on an accelerating merry-go-round [FW26b]. We, more generally, consider 1-forms which are only twisted-periodic in time, and not periodic. This additional twist allows to model electric forces as well [FW26b, §3.3]. In this introduction, for simplicity of exposition, we do not consider this additional twist and also do not consider the case where the 1-form is defined on open subsets of the plane.

Periodic orbits of this problem can be detected variationally as the critical points of the classical Lagrangian action functional on loop space

$$\mathcal{S}\colon \mathcal{L}\mathbb{C}^\times \to \mathbb{R}, \quad q \mapsto \int_0^1 \Big(\tfrac{1}{2} \left|\dot{q}_t\right|^2 + \frac{1}{|q_t|} + \theta_t|_{q_t} \dot{q}_t \Big) dt.$$

Since we have to exclude collisions with the singularity of the Kepler potential $-1/|q_t|$ at the origin, this functional is only defined on loops that avoid the origin, i.e. loops in $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. In order to also allow collisional solutions variationally we regularize this functional. For that purpose we consider a non-local map $\mathcal{Q}\colon \mathcal{L}\mathbb{C}^\times \to \mathcal{L}\mathbb{C}^\times$. This map $\mathcal{Q}$ re-parametrizes the complex squaring map by a circle diffeomorphism $\tau_z\colon \mathbb{S}^1_t \to \mathbb{S}^1_\tau$ which depends on the loop z itself

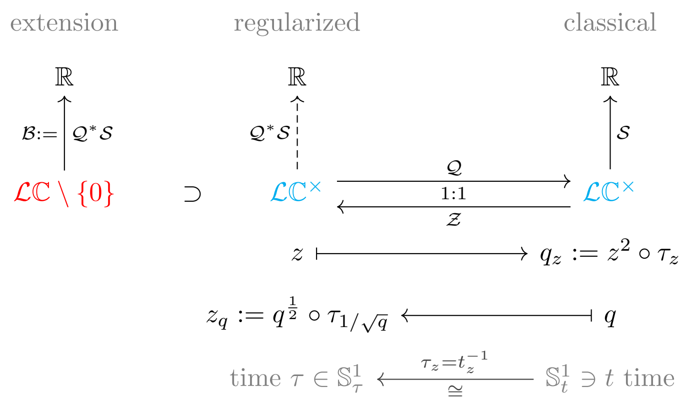

Figure 1: Regularization $\mathcal{B}$ on blow-up of loop space $\mathcal{L}\mathbb{C}^\times$

and is therefore not local. The map $\mathcal{Q}$ was discovered by Barutello, Ortega, and Verzini [BOV21]. It is not smooth in the usual sense, but scale-smooth in the sense of Hofer-Wysocki-Zehnder [FW26a]. Pulling back the functional $\mathcal{S}$ under $\mathcal{Q}$ we obtain the following sum of three terms $\mathcal{B} = \mathcal{K} - \mathcal{U} + \mathcal{M} \colon \mathcal{L}\mathbb{C}^\times \to \mathbb{R}$ where

$$\mathcal{B}(z) = 2\|z\|^2\|z'\|^2 + \frac{1}{\|z\|^2} + \int_0^1 \vartheta_{t_z(\tau)}|_{z(\tau)} z'(\tau)\, d\tau.$$

Here t_z is the inverse of τ_z and $\langle\cdot,\cdot\rangle$ is the L^2-inner product with associated norm $\|\cdot\|$ and ϑ is the pull-back of θ by the complex squaring map. Because the map $\mathcal{Q}$ is non-local, the periodic orbits pulled back by $\mathcal{Q}$ do not satisfy a second order ODE (2.8) any more, but a second order delay differential equation, the DDE (4.46) which actually characterizes the critical points of $\mathcal{B}$.

The functional $\mathcal{B}$ naturally smoothly extends, via the formula provided by $\mathcal{Q}^*\mathcal{S}$ as illustrated in Figure 1, to the space $\mathcal{L}^\times\mathbb{C} = \mathcal{L}\mathbb{C} \setminus \{0\}$, i.e. the loop is allowed to cross the origin, the only thing that is forbidden is that the loop stays for all times at the origin. Times where the loop crosses the origin are interpreted as collision times. After the extension, critical points of the functional consist of two kinds, namely non-collisional solutions (Section 4.3), i.e. solutions which never cross the origin, and solutions admitting collisions (Section 4.4), i.e. solutions which cross the origin.

Non-collisional critical points are in 1-to-1 correspondence with critical points of $\mathcal{S}$, i.e. periodic orbits of a particle subject to firstly Newton's, respectively Coulomb's, force secondly a time-dependent Lorentz, respectively Coriolis, force as well as thirdly the Euler force. If the 1-form is even allowed to be only twisted-periodic in time, we can twist the 1-form with an additional electric potential and therefore the particle can be subject additionally to an electric force.

In Section 4.5 we explain what precisely collisional solutions are on the q-side and show the following.

Theorem A. *Collisional critical points z of $\mathcal{B}$ are in 1-to-1 correspond to classical collisional solutions q.*

Theorem A is proved in Theorem 4.15 by constructing a parcial inverse map $\mathcal{Z}$ to $\mathcal{Q}$ which is illustrated in Figure 1.

The second contribution of this paper is that, in Section 5, we provide as well a Hamiltonian formulation of the regularized solutions. For this purpose we apply a non-local Legendre transformation to the non-local functional $\mathcal{B}$ which is motivated by [FW21]. The functional we obtain is given by

$$\mathcal{A} = i_{\mathcal{V}}\left(\Lambda + \pi^*\Theta\right) - \mathcal{H}\colon T^*\mathcal{L}^\times\mathbb{C} \to \mathbb{R}, \qquad T^*\mathcal{L}^\times\mathbb{C} = \mathcal{L}^\times\mathbb{C} \times \mathcal{L}\mathbb{C}.$$

The ingredients on the right hand side of $\mathcal{A}$ are the following. Firstly there is the canonical vector field $\mathcal{V} = \partial_\tau$ along $T^*\mathcal{L}^\times\mathbb{C}$, namely $\mathcal{V}(z,\eta) = (z', \eta')$. Secondly Λ is a 1-form on $T^*\mathcal{L}^\times\mathbb{C}$ obtained by integrating the Liouville form λ on $T^*\mathbb{C}$. Thirdly, in contrast to Λ, the 1-from Θ on $\mathcal{L}^\times\mathbb{C}$ is non-local. If $z \in \mathcal{L}^\times\mathbb{C}$, then for a tangent vector $\xi \in T_z\mathcal{L}^\times\mathbb{C} = \mathcal{L}\mathbb{C}$ we define

$$\Theta_z\xi := \int_0^1 \vartheta_{t_z(\tau)}|_{z(\tau)}\xi(\tau)\, d\tau.$$

In particular, the last term $\mathcal{M}$ of $\mathcal{B}$ is given by $\mathcal{M}(z) = \Theta_z z'$. The base point projection and a canonical injection are given by

$$\begin{aligned} \pi\colon T^*\mathcal{L}^\times\mathbb{C} &\to \mathcal{L}^\times\mathbb{C} & \qquad \iota\colon \mathcal{L}^\times\mathbb{C} &\to T^*\mathcal{L}^\times\mathbb{C} \\ (z,\eta) &\mapsto z & z &\mapsto \left(z, 4\|z\|^2 z'\right) \end{aligned} \tag{1.1}$$

The non-local Hamiltonian $\mathcal{H}\colon T^*\mathcal{L}^\times\mathbb{C} \to \mathbb{R}$ can be thought of as the non-local Legendre transform of the first two terms $\mathcal{K} - \mathcal{U}$ of $\mathcal{B}$ and reads

$$\mathcal{H}(z,\eta) = \frac{\|\eta\|^2}{8\|z\|^2} - \frac{1}{\|z\|^2}.$$

Our second main result is the following.

Theorem B. *There is a bijection given by*

$$\mathrm{Crit}\,\mathcal{A} \underset{\iota}{\overset{\pi}{\underset{1:1}{\rightleftarrows}}} \mathrm{Crit}\,\mathcal{B}\ .$$

Proof. Section 5.5. □

The critical points of $\mathcal{A}$ are solutions of the following second order delay equation.

Theorem C. *Suppose that $(z,\eta) \in \mathcal{L}^\times\mathbb{C} \times \mathcal{L}\mathbb{C}$ is a critical point of $\mathcal{A}$. Then (z,η) is a solution of the following problem*

$$\begin{aligned} z' &= \tfrac{1}{4\|z\|^2}\, \eta \\ \eta' &= \tfrac{\|\eta\|^2 - 8}{4\|z\|^4} z + \mathrm{grad}\, \mathcal{M}(z) \end{aligned}$$

where the L^2-gradient of $\mathcal{M}$ is of the following form. Writing the 1-form $\vartheta_t = a_t^1\,dx + a_t^2\,dy$ in the form of a pair $\boldsymbol{a}_t = (a_t^1, a_t^2) \in \mathbb{R}^2$ there is the formula

$$
\begin{aligned}
(\operatorname{grad}\mathcal{M}|_z)_\tau = &-\frac{2z_\tau}{\|z\|^4}\int_0^1 \textstyle\int_0^\sigma |z_\rho|^2\,d\rho \cdot \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma - \frac{|z_\tau|^2}{\|z\|^2}\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} \\
&+ \frac{2z_\tau}{\|z\|^2}\int_{\sigma=\tau}^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma - (\operatorname{rot}\boldsymbol{a}_{t_z}|_z)_\tau \, i z'_\tau
\end{aligned}
$$

for each time $\tau \in \mathbb{S}^1$ and abbreviating $z_\tau := z(\tau)$.

Proof. Theorem 5.13 and Lemma 4.3. □

In Appendix B we show that the exterior derivative of the non-local 1-form $\Lambda + \pi^*\Theta$ is a weak symplectic form on $T^*\mathcal{L}^\times\mathbb{C}$. This uses an abstract result on weak symplectic forms in Appendix A which is of independent interest. Hence the critical point equation in Theorem C can be interpreted as the Euler-Hamilton equation of $\mathcal{H}$ with respect to the 1-form $\Lambda + \pi^*\Theta$ as in [FW26b]. This is explained in Theorem 5.13.

The 1-form Θ only exists for time-dependent 1-forms ϑ which are periodic in time. However, the weak symplectic form on $T^*\mathcal{L}^\times\mathbb{C}$ still makes sense in the twisted-periodic case and remains a weak symplectic form, as discussed as well in Appendix B.

Notation. Working with functions on function spaces easily triggers excesses of parentheses, which harms legibility. Therefore we often write variables either as subscripts $\mathcal{M}_z$ or in the form $\mathcal{M}|_z$, as opposed to $\mathcal{M}(z)$. For time-dependence subscript has priority, for example if $t \mapsto q(t)$ is a loop then $\theta_t|_{q_t}\dot{q}_t$ denotes a time-dependent 1-form at time t and at the spatial point $q(t)$ evaluated on the velocity vector $\dot{q}(t)$. Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ be the $L^2(\mathbb{S}^1,\mathbb{R}^2)$ norm and inner product. There are two circles, the quotient circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and the unit circle $\mathbb{S}^1_\mathbb{C} \subset \mathbb{C}$. A map $f\colon \mathbb{R} \to X$ with $f_{t+1} = f_t$ $\forall t \in \mathbb{R}$ is called **periodic**, notation $f\colon \mathbb{S}^1 \to X$. While, as is common, ODE abbreviates ordinary differential equation, **DDE** stands for **delay differential equation**.

Acknowledgements. UF acknowledges support by DFG grant FR 2637/5-1.

2 Twisted Zeeman system

Let $0 \in \mathfrak{Q} \subset \mathbb{C}$ be an open subset containing the origin. Without 0 we write

$$\mathfrak{Q}^\times := \mathfrak{Q} \setminus \{0\} \subset \mathbb{C} \simeq \mathbb{R}^2.$$

We freely identify $\mathbb{C}$ with $\mathbb{R}^2$ using whichever is convenient. For $q \in \mathfrak{Q}$ we write

$$\mathbb{C} \ni q_1 + iq_2 \simeq (q_1, q_2) \in \mathbb{R}^2 \qquad \mathcal{L}_{\mathbb{R}}(\mathbb{C}) \ni i \simeq j_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2). \tag{2.2}$$

For brevity we often write i. If i is in front of an element of $\mathbb{R}^2$ it means j_0. By $\langle \cdot, \cdot \rangle_0$ we denote the Euclidean inner product on $\mathbb{R}^2$, by $|\cdot|$ the induced norm.

Definition 2.1 (twisted-periodic)**.** A **twisted-periodic 1-form** $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is a real smooth family of 1-forms on $\mathfrak{Q}$, notation

$$\theta_t = A_t^1\, dq_1 + A_t^2\, dq_2, \qquad A_t^1, A_t^2 \colon \mathfrak{Q} \to \mathbb{R}, \tag{2.3}$$

with θ **twisted-periodic** in the sense that a) the time-derivative is periodic

$$\text{a) } \dot\theta_{t+1} = \dot\theta_t, \qquad \text{b) } \theta_{t+1} = \theta_t + df, \tag{2.4}$$

and b) there is a smooth function $f \colon \mathfrak{Q} \to \mathbb{R}$ satisfying the above, cf. [FW26b, §5], called a **twist function** of $\theta = \{\theta_t\}$. In the periodic case $f = 0$. As a consequence the exterior derivative is periodic, in symbols $d\theta_{t+1} = d\theta_t$. The coefficients of the twisted 1-form θ yield a vector field along $\mathfrak{Q}$ of the form

$$\boldsymbol{A}_t|_q := \left(A_t^1|_{(q_1,q_2)}, A_t^2|_{(q_1,q_2)}\right), \qquad \dot{\boldsymbol{A}}_{t+1} \overset{(2.4)_{\mathrm{a}}}{=} \dot{\boldsymbol{A}}_t, \tag{2.5}$$

called a **vector potential**.

Remark 2.2. Magnetic fields are described by closed 2-forms σ, where closedness $d\sigma = 0$ encodes the fact that there are no magnetic charges; see e.g. [Web17, § 2.4.1]. For a twisted-periodic 1-form θ the 2-form

$$\sigma_t := d\theta_t = d\left(A_t^1\, dq_1 + A_t^2\, dq_2\right) = \underbrace{\left(\partial_1 A_t^2 - \partial_2 A_t^1\right)}_{=:\operatorname{rot} \boldsymbol{A}_t =: B_t} \underbrace{dq_1 \wedge dq_2}_{=:\omega_0} \tag{2.6}$$

is time-periodic $\sigma_t := d\theta_t = d\theta_{t+1} = \sigma_{t+1}$ and closed $d\sigma_t = dd\theta_t = 0$.

2.1 Classical Lagrangian functional $\mathcal{S}$

Definition 2.3. Let θ be a twisted-periodic 1-form (2.4) and f a twist function. Define the **classical Lagrangian action** functional on the **free loop space** $\mathcal{L}\mathfrak{Q}^\times := C^\infty(\mathbb{S}^1, \mathfrak{Q}^\times)$ of $\mathfrak{Q}^\times := \mathfrak{Q} \setminus \{0\}$ by

$$\begin{gathered} \mathcal{S} = \mathcal{S}_{L^\theta} \colon \mathcal{L}\mathfrak{Q}^\times \to \mathbb{R} \\ q \mapsto \int_0^1 \left(\tfrac{1}{2}\,|\dot q_t|^2 + \tfrac{1}{|q_t|} + \theta_t|_{q_t} \dot q_t\right) dt - f(q_0) = \int_0^1 L_t^\theta(q_t, \dot q_t)\, dt - f(q_0) \end{gathered} \tag{2.7}$$

where $L^\theta_t\colon T\mathfrak{Q}^\times \to \mathbb{R}$ is the function, called **Lagrangian**, defined by

$$L^\theta_t(q,v) := \tfrac{1}{2}\left|v\right|^2 + \tfrac{1}{|q|} + \theta_t|_q v.$$

Remark 2.4 (twist term and integration interval, [FW26b, Rmk. 3.4]). (i) Since we integrate over $[\underline{0},1]$, we subtract specifically the **twist term $\boldsymbol{f(q_{\underline{0}})}$** in (2.7) in order to get as critical points the periodic solutions of the $(\boldsymbol{A},\phi)$-equation (2.8).

(ii) If θ_t is twisted-periodic, and not periodic, then L^θ_t is not periodic in time. Thus the integral (2.7) is not over the circle $\mathbb{S}^1=\mathbb{R}/\mathbb{Z}$, but over the interval $[0,1]$; see [FW26b, Rmk. 3.4] which shows that integration $\int_k^{k+1}$ leads to the same value $\mathcal{S}(q)$ in (2.7) for any integer $k\in\mathbb{Z}$.
(iii) If $\theta_{t+1}=\theta_t$ is periodic, then there is no twist term ($f\equiv 0$) and any integration interval $[r,r+1]$ for $r\in\mathbb{R}$ leads to the same value $\mathcal{S}(q)$ in (2.7).[4]

Remark 2.5 (classical critical point ODE). By [FW26b, Prop. 3.7], the critical points of the functional $\mathcal{S}$ are solutions of the $(\boldsymbol{A},\phi)$-equation with $\phi(q)=-\frac{1}{|q|}$

$$\operatorname{Crit}\mathcal{S} = \left\{q\in\mathcal{L}\mathfrak{Q}^\times \mid \ddot q = -B_t|_q j_0\dot q - \dot{\boldsymbol{A}}_t|_q - \tfrac{q}{|q|^3}\right\},\quad B_t|_q := \operatorname{rot}\boldsymbol{A}_t|_q. \tag{2.8}$$

The critical points q are called **classical** or **physical solutions**. A solution to this ODE describes the motion $t\mapsto q(t)$ of an electron attracted by a proton at the origin and subject to a magnetic field. The origin is a singularity of the potential ϕ and $q(t)$ approaching 0 is called a **collision**. In the classical description the origin is a forbidden locus.

3 Loop space blow-up

3.1 Complex squaring and twisted loop spaces

Standard notions on the Euclidean plane $\mathbb{R}^2\simeq\mathbb{C}$

It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x,y)\mapsto x+iy=z$. We recall how standard notions look like on each side. The complex side lends itself for shorter formulas and quicker calculations.

On $\mathbb{R}^2$ three natural players are the Euclidean inner product, the counter-clockwise quarter rotation, and the natural symplectic form. They are defined by

$$\langle z,\zeta\rangle_0 := x\xi+y\eta,\qquad i\overset{(2.2)}{\simeq} j_0,\qquad \omega_0(z,\zeta):=x\eta-y\xi,$$

for $z=(x,y)$ and $\zeta=(\xi,\eta)$. The three are compatible in the sense that

$$\omega_0(\cdot,\cdot):=\langle j_0\cdot,\cdot\rangle_0 \qquad\text{or, equivalently}\qquad \langle\cdot,\cdot\rangle_0:=\omega_0(\cdot,j_0\cdot). \tag{3.9}$$

As $j_0j_0=-\mathbb{1}$ and $j_0^t=-j_0$ is skew-symmetric, the previous identity tells that

$$\omega_0(j_0\cdot,j_0\cdot)=\omega_0(\cdot,\cdot),\qquad \langle j_0\cdot,j_0\cdot\rangle_0=\langle\cdot,\cdot\rangle_0\,,$$

are both invariant under j_0. In natural coordinates $\omega_0=dx\wedge dy$.

[4] schematically $\int_r^{r+1}\theta_t=\int_r^1\theta_t+\int_1^{r+1}\theta_t=\int_r^1\theta_t+\int_0^r\theta_{t+1}=\int_0^1\theta_t$ as $\theta_{t+1}=\theta_t$ is periodic

Remark 3.1. On $\mathbb{C}$ the three natural players appear as follows[5]

$$\bar{z}\zeta = \langle z,\zeta\rangle_0 + i\omega_0(z,\zeta), \qquad \mathrm{Re}(\bar{z}\zeta) = \langle z,\zeta\rangle_0\,, \quad \mathrm{Im}(\bar{z}\zeta) = \omega_0(z,\zeta), \tag{3.10}$$

and

$$\begin{aligned}\overline{iz}\zeta &= \langle j_0 z,\zeta\rangle_0 + i\omega_0(j_0 z,\zeta)\\ &= \omega_0(z,\zeta) - i\langle z,\zeta\rangle_0\end{aligned}$$

for $z = x+iy$ and $\zeta = \xi+i\eta$ and where $\bar{z} := x-iy$ is the complex conjugate. On the complex side multiplication by i corresponds on the real plane to applying the rotation matrix j_0. Note that $\mathrm{Re}(\overline{iz}\zeta) = \mathrm{Im}(\bar{z}\zeta)$ and $\mathrm{Im}(\overline{iz}\zeta) = -\mathrm{Re}(\bar{z}\zeta)$. Observe that $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/2i$. The one-forms $dz = dx+i\,dy$ and $d\bar{z} = dx - i\,dy$ reproduce $dx = \frac{dz+d\bar{z}}{2}$ and $dy = \frac{dz-d\bar{z}}{2i}$. There are the identities

$$d\bar{z}\wedge dz = (dx+idy)\wedge(dx-idy) = 2i\,dx\wedge dy, \qquad z\bar{z} = x^2+y^2 = |z|^2\,,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$.

Complex squaring map and sign involution $\mathfrak{i}$

In polar coordinates $z = re^{i\varphi}$ on $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$ the complex squaring map is $\varsigma\colon \mathbb{C}^\times \to \mathbb{C}^\times$, $re^{i\varphi} = z \mapsto z^2 = r^2e^{i2\varphi}$. Let $0 \in \mathfrak{Q} \subset \mathbb{C}$ be an open subset containing the origin. Without 0 we write

$$\mathfrak{Z}^\times := \varsigma^{-1}(\mathfrak{Q}^\times) = \{z\in\mathbb{C}^\times \mid z^2 \in \mathfrak{Q}^\times\}, \qquad \varsigma(\mathfrak{Z}^\times) = \mathfrak{Q}^\times := \mathfrak{Q}\setminus\{0\},$$

where the pre-image $\mathfrak{Z}^\times$ is open since ς is continuous. While $\mathfrak{Z}^\times$ does not contain the origin, its closure does. For later use we define

$$\mathfrak{Z} := \varsigma^{-1}(\mathfrak{Q}) = \mathfrak{Z}^\times \cup \{0\} \subset \mathbb{C}. \tag{3.11}$$

A useful observation is that $\mathfrak{Z}^\times$ is invariant under the **sign involution**

$$\mathfrak{i}\colon \mathfrak{Z}^\times \to \mathfrak{Z}^\times, \quad z\mapsto -z \tag{3.12}$$

indeed $z\in\mathfrak{Z}^\times \Leftrightarrow -z\in\mathfrak{Z}^\times$ as $z^2 = (-z)^2$. Hence the **complex squaring map**

$$\varsigma\colon \mathfrak{Z}^\times \xrightarrow{2:1} \mathfrak{Q}^\times, \quad z\mapsto z^2, \qquad \varsigma\circ\mathfrak{i} = \varsigma, \tag{3.13}$$

is a double cover and invariant under sign involution. By linearization

$$T\mathfrak{i}\colon T\mathfrak{Z}^\times \to T\mathfrak{Z}^\times, \quad (z,\xi)\mapsto(-z,-\xi). \tag{3.14}$$

Remark 3.2 (square root of complex numbers)**.** A non-zero complex number $z = re^{i\phi}$ has two square roots $\sqrt{r}e^{i\phi/2}$ and $\sqrt{r}e^{i(\phi/2+\pi)} = \sqrt{r}e^{i\phi/2}e^{i\pi} = -\sqrt{r}e^{i\phi/2}$. Thus, the square root is only well defined modulo sign. We need to take square roots in order to define the inverse $\mathcal{Z}$ of the rescale-square map $\mathcal{Q}$ later on in (3.27), as illustrated by Figure 2. This is why in (3.16) we will divide by the sign involution and work on the quotient space.

[5] Of course, on the left hand sides z stands for $x+iy$ and on the right hand sides for (x,y).

Twisted loop spaces and sign involution I

Considering twisted loop space allows for defining square roots. The open subsets $\mathfrak{Z}^\times \subset \mathbb{C}\setminus\{0\}$ and $\mathfrak{Z} = \mathfrak{Z}^\times \cup \{0\}$ are defined by (3.11).

Definition 3.3 (regularization loop space $\bar{\mathcal{L}}^\times\mathfrak{Z}$)**.** (i) Both the $^\pm$**loop spaces**

$$\mathcal{L}_\pm\mathfrak{Z} := \{z \in C^\infty(\mathbb{R},\mathfrak{Z}) \mid \forall t \in \mathbb{R} \colon z(t+1) = \pm z(t)\},$$

for $\mathfrak{Z}$ from (3.11) are invariant under **sign involution**

$$I = -\mathrm{Id}_1 \colon \mathcal{L}_\pm\mathfrak{Z} \to \mathcal{L}_\pm\mathfrak{Z}, \quad z \mapsto -z, \qquad \mathrm{Id}_k := \mathrm{Id}_{\mathbb{C}^k}, \tag{3.15}$$

because the target $\mathfrak{Z}$ is by (3.12). For simplicity we call the elements of both spaces still **loops**. The intersection of both spaces consists of the zero loop, the unique fixed point of I, in symbols $\mathcal{L}_+\mathfrak{Z}\cap\mathcal{L}_-\mathfrak{Z} = \{0\} = \mathrm{Fix}\, I$. Taking away the zero loop $z \equiv 0$ we get the pointed $^\pm$loop spaces

$$\mathcal{L}_\pm^\times\mathfrak{Z} := \mathcal{L}_\pm\mathfrak{Z}\setminus\{0\}, \qquad \mathcal{L}_+^\times\mathfrak{Z}\cap\mathcal{L}_-^\times\mathfrak{Z} = \emptyset.$$

These are disjoint and the sign involution I acts freely on each one of them, so on their union. We introduce the quotient spaces (indicated by a bar)

$$\bar{\mathcal{L}}_\pm^\times\mathfrak{Z} := \frac{\mathcal{L}_\pm^\times\mathfrak{Z}}{I}, \qquad \bar{\mathcal{L}}^\times\mathfrak{Z} := \bar{\mathcal{L}}_+^\times\mathfrak{Z}\cup\bar{\mathcal{L}}_-^\times\mathfrak{Z}, \tag{3.16}$$

and call their union $\bar{\mathcal{L}}^\times\mathfrak{Z}$ **regularization loop space**.

The elements of quotient space $\bar{\mathcal{L}}^\times\mathfrak{Z}$ are still called **loops** and denoted by z. We keep in mind that, in fact, each element has two representatives $\pm z$ and our constructions must be independent of choosing z or $-z$. The elements t_* of the set $z^{-1}(0)$ are called **collision times** or simply **collisions**.

(ii) The elements z of $\bar{\mathcal{L}}^\times\mathfrak{Z}$ that have no collisions, in symbols $z^{-1}(0) = \emptyset$, form the **non-collisional part** of regularization loop space, notation

$$\bar{\mathcal{L}}\mathfrak{Z}^\times = \bar{\mathcal{L}}_+\mathfrak{Z}^\times \cup \bar{\mathcal{L}}_-\mathfrak{Z}^\times$$

with $\bar{\mathcal{L}}_\pm\mathfrak{Z}^\times$ defined as earlier, just with $\mathfrak{Z}^\times = \mathfrak{Z}\setminus\{0\}$ in place of $\mathfrak{Z}$.

Remark 3.4 (tangent bundles)**.** Both tangent bundles

$$T\mathcal{L}_\pm\mathfrak{Z} := \{(z,\xi) \in C^\infty(\mathbb{R},\mathfrak{Z}\times\mathbb{C}) \mid \forall t \in \mathbb{R} \colon (z_{\tau+1},\xi_{\tau+1}) = \pm(z_\tau,\xi_\tau)\} \tag{3.17}$$

are invariant under **sign involution**

$$TI = -\mathrm{Id}_2 \colon T\mathcal{L}_\pm\mathfrak{Z} \to T\mathcal{L}_\pm\mathfrak{Z}, \quad (z,\xi) \mapsto (-z,-\xi)$$

because the target $\mathfrak{Z}\times\mathbb{C}$ is, see (3.12). The intersection of both tangent bundles consists of the zero element $(0,0)$, the unique fixed point of TI. Taking away the fiber over the zero loop we get the two tangent bundles[6]

$$T\mathcal{L}_\pm^\times\mathfrak{Z} := \{\Xi = (z,\xi) \in T\mathcal{L}_\pm\mathfrak{Z} \colon \|z\| \neq 0\}, \qquad T\mathcal{L}_+^\times\mathfrak{Z}\cap T\mathcal{L}_-^\times\mathfrak{Z} = \emptyset.$$

[6] The uppercase greek letter Ξ is called “Xi”.

To put it differently

$$T\mathcal{L}_+^\times \mathfrak{z} = \mathcal{L}_+^\times \mathfrak{z} \times \mathcal{L}_+\mathbb{C}, \qquad T\mathcal{L}_-^\times \mathfrak{z} = \mathcal{L}_-^\times \mathfrak{z} \times \mathcal{L}_-\mathbb{C}.$$

These are disjoint and the sign involution TI acts freely on each one of them, so on their union. We unite the two quotient spaces (indicated by bars)

$$T\bar{\mathcal{L}}_\pm^\times \mathfrak{z} := \frac{T\mathcal{L}_\pm^\times \mathfrak{z}}{TI}, \qquad T\bar{\mathcal{L}}^\times \mathfrak{z} := T\bar{\mathcal{L}}_+^\times \mathfrak{z} \cup T\bar{\mathcal{L}}_-^\times \mathfrak{z}.$$

The elements of $T\bar{\mathcal{L}}^\times \mathfrak{z}$ are still denoted by

$$\Xi = (z, \xi) \in \mathcal{L}_\pm^\times \mathfrak{z} \times \mathcal{L}_\pm \mathbb{C} \tag{3.18}$$

actually representing $\pm(z, \xi)$. This concludes Remark 3.4.

Canonical vector field on loop space and induced flow

The **canonical vector field** is generated by time derivative

$$\nu\colon \bar{\mathcal{L}}^\times \mathfrak{z} \to T\bar{\mathcal{L}}^\times \mathfrak{z}, \quad z \mapsto (z, z') \tag{3.19}$$

For the principal part we use the same notation $\nu(z) = z'$. Note that both ν is well defined on quotients, indeed $\nu(-z) = (-z, -z') = -(z, z') = -\nu(z)$.

The **flow** induced by ν on $\bar{\mathcal{L}}^\times \mathfrak{z}$ is time shift, in symbols

$$\phi_\nu^r z = r_* z, \qquad (\phi_\nu^r z)_\tau = z_{\tau + r}, \tag{3.20}$$

where $(r_* z)(\tau) = z(\tau + r)$ for every time $\tau \in \mathbb{R}$. Given $\xi \in T_z \bar{\mathcal{L}}^\times \mathfrak{z}$, pick a smooth map $h\colon \mathbb{R} \to \bar{\mathcal{L}}^\times \mathfrak{z}$ with $h(0) = z$ and $h'(0) = \xi$. Then

$$\left(d\phi_\nu^r|_z \xi\right)_\tau = \left(\tfrac{d}{ds}\big|_{s=0} \phi_\nu^r(h(s))\right)_\tau = \tfrac{d}{ds}\big|_{s=0} h(s)_{\tau+r} = \xi_{\tau+r}.$$

3.2 Rescale-square map $\mathcal{Q}$ and inverse $\mathcal{Z}$ (non-collisional)

In this section we consider loops avoiding the origin (no collisions). The constructions do not depend on choosing z or $-z$. We construct two maps $\mathcal{Q}$ and its inverse $\mathcal{Z}$, as illustrated by Figure 2.

Remark 3.5. Pick $z \in \bar{\mathcal{L}}\mathfrak{z}^\times$, so there are no collisions $z^{-1}(0) = \emptyset$, see Figure 2. We call the variable τ of $z\colon \mathbb{S}^1 \to \mathfrak{z}^\times$, equivalently $z\colon \mathbb{R} \to \mathfrak{z}^\times$ with $z_{\tau+1} = \pm z_\tau$, **regularized time**. **Classical time** we call the values of the map $t_z\colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$

$$\forall \tau \in \mathbb{R}\colon \qquad t_z(\tau) := \frac{\int_0^\tau |z(s)|^2 \, ds}{\|z\|^2}, \qquad t_z = t_{-z}. \tag{3.21}$$

Classical time has the following obvious properties

$$t_z'(\tau) = \frac{|z(\tau)|^2}{\|z\|^2} > 0, \quad t_z \in C^\infty, \quad t_{\rho z} = t_z, \quad t_z(0) = 0, \quad t_z(1) = 1, \tag{3.22}$$

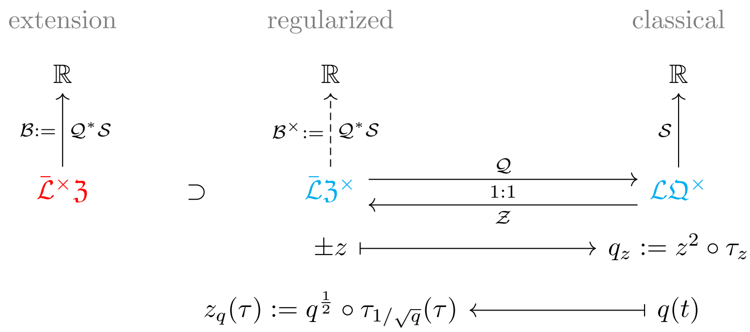

Figure 2: Pull-back of classical action gives a formula $\mathcal{Q}^*\mathcal{S}$ on $\bar{\mathcal{L}}\mathfrak{Z}^\times$ (loops avoiding 0) that makes sense on the larger space $\bar{\mathcal{L}}^\times\mathfrak{Z}$ (all loops except zero loop $\equiv 0$) on which $\mathcal{B} := \mathcal{Q}^*\mathcal{S}$ has extra critical points (rescaled classical collision trajectories)

for every real $\rho \neq 0$. Moreover, classical time is indeed equivariant with respect to the $\mathbb{Z}$-action $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $(k, \tau) \mapsto \tau + k$, in symbols $t_z(\tau + k) = t_z(\tau) + k$. As the map $t_z \colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$ is continuous and strictly monotone increasing (as z avoids the origin), it has a continuous strictly monotone increasing inverse[7]

$$\tau_z := {t_z}^{-1} \colon \mathbb{S}^1_t \to \mathbb{S}^1_\tau, \quad t \mapsto \tau_z(t) \tag{3.23}$$

called **regularized time**. Since z avoids the origin $\dot{\tau}_z$ is continuous. Indeed

$$\dot{\tau}_z(t) = \tfrac{1}{t'_z(\tau_z(t))} = \tfrac{\|z\|^2}{|z(\tau_z(t))|^2} > 0, \quad \tau_z \in C^\infty, \quad \tau_z(0) = 0, \quad \tau_z(1) = 1, \tag{3.24}$$

and $\tau_{\rho z} = \tau_z$ $\forall \rho > 0$. In particular, both t_z and τ_z are <u>smooth</u> diffeomorphisms of $\mathbb{S}^1$. This is due to the fact that z <u>avoids</u> the origin. This concludes Remark 3.5.

Definition 3.6. The **rescale-square operation** is defined by

$$\mathcal{Q} \colon \bar{\mathcal{L}}\mathfrak{Z}^\times \to \mathcal{L}\mathfrak{Q}^\times, \quad z \mapsto q_z := z^2 \circ \tau_z = \varsigma \circ z \circ \tau_z, \quad \forall \rho \neq 0 \colon \mathcal{Q}(\rho z) = \rho^2 \mathcal{Q}(z).$$

Remark 3.7. By definition of q_z and the chain rule we get the identity

$$\dot{q}_z(t) = 2z(\tau_z(t))\, z'(\tau_z(t))\, \dot{\tau}_z(t) \overset{(3.24)}{=} 2\|z\|^2\, \tfrac{z'(\tau_z(t))}{\bar{z}(\tau_z(t))}. \tag{3.25}$$

In equality 2 below we change the variable to $\sigma := \tau_z(t)$, then use (3.22) to get

$$\begin{aligned} \|\dot{q}_z\|^2 &\overset{(3.25)}{=} 4\|z\|^4 \int_0^1 \frac{|z'(\tau_z(t))|^2}{|\bar{z}(\tau_z(t))|^2} dt \overset{2}{=} 4\|z\|^4 \int_0^1 \frac{|z'(\sigma)|^2}{|z(\sigma)|^2} t'_z(\sigma)\, d\sigma \\ &\overset{(3.22)}{=} 4\|z\|^4 \int_0^1 \frac{|z'(\sigma)|^2}{|z(\sigma)|^2} \frac{|z(\sigma)|^2}{\|z\|^2}\, d\sigma = 4\|z\|^2 \langle z', z' \rangle . \end{aligned} \tag{3.26}$$

[7] see e.g. [For11, §12 Satz 1]

Lemma 3.8 (well defined bijection)**.** *For $z \in \bar{\mathcal{L}}\mathfrak{Z}^\times$ the image $\mathcal{Q}(z)$ lies in $\mathcal{L}\mathfrak{Q}^\times$. The map $\mathcal{Q}\colon \bar{\mathcal{L}}\mathfrak{Z}^\times \to \mathcal{L}\mathfrak{Q}^\times$ is a bijection with inverse (3.27).*

Proof. Pick $z \in \bar{\mathcal{L}}\mathfrak{Z}^\times$, in particular $z\colon \mathbb{S}^1 \to \mathfrak{Z}^\times$. Since z and τ_z are smooth, so is their composition q_z. Non-vanishing of z implies non-vanishing of q_z. To see that q_z is 1-periodic note that $q_z(1) = z^2 \circ \tau_z(1) = z(1)^2 = (\pm z(0))^2 = z(0)^2 = z^2 \circ t_z(0) = q_z(0)$. This shows that $\mathcal{Q}(z) = q_z \in \mathcal{L}\mathfrak{Q}^\times = C^\infty(\mathbb{S}^1, \mathfrak{Q} \setminus \{0\})$.

Surjective. Given $q \in \mathcal{L}\mathfrak{Q}^\times$, define $1/\sqrt{q} \in \bar{\mathcal{L}}\mathbb{C}^\times$ and $\mathcal{Z}(q) \in \bar{\mathcal{L}}\mathfrak{Z}^\times$ by

$$\mathcal{Z}(q) := z_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}, \qquad 1/\sqrt{q} = [t \mapsto 1/q(t)^{\frac{1}{2}}], \tag{3.27}$$

where $\tau_{1/\sqrt{q}} = {t_{1/\sqrt{q}}}^{-1}\colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$ is the inverse of the map $t_{1/\sqrt{q}}$ associated by (3.21) to the loop $1/\sqrt{q}$ in $\mathbb{C}^\times$. For $\tau \in \mathbb{R}$ we calculate the identities

$$\begin{aligned} t_{z_q}(\tau) &\stackrel{\text{def.}}{=} \frac{\int_0^\tau |{z_q}^2(\sigma)|\, d\sigma}{\int_0^1 |{z_q}^2(\sigma)|\, d\sigma} = \frac{\int_0^\tau |q \circ \overbrace{\tau_{1/\sqrt{q}}(\sigma)}^{=:s}|\, d\sigma}{\int_0^1 |q \circ \tau_{1/\sqrt{q}}(\sigma)|\, d\sigma} = \frac{\int_0^{\tau_{1/\sqrt{q}}(\tau)} \frac{ds}{\|1/\sqrt{q}\|^2}}{\int_0^1 \frac{ds}{\|1/\sqrt{q}\|^2}} \\ &= \tau_{1/\sqrt{q}}(\tau) \end{aligned} \tag{3.28}$$

using change of variables $s(\sigma) = \tau_{1/\sqrt{q}}(\sigma)$, equivalently $\sigma(s) = t_{1/\sqrt{q}}(s)$, hence

$$d\sigma = t'_{1/\sqrt{q}}(s)\, ds \stackrel{(3.22)}{=} \tfrac{|1/\sqrt{q}(s)|^2}{\|1/\sqrt{q}\|^2}\, ds = \tfrac{1/|q_s|}{\|1/\sqrt{q}\|^2}\, ds.$$

With this result we obtain that

$$\big(\mathcal{Q} \circ \underbrace{\mathcal{Z}(q)}_{z_q}\big)(t) \stackrel{\text{def. } \mathcal{Q}}{=} {z_q}^2 \circ \tau_{z_q}(t) \stackrel{\text{def. } z_q^2}{=} q \circ \underbrace{\tau_{1/\sqrt{q}}}_{t_{z_q}} \circ \underbrace{\tau_{z_q}}_{{t_{z_q}}^{-1}}(t) = q(t).$$

Injective. For $z \in \bar{\mathcal{L}}\mathfrak{Z}^\times$ set $q_z := \mathcal{Q}(z) := z^2 \circ \tau_z$. Then for $t \in \mathbb{R}$ we get

$$\int_0^t \frac{1}{|q_z(s)|}\, ds = \int_0^t \frac{1}{|z \circ \tau_z(s)|^2}\, ds = \int_0^{\tau_z(t)} \frac{1}{|z(\sigma)|^2} \frac{|z(\sigma)|^2\, d\sigma}{\|z\|^2} = \frac{\tau_z(t)}{\|z\|^2}$$

by change of variables $\sigma = \tau_z(s)$ using (3.24). Pick $t = 1$ to obtain

$$\int_0^1 \frac{1}{|q_z(t)|}\, dt = \frac{1}{\|z\|^2}. \tag{3.29}$$

Therefore we get for $t \in \mathbb{R}$ the formula

$$\tau_z(t) = \|z\|^2 \int_0^t \frac{1}{|q_z(s)|}\, ds \stackrel{(3.29)}{=} \frac{\int_0^t \frac{1}{|q_z(s)|}\, ds}{\int_0^1 \frac{1}{|q_z(s)|}\, ds} \stackrel{(3.21)}{=} t_{1/\sqrt{q_z}}(t). \tag{3.30}$$

With this result we obtain

$$\big(\mathcal{Z} \circ \underbrace{\mathcal{Q}(z)}_{q_z}\big)(\tau) \stackrel{\text{def. } \mathcal{Z}}{=} {q_z}^{\frac{1}{2}} \circ \tau_{1/\sqrt{q_z}}(\tau) \stackrel{\text{def. } q_z}{=} z \circ \underbrace{\tau_z}_{t_{1/\sqrt{q_z}}} \circ \underbrace{\tau_{1/\sqrt{q_z}}}_{{t_{1/\sqrt{q_z}}}^{-1}}(\tau) = z(\tau).$$

This proves Lemma 3.8. $\square$

Proposition 3.9 (extendable formula)**.** *For $\mathcal{S}$ in (2.7) and $z \in \bar{\mathcal{L}}\mathfrak{Z}^\times$ it holds*

$$\mathcal{S}(\mathcal{Q}(z)) = 2\|z\|^2\|z'\|^2 + \frac{1}{\|z\|^2} + \int_0^1 (\varsigma^*\theta)_{t_z(\sigma)}|_{z(\sigma)} z'(\sigma)\, d\sigma - f(z_0^2) \tag{3.31}$$

as illustrated by Figure 2. The functional $\mathcal{Q}^*\mathcal{S}$ has the same value on $\pm z$.

Note. Formula (3.31) makes sense even if z takes on the value 0, even along intervals, as long as $\|z\| \neq 0$, i.e. as long as z is not constantly zero.

Proof. Set $q_z := \mathcal{Q}(z) = \varsigma \circ z \circ \tau_z$. By definition (2.7) of $\mathcal{S}$ we have

$$\begin{aligned} \mathcal{S}(q_z) :&= \tfrac{1}{2}\|\dot q_z\|^2 + \int_{[0,1]} q_z^*\theta + \int_0^1 \frac{1}{|q_z(t)|}\, dt - f(q_z(0)) \\ &= \tfrac{1}{2}\, 4\|z\|^2 \langle z', z'\rangle + \int_{[0,1]} q_z^*\theta + \frac{1}{\|z\|^2} - f(z_0^2). \end{aligned}$$

Here equality two is by (3.26) and (3.29). We compute the last two summands

$$\begin{aligned} &\int_{[0,1]} q_z^*\theta - f(q_z(0)) \\ &= \int_0^1 \theta_t|_{q_z(t)} \dot q_z(t)\, dt - f(q_z(0)) \\ &\overset{2}{=} 2\|z\|^2 \int_0^1 \theta_{t_z(\sigma)}|_{z_\sigma^2} \tfrac{z'_\sigma}{\bar z_\sigma} t'_z(\sigma)\, d\sigma - f(z_0^2) \\ &\overset{3}{=} 2\int_0^1 \theta_{t_z(\sigma)}|_{z^2(\sigma)} z_\sigma z'_\sigma\, d\sigma - f(z_0^2) \\ &= 2\int_0^1 \theta_{t_z(\sigma)}|_{{z_\sigma}^2} (x+iy)_\sigma\, (x'+iy')_\sigma\, d\sigma - f(z_0^2) \\ &= 2\int_0^1 \theta_{t_z(\sigma)}|_{{z_\sigma}^2} \Big((xx'-yy')_\sigma + i\,(xy'-yx')_\sigma \Big) d\sigma - f(z_0^2) \\ &\overset{6}{=} 2\int_0^1 \Big(A^1_{t_z(\sigma)}|_{{z_\sigma}^2} (xx'-yy')_\sigma + A^2_{t_z(\sigma)}|_{{z_\sigma}^2} (xy'+yx')_\sigma \Big) d\sigma - f(z_0^2) \\ &\overset{7}{=} 2\int_0^1 \Big(\big(xA^1_{t_z}|_{z^2} + yA^2_{t_z}|_{z^2}\big)\, x' + \big(xA^2_{t_z}|_{z^2} - yA^1_{t_z}|_{z^2}\big)\, y' \Big)_\sigma d\sigma - f(z_0^2) \\ &\overset{8}{=} \int_0^1 (\varsigma^*\theta)_{t_z(\sigma)}|_{z_\sigma} z'_\sigma\, d\sigma - f(z_0^2) \overset{(4.33)}{=:} \mathcal{M}(z) \end{aligned} \tag{3.32}$$

where we identified $\mathbb{C} \simeq \mathbb{R}^2$ via $z = x + iy \mapsto (x, y)$. Equality 2 is by (3.25) in combination with variable substitution $\sigma = \tau_z(t)$ and the identity $t = t_z(\sigma)$ which uses the inverse function $\tau_z(t) = t_z^{-1}(t)$. Equality 3 uses that $t'_z(\tau)$ is given by (3.22). Equality 6 uses that θ is of the form (2.3). Equality 8 is by definition (3.13) of ς and of the pull-back by ς. The result we denote by $\mathcal{M}(z)$. $\mathcal{M}$ has the same value on $\pm z$: By (3.32) it suffices to show $q_z = q_{-z}$. Indeed $q_z = z^2 \circ {t_z}^{-1} = (-z)^2 \circ {t_{-z}}^{-1} = q_{-z}$ for $t_z = t_{-z}$ see (3.21). Alternatively, inspect the right hand side of equality 3 in (3.32). This proves Proposition 3.9. □

4 Non-local Lagrangian mechanics

4.1 Magnetic functional $\mathcal{M}$

Fix a twisted-periodic 1-form θ and a twist function f on $\mathfrak{Q} \subset \mathbb{C}$; see (2.4). Formula (3.31) for the pull-back of the classical action functional motivates

Definition 4.1 (magnetic functional)**.** Define the **magnetic functional** by

$$\mathcal{M}\colon \bar{\mathcal{L}}^\times \mathfrak{Z} \to \mathbb{R}, \quad z \mapsto \int_0^1 (\varsigma^*\theta)_{t_z(\tau)}|_{z_\tau} z'_\tau \, d\tau - f(z_0^2), \quad f(z_0^2) = (\varsigma^* f)|_{z_0} \tag{4.33}$$

with reparametrization $t_z\colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$ defined by (3.21) and $\varsigma(z) = z^2$. For $\mathcal{M}$ written as L^2-inner product see (4.39). That $\mathcal{M}(z) = \mathcal{M}(-z)$ one reads off immediately from formula $(3.32)_7$. Remark 2.4 explains the integration interval.

Remark 4.2 (pull-back $\vartheta := \varsigma^*\theta$)**.** The pull-back of θ from the open subset $\mathfrak{Q} \subset \mathbb{C}$ with coordinates $q = q_1 + iq_2$ to $\mathfrak{Z} = \varsigma^{-1}(\mathfrak{Q}) \subset \mathbb{C}$ with coordinates $z = x + iy$ under the map $z \mapsto \varsigma(z) := z^2$ is a smooth real family of 1-forms

$$\vartheta_t := \varsigma_*\theta_t = a_t^1 \, dx + a_t^2 \, dy, \qquad a_t^1, a_t^2 \colon \mathfrak{Z} \to \mathbb{R}, \qquad \boldsymbol{a}_t = (a_t^1, a_t^2). \tag{4.34}$$

Analogous to (2.6) we get $d\vartheta_t = (\operatorname{rot} \boldsymbol{a}_t)\, \omega_0$. Note that ϑ is twisted-periodic

$$\begin{aligned}
\dot\vartheta_{t+1} &= \varsigma^*\dot\theta_{t+1} = \varsigma^*\dot\theta_t = \dot\vartheta_t \\
\vartheta_{t+1} &= \varsigma^*\theta_{t+1} = \varsigma^*(\theta_t + df) = \vartheta_t + dF, \qquad F := \varsigma^* f = f \circ \varsigma.
\end{aligned}$$

To compute the pull-back under the squaring map ς we write

$$q_1 + iq_2 = q = z^2 = (x + iy)^2) = x^2 - y^2 + 2ixy.$$

Thus the components are $q_1 = x^2 - y^2$ and $q_2 = 2xy$. Hence

$$dq_1 = 2xdx - 2ydy, \qquad dq_2 = 2ydx + 2xdy.$$

Recall from (2.3) that we wrote θ_t in the form $\theta_t|_q = A_t^1|_q dq_1 + A_t^2|_q dq_2$ for $q \in \mathfrak{Q}$. Calculation shows that the pull-back under ς is of the form

$$\begin{aligned}
\vartheta_t|_z :&= (\varsigma^*\theta)_t \,|_z \\
&= A_t^1|_{\varsigma(z)}(2xdx - 2ydy) + A_t^2|_{\varsigma(z)}(2ydx + 2xdy) \\
&= \left(2xA_t^1|_{z^2} + 2yA_t^2|_{z^2}\right) dx + \left(2xA_t^2|_{z^2} - 2yA_t^1|_{z^2}\right) dy.
\end{aligned}$$

By (4.34) the components $a_t^j|_z$ of ϑ and the $A_t^j|_{z^2}$ of θ are related by

$$\begin{aligned}
a_t^1|_z &= 2xA_t^1|_{z^2} + 2yA_t^2|_{z^2} = 2\left\langle z, \boldsymbol{A}_t|_{z^2}\right\rangle_0 \\
a_t^2|_z &= 2xA_t^2|_{z^2} - 2yA_t^1|_{z^2} = 2\omega_0(z, \boldsymbol{A}_t|_{z^2}).
\end{aligned} \tag{4.35}$$

Pointwise at $z = (x, y)$, using the previous two formulas, we calculate

$$\operatorname{rot} \boldsymbol{a}_t|_z = (\partial_x a_t^2 - \partial_y a_t^1)_z = 4\,|z|^2\,(\partial_1 A_t^2 - \partial_2 A_t^1)_{z^2} = 4\bar{z}z \operatorname{rot} \boldsymbol{A}_t|_{z^2}. \tag{4.36}$$

In particular, we see that $\operatorname{rot} \boldsymbol{a}_t|_0 = 0$ vanishes at the origin singularity at all times. The second equality requires a bit of work. Hence

$$(d\vartheta_t)_z = (\operatorname{rot} \boldsymbol{a}_t) \left\langle j_0\cdot,\cdot\right\rangle_0 = 4\left|z\right|^2 (\operatorname{rot} \boldsymbol{A}_t|_{z^2}) \left\langle j_0\cdot,\cdot\right\rangle_0 . \tag{4.37}$$

In complex notation there are the identities

$$2\bar{z}\boldsymbol{A}_t|_{z^2} = 2(x-iy)(A_t^1 + iA_t^2)|_{z^2} = a_t^1|_z + ia_t^2|_z = \boldsymbol{a}_t|_z. \tag{4.38}$$

Step two uses (4.35); similarly $2\bar{z}\dot{\boldsymbol{A}}_t|_{z^2} = \dot{\boldsymbol{a}}_t|_z$. This concludes Remark 4.2.

By (4.33) and (4.34) the magnetic term translates to an inner product

$$\mathcal{M}(z) = \int_0^1 \left(a^1_{t_z(\tau)}|_{z_\tau} x'_\tau + a^2_{t_z(\tau)}|_{z_\tau} y'_\tau \right) d\tau - f(z_0^2) = \langle \boldsymbol{a}_{t_z}|_z, z'\rangle - f(z_0^2) \tag{4.39}$$

for $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$. The task at hand is to calculate the derivative $d\mathcal{M}(z)$.

4.1.1 L^2-gradient

Lemma 4.3 (L^2-gradient of $\mathcal{M}$)**.** *At $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$ for any time τ we have*

$$\begin{aligned}(\operatorname{grad} \mathcal{M}|_z)_\tau = &-\frac{2z_\tau}{\|z\|^4}\int_0^1 \textstyle\int_0^\sigma \left|z_\rho\right|^2 d\rho \cdot \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma - \frac{\left|z_\tau\right|^2}{\|z\|^2}\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} \\ &+ \frac{2z_\tau}{\|z\|^2}\int_{\sigma=\tau}^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma - (\operatorname{rot} \boldsymbol{a}_{t_z}|_z)_\tau \, j_0 z'_\tau.\end{aligned}$$

Proof. We use the representation (4.39) of $\mathcal{M}$ as L^2-inner product to calculate the L^2-gradient of $\mathcal{M}$ at a loop z. Given a smooth vector field $\xi \in T_z\bar{\mathcal{L}}^\times\mathfrak{Z}$ along z, pick a smooth path of loops $\varepsilon \mapsto z_\varepsilon$ with $z_0 = z$ and $\frac{d}{d\varepsilon}\big|_0 z_\varepsilon = \xi$. Then

$$(dt|_z\xi)_\tau := \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} t(z_\varepsilon)_\tau = \frac{2}{\|z\|^2}\int_0^\tau \langle z_\sigma, \xi_\sigma\rangle_0 \, d\sigma - \frac{2\,\langle z,\xi\rangle}{\|z\|^4}\int_0^\tau \left|z_\sigma\right|^2 d\sigma. \tag{4.40}$$

Another preparation for the main calculation is to determine the difference

$$\begin{aligned}\langle \boldsymbol{a}_1|_{z_1}, \xi_1\rangle_0 - \langle \boldsymbol{a}_0|_{z_0}, \xi_0\rangle_0 &= (\varsigma^*\theta)_1|_{z_1}\xi_1 - (\varsigma^*\theta)_0|_{z_0}\xi_0 \\ &= (\mathfrak{i}^*\varsigma^*\theta)_1|_{-z_0}(-\xi_0) - (\varsigma^*\theta)_0|_{z_0}\xi_0 \\ &= (\varsigma^*\theta)_1|_{z_0}\xi_0 - (\varsigma^*\theta)_0|_{z_0}\xi_0 \\ &= (\varsigma^*\theta_0 + \varsigma^* df)|_{z_0}\xi_0 - (\varsigma^*\theta)_0|_{z_0}\xi_0 \\ &= d(\varsigma^* f)|_{z_0}\xi_0.\end{aligned} \tag{4.41}$$

Equality 2 is only for the twisted case, skip it in the untwisted case. Equality 2 uses $\varsigma^* = \mathfrak{i}^*\varsigma^*$, by (3.13), and $(z_1, \xi_1) = -(z_0, \xi_0)$ by the twist hypothesis. Equality 3 uses $Ti(-z_0, -\xi_0) = -(-z_0, -\xi_0)$, by (3.14). Equality 4 is by twisted-periodicity of θ and linearity of pull-back. Equality 5 uses that pull-back and d commute and, moreover, two terms cancel each other.

The principal calculation is as follows

$$
\begin{aligned}
&\langle \operatorname{grad} \mathcal{M}|_z, \xi\rangle = d\mathcal{M}|_z\xi = \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \mathcal{M}(z_\varepsilon)\\
&= \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \left\langle \boldsymbol{a}_{t_{z_\varepsilon}}|_{z_\varepsilon}, z'_\varepsilon\right\rangle - \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} f\circ\varsigma(z_\varepsilon(0))\\
&= \left\langle \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \boldsymbol{a}_{t_{z_\varepsilon}}|_{z_\varepsilon}, z'\right\rangle + \left\langle \boldsymbol{a}_{t_z}|_z, \xi'\right\rangle - d(\varsigma^* f)|_{z_0}\xi_0\\
&\overset{5}{=} \left\langle \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \boldsymbol{a}_{t_{z_\varepsilon}}|_{z_\varepsilon}, z'\right\rangle + \underline{\langle \boldsymbol{a}_1|_{z_1}, \xi_1\rangle_0 - \langle \boldsymbol{a}_0|_{z_0}, \xi_0\rangle_0} - \left\langle (\boldsymbol{a}_{t_z}|_z)', \xi\right\rangle - \underline{d(\varsigma^* f)|_{z_0}\xi_0}\\
&\overset{6}{=} \langle \dot{\boldsymbol{a}}_{t_z}|_z\,(dt|_z\xi) + d\boldsymbol{a}_{t_z}|_z\xi, z'\rangle - \langle \dot{\boldsymbol{a}}_{t_z}|_z (t_z)' + d\boldsymbol{a}_{t_z}|_z z', \xi\rangle\\
&\overset{7}{=} \langle \dot{\boldsymbol{a}}_{t_z}|_z\,(dt|_z\xi), z'\rangle - \langle \dot{\boldsymbol{a}}_{t_z}|_z (t_z)', \xi\rangle + \langle (d\boldsymbol{a}_{t_z}|_z^T - d\boldsymbol{a}_{t_z}|_z) z', \xi\rangle.
\end{aligned}
$$

Identity 5 uses integration by parts. Identity 6 is by the chain and product rules, the underlined terms cancel due to (4.41). Identity 7 moves $d\boldsymbol{a}$ to the other side of the inner product where it arrives as transpose $d\boldsymbol{a}^T$.

Now we are in position to identify the three resulting summands with the summands in Lemma 4.3. Both second summands are equal by formula (3.22) for $(t_z)'$. Abbreviating $\boldsymbol{a} = \boldsymbol{a}_{t_z}|_z$ both final summands correspond, since

$$
(d\boldsymbol{a})^T - d\boldsymbol{a} = \begin{pmatrix} \partial_1\boldsymbol{a}_1 & \partial_1\boldsymbol{a}_2 \\ \partial_2\boldsymbol{a}_1 & \partial_2\boldsymbol{a}_2 \end{pmatrix} - \begin{pmatrix} \partial_1\boldsymbol{a}_1 & \partial_2\boldsymbol{a}_1 \\ \partial_1\boldsymbol{a}_2 & \partial_2\boldsymbol{a}_2 \end{pmatrix} = -\left(\operatorname{rot}\boldsymbol{a}\right) j_0.
$$

It remains to calculate the first summand

$$
\begin{aligned}
\langle \dot{\boldsymbol{a}}_{t_z}|_z\,(dt|_z\xi), z'\rangle &= \int_0^1 \left\langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}\,(dt|_z\xi)_\tau, z'_\tau\right\rangle_0 d\tau\\
&\overset{1}{=} \frac{2}{\|z\|^2}\int_0^1 \textstyle\int_0^\tau \langle z_\sigma, \xi_\sigma\rangle_0\, d\sigma\cdot \left\langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}, z'_\tau\right\rangle_0 d\tau\\
&\quad - \frac{2\langle z,\xi\rangle}{\|z\|^4}\int_0^1 \textstyle\int_0^\tau |z_\sigma|^2\, d\sigma \left\langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}, z'_\tau\right\rangle_0 d\tau\\
&\overset{2}{=} \frac{2}{\|z\|^2}\int_{\tau=0}^1 \langle z_\tau, \xi_\tau\rangle_0 \int_{\sigma=\tau}^1 \left\langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\right\rangle_0 d\sigma\, d\tau\\
&\quad - \frac{2\langle z,\xi\rangle}{\|z\|^4}\int_0^1 \textstyle\int_0^\tau |z_\sigma|^2\, d\sigma \left\langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}, z'_\tau\right\rangle_0 d\tau.
\end{aligned}
$$

Identity 1 spells out the L^2-inner product. Identity 2 inserts formula (4.40) for $dt_z\xi$ and pulls real factors out of the integral. In identity three we interchanged the order of integration using Fubini's theorem. Take the L^2-inner product of the identity in Lemma 4.3 with ξ to see equality. This proves Lemma 4.3. □

In (C.101) we calculate $d\mathcal{M}$, equivalently $\operatorname{grad}\mathcal{M}$, applying Cartan's formula from finite dimensions formally on the loop space.[8] This yields the same formula as the rigorously proved Lemma 4.3 above.

[8] The final summand in Lemma 4.3 arises from the final summand in (C.101), via the identity $-(d\vartheta_{t_z(\tau)})|_{z_\tau}(z'_\tau, \xi_\tau) = -\left(\operatorname{rot}\boldsymbol{a}_{t_z(\tau)}|_{z_\tau}\right)\langle j_0 z'_\tau, \xi_\tau\rangle_0$ in (4.37).

4.2 Lagrangian action functional $\mathcal{B}$

Formula (3.31) for $\mathcal{B}^\times := \mathcal{Q}^*\mathcal{S}\colon \bar{\mathcal{L}}\mathfrak{Z}^\times \to \mathbb{R}$ makes sense on the larger space $\bar{\mathcal{L}}^\times\mathfrak{Z}$ that consists of all smooth loops in $\mathfrak{Z}$ not identically zero. This motivates

Definition 4.4. The **non-local Lagrangian action functional** is defined by

$$\begin{aligned} &\mathcal{B}\colon \bar{\mathcal{L}}^\times\mathfrak{Z} \to \mathbb{R} \\ &z \mapsto \underbrace{2\|z\|^2\|z'\|^2}_{\mathcal{K}(z)} + \underbrace{\frac{1}{\|z\|^2}}_{-\mathcal{U}(z)} + \underbrace{\int_0^1 (\varsigma^*\theta)_{t_z(\tau)}|_{z_\tau} z'_\tau \, d\tau - f(z_0^2)}_{\mathcal{M}(z) = \Theta_z z' \text{ (5.79), (C.94)}} . \end{aligned} \tag{4.42}$$

Here θ is a twisted-periodic 1-form with twist function f along $\mathfrak{Q}$, see (2.3), and $\varsigma\colon \mathbb{C} \supset \mathfrak{Z} \to \mathfrak{Q}$ is the complex squaring map (3.13).

4.2.1 L^2-gradient

Definition 4.5. The **L^2-gradient** of the action functional $\mathcal{B} = \mathcal{K} - \mathcal{U} + \mathcal{M}$ with respect to the standard L^2-inner product at a loop $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$ is determined by $\langle \operatorname{grad}\mathcal{B}(z), \xi\rangle = d\mathcal{B}|_z\xi$ for every ξ with $(z,\xi) \in \mathcal{L}^\times_\pm\mathfrak{Z} \times \mathcal{L}_\pm\mathbb{C}$; see (3.18). Thus

$$\operatorname{grad}\mathcal{B}(z) = \operatorname{grad}\mathcal{K}(z) - \operatorname{grad}\mathcal{U}(z) + \operatorname{grad}\mathcal{M}(z) \tag{4.43}$$

where $\mathcal{K}$, $\mathcal{U}$, $\mathcal{M}$ are defined in (4.42). Points where the differential, equivalently the gradient, of a function vanishes are called **critical points**.

With the chain and product rule we calculate

$$\begin{aligned} d\mathcal{K}|_z\xi :&= \tfrac{d}{d\varepsilon}\big|_0 \mathcal{K}(z+\varepsilon\xi) \\ &= \tfrac{d}{d\varepsilon}\big|_0 2\langle z+\varepsilon\xi, z+\varepsilon\xi\rangle \langle z'+\varepsilon\xi', z'+\varepsilon\xi'\rangle \\ &= 4\|z'\|^2\langle z,\xi\rangle + 4\|z\|^2\langle z',\xi'\rangle \\ &= 4\|z'\|^2\langle z,\xi\rangle + \cancel{4\|z\|^2}0 - 4\|z\|^2\langle z'',\xi\rangle \end{aligned}$$

where the last equality (integration by parts) holds since z is sufficiently regular, i.e. has at least two weak derivatives and since the boundary term vanishes[9]

$$\langle z_1', \xi_1\rangle_0 - \langle z_0', \xi_0\rangle_0 = \langle \pm z_0', \pm\xi_0\rangle_0 - \langle z_0', \xi_0\rangle_0 = \big((\pm 1)^2 - 1\big)\langle z_0', \xi_0\rangle_0 = 0.$$

Vanishing is independent of choosing (z,ξ) or $-(z,\xi)$. Furthermore, we calculate

$$d\mathcal{U}|_z\xi := \tfrac{d}{d\varepsilon}\big|_0 \mathcal{U}(z+\varepsilon\xi) = \tfrac{d}{d\varepsilon}\big|_0 \langle z+\varepsilon\xi, z+\varepsilon\xi\rangle^{-1} = \frac{2\langle z,\xi\rangle}{\|z\|^4}.$$

From these differentials one immediately reads off the L^2-gradients

$$\operatorname{grad}\mathcal{K}(z) = 4\|z'\|^2 z - 4\|z\|^2 z'', \qquad \operatorname{grad}\mathcal{U}(z) = \frac{2z}{\|z\|^4}. \tag{4.44}$$

The magnetic gradient is much more subtle. We calculated it in Lemma 4.3.

[9] The derivative of $(z_{\tau+1}, \xi_{\tau+1}) = \pm(z_\tau, \xi_\tau)$ in (3.17) at $\tau = 0$ tells $(z_1', \xi_1') = (\pm z_0', \pm\xi_0')$. The identity $(z_{\tau+1}, \xi_{\tau+1}) = \pm(z_\tau, \xi_\tau)$ in (3.17) at $\tau = 0$ tells $(z_1, \xi_1) = (\pm z_0, \pm\xi_0)$.

Lemma 4.6 (L^2-gradient of $\mathcal{B}$)**.** *At $z \in \bar{\mathcal{L}}^\times \mathfrak{Z}$ the L^2-gradient is of the form*

$$\begin{aligned} \operatorname{grad}\mathcal{B}|_z &= 4\|z'\|^2 z - 4\|z\|^2 z'' - \frac{2z}{\|z\|^4} + \operatorname{grad}\mathcal{M}|_z \\ &= 4\|z'\|^2 z - 4\|z\|^2 z'' - \frac{2z}{\|z\|^4} \\ &\quad - \frac{2z}{\|z\|^4}\int_0^1 \textstyle\int_0^\sigma |z_\rho|^2\, d\rho \cdot \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma - \dfrac{|z|^2}{\|z\|^2}\dot{\boldsymbol{a}}_{t_z}|_z \\ &\quad + \frac{2z}{\|z\|^2}\int_{\sigma=\tau}^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma - \left(\operatorname{rot}\boldsymbol{a}_{t_z}|_z\right) j_0 z'. \end{aligned} \tag{4.45}$$

Proof. (4.43), (4.44), and Lemma 4.3. □

4.2.2 Critical points – finitely many collision times

As the identity $d\mathcal{B}|_z = \langle \operatorname{grad}\mathcal{B}(z), \cdot\rangle$ determines the gradient, an equation for the critical points of $\mathcal{B}$ is obtained by setting (4.45) equal zero, then solve for z''.

Theorem 4.7 (regularized critical point DDE)**.** *The critical points of the non-local action $\mathcal{B}\colon \bar{\mathcal{L}}^\times \mathfrak{Z} \to \mathbb{R}$ in (4.42) are the solutions of the delay equation*

$$\begin{aligned} z''(\tau) &= z(\tau)\left(\frac{\|z'\|^2}{\|z\|^2} - \frac{1}{2\|z\|^6}\right) + \frac{1}{4\|z\|^2}(\operatorname{grad}\mathcal{M}|_z)(\tau) \\ &= z(\tau)\Bigg(\frac{\|z'\|^2}{\|z\|^2} - \frac{1}{2\|z\|^6} - \frac{1}{2\|z\|^6}\int_0^1 \textstyle\int_0^\sigma |z_\rho|^2 d\rho\, \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma \\ &\qquad\qquad + \frac{1}{2\|z\|^4}\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma\Bigg) \\ &\quad - \frac{|z(\tau)|^2}{4\|z\|^4}\cdot \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z(\tau)} - \frac{\operatorname{rot}\boldsymbol{a}_{t_z(\tau)}|_{z(\tau)}}{4\|z\|^2}\, j_0 z'(\tau) \end{aligned} \tag{4.46}$$

for smooth loops $z \not\equiv 0$ in $\mathfrak{Z}$. In symbols, the set of critical points is given by

$$\operatorname{Crit}\mathcal{B} := \{z \in \bar{\mathcal{L}}^\times \mathfrak{Z} \mid \operatorname{grad}\mathcal{B}|_z = 0\} = \{z \in \bar{\mathcal{L}}^\times \mathfrak{Z} \text{ solving (4.46)}\}.$$

*The elements $z \in \operatorname{Crit}\mathcal{B}$ are called **regularized collision solutions**.*[10]

Proof. Divide (4.45) by $4\|z\|^2$, resolve for z''. □

Lemma 4.8. *For $z \in \operatorname{Crit}\mathcal{B}$ **regularization collision times** form a finite set*

$$\mathcal{T}_z := \{\tau_* \in [0,1)\colon z(\tau_*) = 0\} = \{\tau_1, \dots, \tau_N\}$$

enumerated increasingly $\tau_j < \tau_{j+1}$. Hence $[0,1)\setminus\mathcal{T}_z$ is a union of N intervals[11]

$$[0,1)\setminus\mathcal{T}_z = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_N, \qquad \mathcal{I}_j := (\tau_j, \tau_{j+1})), \quad \tau_{N+1} := \tau_1.$$

[10] away from collisions, they correspond to classical/physical solutions, i.e. of the ODE (2.8)

[11] In case $\tau_1 = 0$ the formal final interval (τ_N, τ_1) abbreviates $(\tau_N, 1)$. In case $\tau_1 > 0$ the formal final interval (τ_N, τ_1) abbreviates $(\tau_N, 1) \cup [0, \tau_1)$.

Furthermore, at collisions velocities are non-zero while accelerations vanish

$$\forall \tau_* \in \mathcal{T}_z = z^{-1}(0)\colon \qquad z'(\tau_*) \neq 0, \qquad z''(\tau_*) = 0.$$

Proof. We define three maps pointwise at $\tau \in \mathbb{S}^1$, namely $a \in \mathcal{L}\mathbb{R}$ by

$$a(\tau) := \frac{\|z'\|^2}{\|z\|^2} - \frac{(1 + \int_0^1 \int_0^s |z_\sigma|^2 \, d\sigma \, \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, z_s'\rangle_0 ds)}{2\|z\|^6} + \frac{\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z_\sigma'\rangle_0 d\sigma}{2\|z\|^4},$$

the map $b \in \mathcal{L}\mathbb{C}$ by $b(\tau) := -\frac{1}{4\|z\|^4} \, \bar{z}(\tau)\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z(\tau)}$, and the map $c \in \mathcal{L}i\mathbb{R}$ by $c(\tau) := -\frac{1}{4\|z\|^2} \, i\tilde{B}_{t_z\tau}|_{z_\tau}$. Because z solves the regularized DDE (4.46), it is as well a solution of the second order linear homogeneous ODE with continuous coefficient functions $z''(\tau) = a(\tau)z(\tau) + b(\tau)\bar{z}(\tau) + c(\tau)z'(\tau)$. This implies that

$$z'(\tau_*) \neq 0, \quad \forall \tau_* \in \mathcal{T}_z.$$

Suppose by contradiction that z is a solution of the second order ODE with initial condition $z(\tau_*) = z'(\tau_*) = 0$. Hence $z \equiv 0$ and $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$. Contradiction. So $\mathcal{T}_z$ is a discrete subset of the compact space $\mathbb{S}^1$, hence finite.

That $z(\tau_*) = 0 \Rightarrow z''(\tau_*) = 0$ follows from the critical point equation (4.46) together with $\operatorname{rot} \boldsymbol{a}_t|_0 = 0$ by (4.36). This proves Lemma 4.8. □

4.3 Non-collisional regularized solutions – Crit $\mathcal{B}^\times$

We show that those solutions z of the regularized critical point DDE (4.46) which actually avoid the singularity at the origin, in symbols $z \in (\operatorname{Crit} \mathcal{B}) \cap \bar{\mathcal{L}}\mathfrak{Z}^\times$, correspond simultaneously to *physical solutions*, i.e. solutions of the ODE (2.8), namely via the map $z \mapsto \mathcal{Q}(z) =: q \in \mathcal{L}\mathfrak{Q}^\times$. Abstractly this is clear, since solutions of (2.8) are critical points of the classical functional $\mathcal{S}\colon \mathcal{L}\mathfrak{Q}^\times \to \mathbb{R}$ and the restriction $\mathcal{B}^\times\colon \bar{\mathcal{L}}\mathfrak{Z}^\times \to \mathbb{R}$ of the regularized functional $\mathcal{B}\colon \bar{\mathcal{L}}^\times\mathfrak{Z} \to \mathbb{R}$ coincides with the pull-back $\mathcal{S} \circ \mathcal{Q}\colon \bar{\mathcal{L}}\mathfrak{Z}^\times \to \mathbb{R}$, as illustrated by Figure 2. Although clear, it is not obvious by looking at the equations (4.46) and (2.8).

Hence in this section we show this explicitly. In doing so we find in an intermediate step a second order DDE for q which will help us in the subsequent Section 4.4 to see how solutions of the regularized DDE (4.46) in the complement of $\bar{\mathcal{L}}\mathfrak{Z}^\times$ in $\bar{\mathcal{L}}^\times\mathfrak{Z}$ correspond to *collisional* solutions of the classical ODE (2.8).

Proposition 4.9. *Let $z \in \bar{\mathcal{L}}\mathfrak{Z}^\times$ solve the regularized critical point DDE (4.46). Then $q := z^2 \circ \tau_z$, Definition 3.6, solves the classical critical point ODE (2.8).*

Proof. The proof has two steps.

Step 1. *If $z \in \bar{\mathcal{L}}\mathfrak{Z}^\times$ is a solution of the regularized critical point DDE (4.46), then $q := \mathcal{Q}(z) := z^2 \circ \tau_z$ is a solution of the 2^{nd} order **intermediate DDE***

$$\begin{aligned} &\ddot{q}_t + B_t(q_t)j_0\dot{q}_t + \dot{\boldsymbol{A}}_t(q_t) \\ &= \frac{\frac{1}{2}\|\dot{q}\|^2 - \int_0^1 \frac{ds}{|q_s|} - \int_0^1 s\langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot{q}_s\rangle_0 \, ds - \frac{1}{2}|\dot{q}_t|^2 + \int_t^1 \langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot{q}_s\rangle_0 \, ds}{\bar{q}_t}. \end{aligned} \tag{4.47}$$

Proof of Step 1. We need to calculate the first two derivatives of $\mathcal{Q}(z)\colon \mathbb{S}^1 \to \mathfrak{Q}^\times$. It is convenient to abbreviate and use complex notation

$$q := q_z := \mathcal{Q}(z), \qquad q = q_1 + iq_2, \qquad q_t := q(t), \qquad q_t = q_{1,t} + iq_{2,t}.$$

Hence $\bar q := \overline{q_z} = \bar z^2 \circ \tau_z$. The derivative $\dot q$ is given by (3.25) and the derivative of $\tau_z(t)$ by (3.24). So, following [Fra25, §5], the second derivative is of the form

$$\ddot q_t = \frac{2\|z\|^4}{\bar z_{\tau_z(t)}\bar z_{\tau_z(t)} z_{\tau_z(t)}} \left(z''_{\tau_z(t)} - \frac{\left|z'_{\tau_z(t)}\right|^2}{\bar z_{\tau_z(t)}} \right) = \frac{1}{\bar q_t} \left(\frac{2\|z\|^4 z''_{\tau_z(t)}}{z_{\tau_z(t)}} - \frac{1}{2} \left|\dot q_t\right|^2 \right)$$

for every $t \in \mathbb{S}^1$. To calculate $\left|\dot q_t\right|^2 = \dot q_t \bar{\dot q}_t$ we used (3.25) and its complex conjugate form. Continuing the calculation of $\ddot q_t$, we change first the order of the two summands, then in equality 2 we replace $z''_{\tau_z(t)}$ by (4.46), hence

$$\begin{aligned}
\ddot q(t) &= -\frac{|\dot q_t|^2}{2\bar q_t} + \frac{\|z\|^2}{2\bar q_t} \frac{4\|z\|^2 z''_{\tau_z(t)}}{z_{\tau_z(t)}} \\
&\overset{2}{=} -\frac{|\dot q_t|^2}{2\bar q_t} + \frac{2}{\bar q_t}\|z\|^2\|z'\|^2 - \frac{1}{\bar q_t \|z\|^2} \\
&\quad - \frac{1}{\bar q_t\|z\|^2} \int_0^1 \underbrace{\textstyle\int_0^\sigma |z_\rho|^2\, d\rho}_{\overset{(3.21)}{=} t_z(\sigma)\|z\|^2} \langle \dot{\boldsymbol a}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma \\
&\quad + \frac{1}{\bar q_t} \int_{\sigma=\tau_z(t)}^1 \langle \dot{\boldsymbol a}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0 \, d\sigma - \frac{\left|z_{\tau_z(t)}\right|^2}{2\bar q_t z_{\tau_z(t)}} \dot{\boldsymbol a}_t|_{z_{\tau_z(t)}} \\
&\quad - \frac{1}{2\bar q_t}\|z\|^2 \left(\operatorname{rot} \boldsymbol a_t|_{z_{\tau_z(t)}}\right) i\frac{z'_{\tau_z(t)}}{z_{\tau_z(t)}} \\
&\overset{3}{=} -\frac{|\dot q_t|^2}{2\bar q_t} + \frac{1}{2\bar q_t}\|\dot q\|^2 - \frac{1}{\bar q_t}\int_0^1 \frac{1}{|q_s|}\, ds \\
&\quad - \frac{1}{\bar q_t} \int_{s=t_z(0)=0}^{t_z(1)=1} s \cdot \operatorname{Re}\left(\overline{2\bar z_\sigma \dot{\boldsymbol A}_s} \cdot \frac{\dot q_s \bar z_\sigma}{2\|z\|^2} \right) \frac{\|z\|^2 ds}{z_\sigma \bar z_\sigma} \\
&\quad + \frac{1}{\bar q_t} \int_{s=t_z(\tau_z)=t}^{t_z(1)=1} \operatorname{Re}\left(\overline{2\bar z_\sigma \dot{\boldsymbol A}_s} \cdot \frac{\dot q_s \bar z_\sigma}{2\|z\|^2} \right) \frac{\|z\|^2 ds}{z_\sigma \bar z_\sigma} - \dot{\boldsymbol A}_t|_{z^2_{\tau_z(t)}} \\
&\quad - \frac{1}{2\bar q_t}\|z\|^2 4\bar z_{\tau_z(t)} z_{\tau_z(t)} \underbrace{\left(\operatorname{rot} \boldsymbol A_t|_{q_t}\right)}_{=B_t(q_t)} i\frac{\dot q_t \bar z_{\tau_z(t)}}{z_{\tau_z(t)} 2\|z\|^2} \\
&\overset{4}{=} -\frac{|\dot q_t|^2}{2\bar q_t} + \frac{1}{2\bar q_t}\|\dot q\|^2 - \frac{1}{\bar q_t}\int_0^1 \frac{1}{|q_s|}\, ds - \frac{1}{\bar q_t}\int_0^1 s\cdot \langle \dot{\boldsymbol A}_s|_{q_s}, \dot q_s\rangle_0 \, ds \\
&\quad + \frac{1}{\bar q_t}\int_t^1 \langle \dot{\boldsymbol A}_s|_{q_s}, \dot q_s\rangle_0 \, ds - \dot{\boldsymbol A}_t|_{q_t} - B_t(q_t) j_0 \dot q_t.
\end{aligned}$$

Equality 2 also uses that $t_z(\tau_z(t)) = t$ by (3.23).
Equality 3: There are seven summands. Summand one remains unchanged. In

summand two replace $\|z\|^2\|z'\|^2$ by $\frac{1}{4}\|\dot{q}\|^2$ according to (3.26). In summand three replace $1/\|z\|^2$ by (3.29). In summand four we replaced the $\int_0^\sigma$-term according to (3.21). Then we changed the integration variable to $s(\sigma) := t_z(\sigma)$. The substitution of the inner product, of z'_σ, and of $d\sigma$ is explained next in summand five. In summand five we rewrote the inner product in the $\mathbb{R}^2$ picture in terms of the real part (3.10) in the $\mathbb{C}$ picture and we substituted the vector potential according to (4.38) with dots (time derivative). Then we changed the integration variable to $s := t_z(\sigma)$: We replaced $d\sigma$ with the help of (3.24) and z'_σ according to (3.25). Now many factors annulate in pairs and (in equality 4) we go back to the $\mathbb{R}^2$ picture and inner product. In summand six we also used (4.38) for time derivatives and that $\bar{q}_t = \bar{z}^2_{\tau_z(t)}$. To the final summand seven the following happened: We used that $z^2_{\tau_z(t)} = q_t$. We replaced the rotational according to (4.36) and $z'_{\tau_z(t)}$ according to (3.25). Now certain factors annulate in pairs. Equality 4 writes down the cleaned up result. This proves Step 1. □

Step 2. *If $q \in \mathcal{L}\mathfrak{Q}^\times$ is a solution of the intermediate DDE (4.47), then it is also a solution of the classical ODE (2.8).*

Proof of Step 2. Given a solution $q \in \mathcal{L}\mathfrak{Q}^\times$ of (2.8), we define

$$\beta_t := \frac{\ddot{q}_t + B_t(q_t)i\dot{q}_t + \dot{\boldsymbol{A}}_t(q_t)}{q_t}, \quad \forall t \in \mathbb{R}. \tag{4.48}$$

By definition of β and since $|q_t|^2 = \bar{q}_t q_t$, using (4.47) in equality 2, we obtain

$$\begin{aligned}\beta_t\,|q_t|^2 &= \bar{q}_t\left(\ddot{q}_t + B_t|_{q_t} i\dot{q}_t + \dot{\boldsymbol{A}}_t|_{q_t}\right)\\ &\stackrel{2}{=} \underbrace{\tfrac{1}{2}\|\dot{q}\|^2 - \int_0^1 \tfrac{ds}{|q_s|} - \int_0^1 s\langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot{q}_s\rangle_0\, ds}_{\text{independent of } t} - \tfrac{1}{2}\,|\dot{q}_t|^2 + \int_t^1 \langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot{q}_s\rangle_0\, ds.\end{aligned} \tag{4.49}$$

As the right hand side of (4.49) is real, so is β, in symbols $\beta_t \in \mathbb{R}$, $\forall t \in \mathbb{R}$. Differentiate (4.49) to get the first equality, the definition of β gives the second

$$\begin{aligned}&2\,\langle q_t, \dot{q}_t\rangle_0\,\beta_t + |q_t|^2\,\dot{\beta}_t\\ &= -\,\langle \dot{q}_t, \ddot{q}_t\rangle_0 - \langle \dot{\boldsymbol{A}}_t|_{q_t}, \dot{q}_t\rangle_0\\ &= -\langle \dot{q}_t, \beta_t q_t\rangle_0 + \langle \dot{q}_t, B_t(q_t) j_0 \dot{q}_t\rangle_0 + \langle \dot{q}_t, \dot{\boldsymbol{A}}_t(q_t)\rangle_0 - \langle \dot{\boldsymbol{A}}_t|_{q_t}, \dot{q}_t\rangle_0\\ &= -\beta_t\langle q_t, \dot{q}_t\rangle_0.\end{aligned}$$

Equality three uses that β is real and that, since B is real and by (3.9), the term $\langle \dot{q}_t, B_t(q_t)j_0\dot{q}_t\rangle_0 = B_t(q_t)\omega_0(\dot{q}_t, j_0 j_0 \dot{q}_t) = -B_t(q_t)\omega_0(\dot{q}_t, \dot{q}_t) = 0$ vanishes. By rearrangement $|q|^2\,\dot{\beta} = -3\langle q, \dot{q}\rangle\beta$, thus $\frac{\dot{\beta}}{\beta} = -\frac{3}{2}\frac{\partial_t |q|^2}{|q|^2}$. A solution of this first order ODE is of the form $\ln|\beta| = -\frac{3}{2}\ln|q|^2 + c = -\ln|q|^3 + c$ for some constant $c \in \mathbb{R}$. Equivalently $\beta = \frac{\mu}{|q|^3}$ for $\mu = \pm e^c$. Definition (4.48) of β shows that q solves the second order ODE

$$\ddot{q} = \tfrac{\mu q}{|q|^3} - B_t(q)i\dot{q} - \dot{\boldsymbol{A}}_t(q). \tag{4.50}$$

To prove $\mu=-1$, thus proving Step 2, combine $\beta=\frac{\mu}{|q|^3}$ and (4.49) to get

$$\frac{\mu}{|q_t|}=\underbrace{\tfrac{1}{2}\|\dot q\|^2-\int_0^1\tfrac{ds}{|q_s|}-\int_0^1 s\langle\dot{\boldsymbol{A}}_s|_{q_s},\dot q_s\rangle_0\,ds}_{\text{independent of } t}-\tfrac{1}{2}\left|\dot q_t\right|^2+\int_{s=t}^1\langle\dot{\boldsymbol{A}}_s|_{q_s},\dot q_s\rangle_0\,ds.$$

Integrate this equation and in equality 2 change the order of integration to get

$$\begin{aligned}\mu\int_0^1\frac{dt}{|q_t|}&=\tfrac{1}{2}\|\dot q\|^2-\int_0^1\frac{ds}{|q_s|}-\int_0^1 s\langle\dot{\boldsymbol{A}}_s|_{q_s},\dot q_s\rangle_0\,ds\\&\quad-\tfrac{1}{2}\int_0^1\left|\dot q_t\right|^2\,dt+\int_{t=0}^1\int_{s=t}^1\langle\dot{\boldsymbol{A}}_s|_{q_s},\dot q_s\rangle_0\,dsdt\\&\overset{2}{=}-\int_0^1\frac{dt}{|q_t|}-\int_0^1 s\langle\dot{\boldsymbol{A}}_s|_{q_s},\dot q_s\rangle_0\,ds+\int_{s=0}^1\int_{t=0}^s\langle\dot{\boldsymbol{A}}_s|_{q_s},\dot q_s\rangle_0\,dtds\\&=-\int_0^1\frac{dt}{|q_t|}.\end{aligned}\tag{4.51}$$

Thus $\mu=-1$ and this proves Step 2. □

Step 1 and Step 2 together conclude the proof of Proposition 4.9. □

4.4 Collisional regularized solutions – Crit $\mathcal{B}$

First we extend the reparametrization map t_z to collisional loops $z\in\bar{\mathcal{L}}^{\times}\mathfrak{Z}$ with finitely many collisions of non-zero speed each. While such extension is useful later on, in this sections we focus on critical points z of $\mathcal{B}$; cf. Lemma 4.8.

The goal is to show that applying the rescale-square operation $\mathcal{Q}$ to $z\in\operatorname{Crit}\mathcal{B}$ yields a continuous map $q_z:=z^2\circ\tau_z\colon\mathbb{S}^1\to\mathfrak{Q}$ which, away from **classical collision times** $T_{q_z}:=q_z^{-1}(0)$, is smooth and solves the classical ODE (2.8). As a byproduct we get that at all collision times the collision velocity is equal $|z'(\tau_*)|=1/\sqrt{2}\|z\|^2$ as seen in (4.58) since $\mu_j=-1$. (Later on, in (4.67), we show that actually the collision acceleration $|z''(\tau_*)|=0$ vanishes.)

4.4.1 Collisional reparametrization map t_z

Lemma 4.10. *Let $z\in\bar{\mathcal{L}}^{\times}\mathfrak{Z}$ be a loop with finite **regularization collision times set** $\mathcal{T}_z:=z^{-1}(0)$ and $z'(\tau_*)\neq 0$ $\forall\tau_*\in\mathcal{T}_z$. Then the following is true.*

(i) *The map $t_z\colon\mathbb{S}^1_\tau\to\mathbb{S}^1_t$ defined (as in the non-collisional case) by*

$$\forall\tau\in\mathbb{R}:\qquad t_z(\tau):=\frac{\int_0^\tau\left|z(\sigma)\right|^2d\sigma}{\|z\|^2},\qquad t_z'(\tau)=\frac{\left|z(\tau)\right|^2}{\|z\|^2}\geq 0,\tag{4.52}$$

is smooth and a strictly increasing homeomorphism with $t_z(0)=0$.

(ii) *The inverse homeomorphism $\tau_z:=t_z^{-1}\colon\mathbb{S}^1_t\to\mathbb{S}^1_\tau$ is smoothly differentiable away from collision times $t_z(\mathcal{T}_z)$ where $\dot\tau_z(t)=\frac{1}{t_z'(\tau_z(t))}=\frac{\|z\|^2}{|z(\tau_z(t))|^2}>0$.*

Proof. That t_z maps $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ to $\mathbb{S}^1$ is true, because $t_z(\tau+1) = t_z(\tau)+1\ \forall \tau \in \mathbb{R}$, which follows from $\int_0^{\tau+1} = \int_0^1 + \int_1^{\tau+1}$ and since $z_{\sigma+1} = z_\sigma$ is periodic. Clearly t_z is smooth and its derivative is of the form (4.52) which also shows that collision times precisely coincide with critical points of t_z, in symbols $\mathcal{T}_z = \operatorname{Crit} t_z$. The proof completes in two bullets.

• The critical points of t_z are inflection points of t_z, more precisely

$$\forall \tau_* \in \mathcal{T}_z \qquad t_z'(\tau_*) = t_z''(\tau_*) = 0, \qquad t_z'''(\tau_*) > 0. \tag{4.53}$$

To see this we compute the following derivatives

$$t_z''(\tau) = \tfrac{2\langle z(\tau), z'(\tau)\rangle_0}{\|z\|^2}, \qquad t_z'''(\tau) = 2\tfrac{|z'(\tau)|^2 + \langle z(\tau), z''(\tau)\rangle}{\|z\|^2},$$

for $\tau \in \mathbb{R}$. At any collision $z(\tau_*) = 0$, as $|z'_{\tau_*}| = \frac{1}{\sqrt{2}\|z\|^2} \neq 0$, we conclude that

$$t_z'(\tau_*) = 0, \qquad t_z''(\tau_*) = 0, \qquad t_z'''(\tau_*) = \tfrac{2|z'(\tau_*)|^2}{\|z\|^2} > 0.$$

• By (4.52) and (4.53) the continuous map t_z is strictly monotonically increasing. So, since $t_z(0) = 0$ and $t_z(1) = 1$, as in the regular case of Remark 3.5, the map t_z has a continuous inverse $\tau_z := t_z^{-1} \colon \mathbb{S}^1_t \to \mathbb{S}^1_\tau$ strictly monotonically increasing. Thus τ_z is still a homeomorphism of the circle, but in contrast to the regular case τ_z is not everywhere differentiable. Away from the finite set $t_z(\mathcal{T}_z)$ it is still smoothly differentiable, the derivative still given by (3.24). □

4.4.2 Collisional rescale-square map $\mathcal{Q}$

Proposition 4.11 ($\mathcal{Q}$ on $\operatorname{Crit}\mathcal{B}$). *A solution $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$ to the regularized DDE (4.46), equivalently $z \in \operatorname{Crit}\mathcal{B}$, yields a $W^{1,2}$-map*

$$q_z := \mathcal{Q}(z) = z^2 \circ \tau_z \colon \mathbb{S}^1 \to \mathfrak{Q} \tag{4.54}$$

smoothly differentiable away from classical collision times $t_j := t_z(\tau_j)$. Set

$$T_{q_z} := q_z^{-1}(0) = t_z(\mathcal{T}_z) = \{t_1, \dots, t_N\}, \quad I_j := t_z(\mathcal{I}_j) = (t_j, t_{j+1}).$$

Restricted to non-collision times $q_z \colon \mathbb{S}^1 \setminus T_{q_z} \to \mathfrak{Q}$ solves the classical ODE (2.8). Furthermore, at all collision times of z the collision velocity is equal

$$\forall z \in \operatorname{Crit}\mathcal{B} \quad \forall \tau_* \in \mathcal{T}_z = z^{-1}(0) \colon \qquad |z'(\tau_*)| = \frac{1}{\sqrt{2}\|z\|^2}.$$

Proof. Pick $z \in \operatorname{Crit}\mathcal{B}$. Lemma 4.10 applies by Lemma 4.8. That the composition $q_z = z^2 \circ \tau_z$ is continuous and smoothly differentiable away from $t_z(\mathcal{T}_z)$ holds since t_z has these properties, see Lemma 4.10, and z is smooth anyway. Since t_z fixes both 0 and 1, and it is strictly monotonically increasing the order $\tau_1 < \dots < \tau_N$ is inherited by the images $t_1 < \dots < t_N$.

It remains to show that $q := q_z$ lies in $W^{1,2}$ and restricted to any of the collision free intervals I_j solves the classical ODE (2.8). This takes three steps.

STEP 1. The same argument as in the regular case—in the proof of Proposition 4.9 replace $t \in \mathbb{S}^1$ by $t \in I_j$—shows a) that $q_z|_{I_j}$ solves the intermediate DDE (4.47) and b) that there exists $\mu_j \in \mathbb{R}$ such that

$$\ddot{q}_t = \frac{\mu_j q_t}{|q_t|^3} - B_t(q_t) i\dot{q}_t - \dot{\boldsymbol{A}}_t(q_t), \quad \forall t \in I_j. \tag{4.55}$$

To see this note the following. a) The DDE (4.47) still makes sense since $1/\bar{q}_t$ is finite for $t \in I_j$ and so is the mean value $\int_0^1 \frac{ds}{|q_s|} < \infty$, by (3.29). b) Since $q_t \neq 0$ the function β_t in (4.48) is well defined and so is (4.50), respectively (4.55), and the argument to get there.

STEP 2. We show that $\mu_j = -1$ in (4.55) for any j. To see this pick $t \in I_j$, then

$$\begin{aligned}
\mu_j &\overset{(4.55)}{=} |q_t|\, \bar{q}_t \left(\ddot{q}_t + B_t(q_t) i\dot{q}_t + \dot{\boldsymbol{A}}_t(q_t) \right) \\
&\overset{(4.47)}{=} -\frac{|q_t| \cdot |\dot{q}_t|^2}{2} \\
&\quad + \underbrace{|q_t| \left(\tfrac{1}{2}\|\dot{q}\|^2 - \textstyle\int_0^1 \frac{ds}{|q_s|} - \int_0^1 s \langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot{q}_s\rangle_0 \, ds + \int_t^1 \langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot{q}_s\rangle_0 \, ds \right)}_{\to 0, \text{ as } t \to t_j \text{ or } t \to t_{j+1}} .
\end{aligned} \tag{4.56}$$

The limit is zero as the first three summands in the bracket are constants and summand four remains finite as t goes to t_j or t_{j+1}. By definition of q_z we get

$$\frac{|q_t| \cdot |\dot{q}_t|^2}{2} \overset{(3.25)}{=} \frac{\left|z^2(\tau_z(t))\right| \cdot 4\|z\|^4 \left|z'(\tau_z(t))\right|^2}{2\left|\bar{z}(\tau_z(t))\right|^2} = 2\|z\|^4 \left|z'(\tau_z(t))\right|^2 . \tag{4.57}$$

Hence, if we take in (4.56) the limit $t \to t_j$, respectively $t \to t_{j+1}$, we obtain

$$\mu_j = -2\|z\|^4 \left|z'(\tau_z(t_j))\right|^2 = -2\|z\|^4 \left|z'(\tau_z(t_{j+1}))\right|^2 . \tag{4.58}$$

Since this is true for all j, the right hand side is actually equal to μ_{j+1} viewed as the left boundary limit of $I_{j+1} = (t_{j+1}, t_{j+2})$. This means that all the μ_j are equal to the same real number, say μ. Now (4.55) tells that

$$\ddot{q}_t = \tfrac{\mu q_t}{|q_t|^3} - B_t(q_t) i\dot{q}_t - \dot{\boldsymbol{A}}_t(q_t), \quad \forall t \in \mathbb{S}^1 \setminus t_z(\mathcal{T}_z).$$

As in the regular case (4.51), using in addition that $t_z(\mathcal{T}_z)$ is a finite set, integration implies that $\mu = -1$. Hence, by (4.58), it holds $|z'(\tau_*)| = 1/\sqrt{2}\|z\|^2$ whenever $\tau_* \in z^{-1}(0)$. This proves Step 2.

STEP 3. Since q is continuous it is L^2. To see that $q := q_z \in W^{1,2}(\mathbb{S}^1, \mathfrak{Q})$ it remains to show finiteness of

$$\int_0^1 |\dot{q}_t|^2 \, dt \overset{1}{=} 4\|z\|^4 \int_0^1 \frac{|z'_{\tau_z(t)}|^2}{|z_{\tau_z(t)}|^2} \, dt \overset{2}{=} 4\|z\|^2 \int_0^1 |z'_\sigma|^2 \, d\sigma < \infty.$$

Equality 1 is by (4.57) with q_t replaced by $z^2_{\tau_z(t)}$. Equality 2 is by variable substitution $\sigma := \tau_z(t)$, thus $dt = \frac{\|z\|^2}{|z_{\tau_z(t)}|^2} d\tau$. Finiteness is true since z' is smooth and its domain $\mathbb{S}^1$ is compact. This proves Step 3 and Proposition 4.11. □

4.5 Correspondence of regularized and classical collisions

4.5.1 Classical collision spaces and main theorem

Definition 4.12 ($\Lambda^\times_{\mathrm{coll}}\mathfrak{Q}$)**.** The **classical collision loop space** $\Lambda^\times_{\mathrm{coll}}\mathfrak{Q}$ consists of all loops $q \in W^{1,2}(\mathbb{S}^1, \mathfrak{Q})$ such that

(i) there are at most finitely many collision times $T_q := q^{-1}(0) = \{t_1, \dots, t_N\}$;

(ii) there exist (unique) [12] continuous maps

$$\alpha_q \colon \mathbb{S}^1_t \to \mathbb{S}^1_{\mathbb{C}}, \qquad e_q \colon \mathbb{S}^1_t \to \mathbb{R}, \qquad \beta_q \colon \mathbb{S}^1_t \to \mathbb{C}, \quad \beta|_{T_q} \equiv 0,$$

which away from collisions are solution angle and energy, more precisely

$$\alpha|_{\mathbb{S}^1 \setminus T_q} = \frac{q}{|q|}, \qquad e|_{\mathbb{S}^1 \setminus T_q} = \tfrac{1}{2}|\dot q|^2 - \frac{1}{|q|}, \qquad \beta|_{\mathbb{S}^1 \setminus T_q} = \dot\alpha_q |q|. \tag{4.59}$$

Definition 4.13 ($\mathrm{Coll}\,\mathcal{S}$)**.** The **space of classical periodic collision orbits** $\mathrm{Coll}\,\mathcal{S}$ consists of all classical collision loops $q \in \Lambda^\times_{\mathrm{coll}}\mathfrak{Q}$ such that

(iii) away from collision times $q \colon \mathbb{S}^1 \setminus T_q \to \mathfrak{Q}$ satisfies the classical ODE (2.8).

Remark 4.14 (Motivation for (4.59))**.** If one of the first two conditions (4.59) fails, then already in the Kepler problem the correspondence with critical points of the regularized functional will not be true. In the Kepler problem collisional orbits are just rays. By continuity of α, after collision the particle will stay on the same ray. Moreover, in the Kepler problem the energy is constant on each ray, by continuity of the energy it will be globally constant.
Without the continuity assumption on α and e one could consider rays, parametrized according to the Kepler equation, which after collision jump to another ray or change the energy.

Theorem 4.15 (bijection)**.** *The rescale-square map $\mathcal{Q}$ in (4.54) is a bijection between* $\mathrm{Crit}\,\mathcal{B}$ *and* $\mathrm{Coll}\,\mathcal{S}$*.*

4.5.2 Regularized collision spaces

Definition 4.16 ($\bar{\mathcal{L}}^\times_{\mathrm{coll}}\mathfrak{Z}$)**.** Define the **regularized collision loop space** by

$$\bar{\mathcal{L}}^\times_{\mathrm{coll}}\mathfrak{Z} := \left\{ z \in \bar{\mathcal{L}}^\times\mathfrak{Z} \;\middle|\; \forall \tau_* \in z^{-1}(0) \colon |z'_{\tau_*}| = \tfrac{1}{\sqrt{2}\|z\|^2} \text{ and } z''_{\tau_*} = 0 \right\}.$$

The elements are called **regularized collision loops**. Each has finitely many zeroes: first derivatives at zeroes of z have same length and $\mathbb{S}^1$ is compact.

Lemma 4.17. $\mathrm{Crit}\,\mathcal{B} \subset \bar{\mathcal{L}}^\times_{\mathrm{coll}}\mathfrak{Z} \subset \bar{\mathcal{L}}^\times\mathfrak{Z}$.

Proof. Proposition 4.11 and Lemma 4.8. □

[12] Since the set of collision times is finite, the functions $\alpha = \alpha_q$ and $e = e_q$ are uniquely determined by continuity of q.

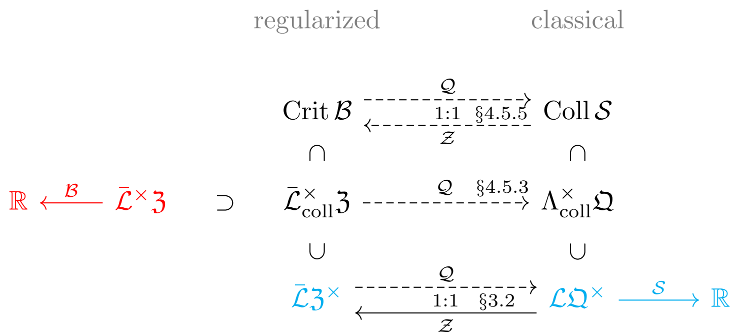

Figure 3: Correspondence 1:1 of regularized and classical collision orbits

4.5.3 Extending $\mathcal{Q}$

The following proposition served us to identify the proper conditions to define the space $\Lambda^\times_{\rm coll}\mathfrak{Q}$ to accommodate the critical point equation on the q-side.

Proposition 4.18 ($\mathcal{Q}$)**.** *The rescale-square map $\mathcal{Q}$ defined on* $\operatorname{Crit}\mathcal{B}$ *by (4.54) extends to $\bar{\mathcal{L}}^\times_{\rm coll}\mathfrak{Z}$ and takes values in $\Lambda^\times_{\rm coll}\mathfrak{Q}$, as illustrated by Figure 3.*

Proof. Pick $z \in \bar{\mathcal{L}}^\times_{\rm coll}\mathfrak{Z}$ and abbreviate $q = q_z$ where $q_z := z^2 \circ \tau_z \colon \mathbb{S}^1 \to \mathfrak{Q}$. The zero sets $\mathcal{T}_z := z^{-1}(0)$ and $T_q := q^{-1}(0)$ are, respectively, called regularized and classical collision times. By Lemma 4.10 the map $t_z \colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by (3.21) is still smooth, but if collisions exist only a homeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$, since the inverse homeomorphism τ_z has a derivative (3.24) with singularities at the classical collision times T_q. The relation (4.57) is still valid on the finitely many intervals $\mathbb{S}^1 \setminus T_q$.

Note that q is periodic and T_q is finite: As in Lemma 3.8 one checks that $q_z(1) = q_z(0)$ is 1-periodic. Since $q = z^2 \circ \tau_z$ and τ_z is a bijection the zeroes of q are in bijection to those of z, but z^{-1} is a finite set.

Step 1. Existence of e_q.

Proof. Since z is smooth and vanishes together with its second derivative at $\tau_* \in \mathcal{T}_z$, in symbols $z''_{\tau_*} = 0$, by Taylor's theorem there exists a smooth function $\zeta \colon \mathbb{R} \to \mathbb{C}$ such that for every τ the following formula for z holds

$$\begin{aligned} z_\tau &= z'_{\tau_*}(\tau - \tau_*) + \zeta_\tau(\tau - \tau_*)^3 \\ z'_\tau &= z'_{\tau_*} + 3\zeta_\tau(\tau - \tau_*)^2 + \zeta'_\tau(\tau - \tau_*)^3. \end{aligned} \tag{4.60}$$

Away from collisions, for $t \in \mathbb{S}^1 \setminus T_q$, we define e_q as follows, then in equality 3 we use (4.57) to get

$$
\begin{aligned}
e_q(t) &:= \tfrac{1}{2}|\dot q_t|^2 - \tfrac{1}{|q_t|}\\
&= \tfrac{1}{2|q_t|}\left(|q_t|\cdot|\dot q_t|^2 - 2\right)\\
&\overset{3}{=} \tfrac{1}{2|z^2_{\tau_z(t)}|}\left(4\|z\|^4|z'_{\tau_z(t)}|^2 - 2\right)\\
&\overset{4}{=} \frac{4\|z\|^4\left|z'_{\tau_*} + 3\zeta_{\tau_z(t)}(\tau_z(t)-\tau_*)^2 + \zeta'_{\tau_z(t)}(\tau_z(t)-\tau_*)^3\right|^2 - 2}{2\left|\left(z'_{\tau_*}(\tau_z(t)-\tau_*) + \zeta_{\tau_z(t)}(\tau_z(t)-\tau_*)^3\right)^2\right|}\\
&\overset{5}{=} \frac{4\|z\|^4|z'_{\tau_*}|^2 + (\tau_z(t)-\tau_*)^2\gamma_t - 2}{2(\tau_z(t)-\tau_*)^2|z'_{\tau_*}|^2 + (\tau_z(t)-\tau_*)^4\delta_t}\\
&\overset{6}{=} \frac{2 + (\tau_z(t)-\tau_*)^2\gamma_t - 2}{2(\tau_z(t)-\tau_*)^2|z'_{\tau_*}|^2 + (\tau_z(t)-\tau_*)^4\delta_t}\\
&\overset{7}{=} \frac{\gamma_t\|z\|^4}{1 + (\tau_z(t)-\tau_*)^2\delta_t\|z\|^4}.
\end{aligned}
$$

Equality 4 uses (4.60). Equality 5 abbreviates two continuous maps $\mathbb{R}\to\mathbb{R}$ by

$$
\begin{aligned}
\gamma_t := 4\|z\|^4\Bigg(&6\langle z'_{\tau_*},\zeta_{\tau_z(t)}\rangle_0 + 2(\tau_z(t)-\tau_*)\langle z'_{\tau_*},\zeta'_{\tau_z(t)}\rangle_0\\
&+ 9(\tau_z(t)-\tau_*)^2\left|\zeta_{\tau_z(t)}\right|^2 + 6(\tau_z(t)-\tau_*)^3\langle\zeta_{\tau_z(t)},\zeta'_{\tau_z(t)}\rangle_0\\
&+ (\tau_z(t)-\tau_*)^4\left|\zeta'_{\tau_z(t)}\right|^2\Bigg)
\end{aligned}
$$

and

$$
\delta_t := 4\langle z'_{\tau_*},\zeta_{\tau_z(t)}\rangle_0 + 2(\tau_z(t)-\tau_*)^2\left|\zeta_{\tau_z(t)}\right|.
$$

Equality 6 uses that $4\|z\|^4|z'_{\tau_*}|^2 = 2$ since $z\in\bar{\mathcal{L}}^{\times}_{\mathrm{coll}}\mathfrak{Z}$. Equation 7 in the e_q calculation above extends continuously to t_*. This proves Step 1. □

Step 2. Existence of α_q.

Proof. Away from collisions, for $t\in\mathbb{S}^1\setminus T_q$, we define α_q as follows

$$
\begin{aligned}
\alpha_q(t) &:= \frac{q_t}{|q_t|} = \frac{z^2_{\tau_z(t)}}{\left|z^2_{\tau_z(t)}\right|}\\
&\overset{3}{=} \frac{2\left(z'_{\tau_*}(\tau_z(t)-\tau_*) + \zeta_{\tau_z(t)}(\tau_z(t)-\tau_*)^3\right)^2}{2\left|\left(z'_{\tau_*}(\tau_z(t)-\tau_*) + \zeta_{\tau_z(t)}(\tau_z(t)-\tau_*)^3\right)^2\right|}\\
&\overset{4}{=} \frac{(\tau_z(t)-\tau_*)^2}{(\tau_z(t)-\tau_*)^2}\cdot\frac{2\left(z'_{\tau_*} + \zeta_{\tau_z(t)}(\tau_z(t)-\tau_*)^2\right)^2}{2|z'_{\tau_*}|^2 + (\tau_z(t)-\tau_*)^2\delta_t}\\
&\overset{5}{=} \frac{2\left(z'_{\tau_*} + \zeta_{\tau_z(t)}(\tau_z(t)-\tau_*)^2\right)^2}{\|z\|^{-4} + (\tau_z(t)-\tau_*)^2\delta_t}.
\end{aligned}
$$

Equation 2 is by definition of q. Equation 3 uses (4.60). In equation 4 for the denominator we used the previous definition of δ_t. Equality 5 uses that $4\|z\|^4|z'_{\tau_*}|^2 = 2$ since $z \in \bar{\mathcal{L}}^\times_{\mathrm{coll}}\mathfrak{Z}$. The right hand side extends continuously to t_*. This proves Step 2. ☐

Step 3. Existence of β_q.

Proof. Away from collisions, for $t \in \mathbb{S}^1 \setminus T_q$, we define β_q as follows

$$\begin{aligned}
\beta_q(t) :&= |q_t|\, \partial_t \tfrac{q_t}{|q_t|} = |z^2_{\tau_z(t)}| \left(\partial_\tau \frac{z_{\tau_z(t)}}{\bar{z}_{\tau_z(t)}}\right) \dot{\tau}_z(t) \\
&\stackrel{3}{=} |z_{\tau_z(t)}|^2 \cdot \tfrac{\|z\|^2}{|z_{\tau_z(t)}|^2} \cdot \tfrac{z'_{\tau_z(t)}\bar{z}_{\tau_z(t)} - z_{\tau_z(t)}\bar{z}'_{\tau_z(t)}}{\bar{z}^2_{\tau_z(t)}} \\
&\stackrel{4}{=} \|z\|^2 \cdot \tfrac{(\tau-\tau_*)^3 2i}{(\tau-\tau_*)^2} \tfrac{2\mathrm{Im}(\bar{z}'_*\zeta_\tau) + (\tau-\tau_*)\mathrm{Im}(\bar{z}'_*\zeta'_\tau) + (\tau-\tau_*)^2\mathrm{Im}(\bar{\zeta}_\tau\zeta'_\tau)}{\bar{z}'_*\bar{z}'_* + 2\bar{z}'_*\zeta_\tau(\tau-\tau_*)^2 + \zeta_\tau\zeta_\tau(\tau-\tau_*)^4} \\
&\stackrel{5}{=} 2i\,(\tau-\tau_*)\,\|z\|^2 \cdot \frac{2\mathrm{Im}(\bar{z}'_*\zeta_\tau) + (\tau-\tau_*)\mathrm{Im}(\bar{z}'_*\zeta'_\tau) + (\tau-\tau_*)^2\mathrm{Im}(\bar{\zeta}_\tau\zeta'_\tau)}{\bar{z}'_*\bar{z}'_* + 2\bar{z}'_*\bar{\zeta}_\tau(\tau-\tau_*)^2 + \bar{\zeta}_\tau\bar{\zeta}_\tau(\tau-\tau_*)^4}.
\end{aligned}$$

Equation 2 is by definition of q. Equality 3 is by the quotient rule and (3.24). In equality 4 we replace z and z' according to (4.60), multiply out and observe that the terms linear in $\tau - \tau_*$ cancel and so do those of fifth power. To simplify we used that $(a+ib)-(a-ib) = 2ib = 2i\mathrm{Im}(a+ib)$. The right hand side extends continuously to t_*. This proves Step 3. ☐

Step 4. Given $z \in \bar{\mathcal{L}}^\times_{\mathrm{coll}}\mathfrak{Z}$, let $q := \mathcal{Q}(z) := z^2 \circ \tau_z$, then

$$\int_0^1 \tfrac{1}{|q_t|}\, dt = \frac{1}{\|z\|^2}. \tag{4.61}$$

Proof. The open intervals $I_j = (t_j, t_{j+1})$ between collision times, Proposition 4.11, cover $\mathbb{S}^1 \setminus T_q$. The corresponding intervals $\mathcal{I}_j = (\tau_j, \tau_{j+1})$ are given in Lemma 4.8 where $\tau_j = \tau(t_j)$. Then

$$\int_0^1 \tfrac{1}{|q_t|}\, dt = \sum_{j=1}^N \int_{I_j} \tfrac{1}{|q_t|}\, dt = \sum_{j=1}^N \int_{I_j} \tfrac{1}{|z\circ\tau_z(t)|^2}\, dt \stackrel{3}{=} \sum_{j=1}^N \int_{\mathcal{I}_j} \tfrac{1}{|z(\tau)|^2} \tfrac{|z(\tau)|^2\, d\tau}{\|z\|^2} = \tfrac{1}{\|z\|^2}.$$

Equality 3 is change of variables $\tau = \tau_z(t)$ with (3.24). This proves Step 4. ☐

Step 5. The map $q\colon \mathbb{S}^1 \to \mathfrak{Q}$ is of Sobolev class $W^{1,2}$.

Proof. Note that $q = z^2 \circ \tau_z$ is continuous since z is smooth and τ_z is a homeomorphism by Lemma 4.10. As q is continuous, it suffices to show that $\dot{q}$ is in L^2. Using the definition of e_q and Step 4 we compute

$$\|\dot{q}\|^2 = 2\int_0^1 e_q(t)\, dt + 2\int_0^1 \tfrac{1}{|q_t|}\, dt \stackrel{(4.61)}{=} 2\int_0^1 e_q(t)\, dt + \tfrac{2}{\|z\|^2}.$$

As e_q is C^0 by Step 1, the right hand side is finite. This proves Step 5. ☐

This proves Proposition 4.18. ☐

Corollary 4.19. *The rescale-square map $\mathcal{Q}$ in (4.54) is well defined as a map*

$$\mathcal{Q}\colon \operatorname{Crit}\mathcal{B} \to \operatorname{Coll}\mathcal{S}.$$

Proof. Lemma 4.17, Proposition 4.11, Proposition 4.18. □

In view of Corollary 4.19, in order to prove Theorem 4.15 it suffices to construct an inverse map, notation $\mathcal{Z}\colon \operatorname{Coll}\mathcal{S} \to \operatorname{Crit}\mathcal{B}$.

4.5.4 Inverse map $\mathcal{Z}$

Definition 4.20 ($\mathcal{Z}$)**.** For a classical periodic collision orbit $q \in \operatorname{Coll}\mathcal{S}$ define

$$\mathcal{Z}(q) := z_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}, \qquad 1/\sqrt{q} = [t \mapsto 1/\sqrt{q_t}],$$

where $\tau_{1/\sqrt{q}}\colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$ is to be defined yet in Corollary 4.24 below. Taking square root is well defined on the quotient space $\bar{\mathcal{L}}^\times\mathfrak{Z}$, see Remark 3.2.

The goal of Subsection 4.5.4 is to show that $z_q \in \operatorname{Crit}\mathcal{B}$.

Rescaling of collisional loops $1/\sqrt{q}$

Lemma 4.21. *Let $q \in \Lambda^\times_{\mathrm{coll}}\mathfrak{Q}$. Then the integral $\int_0^1 \frac{1}{|q_t|}\,dt$ is finite.*

Proof. Since $q \in W^{1,2}$ we have that $\dot q \in L^2$, hence the following integral (obtained by integrating the restriction of e_q)

$$\int_0^1 \tfrac{1}{|q_t|}\,dt = \tfrac{1}{2}\|\dot q\|^2 - \int_0^1 e_q(t)\,dt$$

is finite since the function $e_q\colon \mathbb{S}^1 \to \mathbb{R}$ is continuous. □

In analogy with the non-collisional case (3.27), although $t \mapsto 1/q(t)^{\frac{1}{2}}$ is not defined at collision times, hence it is only a 'loop' up to finitely many points, integration still makes sense and leads to a strictly monotone increasing function. Indeed we still rescale time according to this 'loop', notation $1/\sqrt{q}$, by defining

$$t_{1/\sqrt{q}}\colon \mathbb{S}^1_t \to \mathbb{S}^1_t, \quad t \mapsto \frac{\int_0^t \frac{1}{|q_s|}\,dt}{\int_0^1 \frac{1}{|q_s|}\,ds} \tag{4.62}$$

for every $t \in \mathbb{R}$. It is well defined on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ since, by integral properties, the map $t_{1/\sqrt{q}}$ takes 0 to 0 and 1 to 1. Also $t_{1/\sqrt{q}}$ strictly monotone increases: the integrand is > 0, even ∞ at the finitely many collisions, and $\int_0^1 \frac{1}{|q_s|}\,ds < \infty$.

Lemma 4.22. *Suppose that $q \in \Lambda^\times_{\mathrm{coll}}\mathfrak{Q}$, notation $q_t = r_t e^{i\vartheta_t}$, has a collision at time $t = 0$, i.e. $q_0 = 0$. Then for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that*

$$(\tfrac{3}{2}\sqrt{2-\varepsilon})^{\frac{2}{3}} t^{\frac{2}{3}} \le r_t \le (\tfrac{3}{2}\sqrt{2+\varepsilon})^{\frac{2}{3}} t^{\frac{2}{3}}$$

for $t \in [-\delta_\varepsilon, \delta_\varepsilon]$.

Proof. By definition of the energy in polar coordinates $q_t = r_t e^{i\vartheta_t}$, $e^{i\vartheta_t} = \alpha_t$, we have the energy equation $\frac{1}{2}\dot r^2 + \frac{1}{2} r^2 \dot\vartheta^2 - \frac{1}{r} = e$.

Since $r_0 = 0$ and $r > 0$, there exists $\delta_0 > 0$ such that $\dot r_t \geq 0$ for every $t \in [0, \delta_0]$. Thus, solving for $\dot r$ we obtain the ODE

$$\dot r = \sqrt{\dot r^2} = \sqrt{2e + \tfrac{2}{r} - r^2 \dot\vartheta^2} = \sqrt{2e + \tfrac{2}{r} - |\beta|^2}.$$

As e and β are continuous, for every $\varepsilon \in (0, 2)$, we can further choose $\delta_\varepsilon > 0$ small enough such that

$$-\frac{\varepsilon}{r_t} \leq 2e_t - |\beta_t|^2 \leq \frac{\varepsilon}{r_t}$$

whenever $t \in [0, \delta_\varepsilon]$. So from the above inequality, on $[0, \delta_\varepsilon]$ we get the inequality

$$\sqrt{\tfrac{2-\varepsilon}{r}} \leq \dot r \leq \sqrt{\tfrac{2+\varepsilon}{r}}.$$

For $a > 0$ the ODE $\dot R = \frac{a}{\sqrt{R}}$ on $[0, \infty)$ with $R_0 = 0$ has the solution $R^a_t = (\frac{3}{2} a t)^{\frac{2}{3}}$. The theory of sub- and super-solutions for $R_t^{\sqrt{2-\varepsilon}}$ and $R_t^{\sqrt{2+\varepsilon}}$ tells that

$$(\tfrac{3}{2}\sqrt{2-\varepsilon})^{\frac{2}{3}} t^{\frac{2}{3}} = R_t^{\sqrt{2-\varepsilon}} \leq r_t \leq R_t^{\sqrt{2+\varepsilon}} = (\tfrac{3}{2}\sqrt{2+\varepsilon})^{\frac{2}{3}} t^{\frac{2}{3}}$$

whenever $t \in [0, \delta_\varepsilon]$; see e.g. [Tes12, Le. 1.2].

A similar estimate holds for negative times where $\dot r_t \leq 0$ so that, maybe after shrinking $\varepsilon > 0$, we can assume that the previous inequalities hold for $t \in [-\delta_\varepsilon, \delta_\varepsilon]$. This proves Lemma 4.22. □

Proposition 4.23. *Let $q \in \Lambda^\times_{\mathrm{coll}}\mathfrak{Q}$. Then $t_{1/\sqrt{q}} \colon \mathbb{S}^1_t \to \mathbb{S}^1_t$ is continuous.*

Proof of Proposition 4.23. Away from collisions, namely for $t \in \mathbb{S}^1 \setminus \mathcal{T}_q$, the map $t_{1/\sqrt{q}}$ is differentiable at t, hence continuous. Indeed the derivative is given by

$$\dot t_{1/\sqrt{q}}(t) = \frac{1}{|q_t| \cdot \|\frac{1}{q}\|_{L^1}}. \qquad \text{Thus } \tau'_{1/\sqrt{q}}(\tau) = \left|q \circ \tau_{1/\sqrt{q}}(\tau)\right| \cdot \|\tfrac{1}{q}\|_{L^1}. \tag{4.63}$$

It remains to discuss continuity of $t_{1/\sqrt{q}}$ at collision times. After shifting time we can assume without loss of generality that the collision occurs at time 0.

We next show continuity of $f(t) := \int_0^t \frac{1}{|q_s|}\, ds$ at zero from the right. This means that for every $\varepsilon > 0$ there exists $\mu_\varepsilon > 0$ such that $f(t) \leq \varepsilon$ whenever $t \in [0, \mu_\varepsilon]$.

By Lemma 4.22 there exists $\delta > 0$ such that $r_t \geq (\frac{3}{2})^{\frac{2}{3}} t^{\frac{2}{3}} \geq t^{\frac{2}{3}}$ whenever $t \in [0, \delta]$. Hence if $t \in [0, \delta]$ we estimate

$$f(t) = \int_0^t \frac{1}{r_s}\, ds \leq \int_0^t s^{-\frac{2}{3}}\, ds = 3 s^{\frac{1}{3}} \Big|_{s=0}^{s=t} = 3 t^{\frac{1}{3}}.$$

Choosing $\mu_\varepsilon = (\varepsilon/3)^3$ completes the proof of continuity from the right. Continuity from the left follows similarly. This proves Proposition 4.23. □

Corollary 4.24 (inverse)**.** *Let $q \in \Lambda^{\times}_{\mathrm{coll}}\mathfrak{Q}$. Then $t_{1/\sqrt{q}}$ in (4.62) has an inverse*

$$\tau_{1/\sqrt{q}} = {t_{1/\sqrt{q}}}^{-1} \colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t, \quad \tau \mapsto \tau_{1/\sqrt{q}}(\tau)$$

which is strictly monotone increasing, too.

Proof. As $t_{1/\sqrt{q}}$ is strictly monotone increasing, C^0 by Proposition 4.23, it admits an inverse with the same properties; see footnote to (3.23). □

Loops q: Speed $|z'_q|$ at collisions – equal non-zero speed

Lemma 4.25. *Pick a collision loop $q \in \Lambda^{\times}_{\mathrm{coll}}\mathfrak{Q}$. Set $z_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}$. Then*

$$\|\tfrac{1}{q}\|_{L^1} = \int_0^1 \frac{1}{|q_t|}\, dt = \frac{1}{\|z_q\|^2}. \tag{4.64}$$

Proof. Variable substitution $\sigma(s) = t_{1/\sqrt{q}}(s)$ and (4.63) for $t'_{1/\sqrt{q}}(s)$ yield

$$\|z_q\|^2 = \int_0^1 |q \circ \underbrace{\tau_{1/\sqrt{q}}(\sigma)}_{=s}|\, d\sigma = \int_{s(0)=0}^{s(1)=1} |q_s| \frac{ds}{|q_s| \cdot \|\frac{1}{q}\|_{L^1}} = \frac{1}{\|\frac{1}{q}\|_{L^1}}$$

and this proves Lemma 4.25. □

Lemma 4.26 (first derivative)**.** *Pick a collision loop $q \in \Lambda^{\times}_{\mathrm{coll}}\mathfrak{Q}$ and set $z_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}$. Then all collisions τ_* of z_q happen at equal speed*

$$\forall \tau_* \in z_q^{-1}(0)\colon \qquad \lim_{\tau \to \tau_*} \left|z'_q(\tau)\right| = \frac{1}{\sqrt{2}\|z_q\|^2} \overset{(4.64)}{=} \frac{1}{\sqrt{2}} \|\tfrac{1}{q}\|_{L^1}.$$

Proof. At any non-collision time τ, simplifying $f(t)g(t) =: (fg)_t$, we have

$$\begin{aligned} \left|z'_q(\tau)\right| &= \left| \frac{\dot q_{\tau_{1/\sqrt{q}}(\tau)} \cdot \dot\tau_{1/\sqrt{q}}(\tau)}{2 q^{\frac{1}{2}}_{\tau_{1/\sqrt{q}}(\tau)}} \right| \overset{2}{=} \left(\frac{\left|\sqrt{2e_q + \frac{2}{|q|}}\right| \cdot |q|}{2|q|^{\frac{1}{2}}} \right)_{\tau_{1/\sqrt{q}}(\tau)} \cdot \|\tfrac{1}{q}\|_{L^1} \\ &\overset{3}{=} \left(\frac{\sqrt{e_q\,|q| + 1}}{\sqrt{2}} \right)_{\tau_{1/\sqrt{q}}(\tau)} \cdot \frac{1}{\|z_q\|^2}. \end{aligned} \tag{4.65}$$

Equality 2, also 3, uses that $|\sqrt{\cdot}| = \sqrt{|\cdot|}$ on $\mathbb{C}$. Equality 2 also uses the energy identity (4.59) resolved for $|\dot q|$ and (4.63) for $\dot\tau_{1/\sqrt{q}}$. In equality 3 the common factor $|q|^{\frac{1}{2}}$ in nominator and denominator cancels and the remaining $|q|^{\frac{1}{2}}$ we multiply with the sum under the square root. We replaced $\|\frac{1}{q}\|_{L^1}$ by (4.64).

In the limit as time τ approximates a collision time τ_*, while the energy is bounded q approaches zero, hence

$$e_q \circ \tau_{1/\sqrt{q}}(\tau) \cdot \left|q \circ \tau_{1/\sqrt{q}}(\tau)\right| \overset{\tau \to \tau_*}{\longrightarrow} 0 \tag{4.66}$$

and this proves Lemma 4.26. □

Solutions q: Second derivative z_q'' vanishes at collisions

Lemma 4.27 (second derivative)**.** *If $q \in \operatorname{Coll}\mathcal{S}$ and $z_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}$, then*

$$\lim_{\tau \to \tau_*} z_q''(\tau) = 0$$

at any collision time τ_*.

Proof. Let τ be a non-collision time. We simplify notation $f(t)g(t) =: (fg)_t$. For z_q' use the first line in (4.65) to get equality one

$$\begin{aligned}
z_q''(\tau) &= \frac{d}{d\tau}\left(\frac{\dot q_{\tau_{1/\sqrt{q}}(\tau)} \cdot \dot\tau_{1/\sqrt{q}}(\tau)}{2q^{\frac{1}{2}}_{\tau_{1/\sqrt{q}}(\tau)}}\right) \stackrel{2}{=} \frac{1}{2}\frac{d}{d\tau}\left(\frac{\dot q\,|q|}{q^{\frac{1}{2}}}\right)_{\tau_{1/\sqrt{q}}(\tau)} \|\tfrac{1}{q}\|_{L^1} \\
&\stackrel{3}{=} \tfrac{1}{2}\|\tfrac{1}{q}\|_{L^1}\tfrac{d}{d\tau}\left(\dot q \circ \tau_{1/\sqrt{q}} \cdot (\bar q)^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}\right)_\tau \\
&\stackrel{4}{=} \tfrac{1}{2}\|\tfrac{1}{q}\|_{L^1}\Big(\ddot q_{\tau_{1/\sqrt{q}}(\tau)} \overbrace{|q_{\tau_{1/\sqrt{q}}(\tau)}| \cdot \|\tfrac{1}{q}\|_{L^1}}^{=\dot\tau_{1/\sqrt{q}}(\tau)} (\bar q)^{\frac{1}{2}}_{\tau_{1/\sqrt{q}}(\tau)} \\
&\qquad + \dot q_{\tau_{1/\sqrt{q}}(\tau)} \tfrac{1}{2}(\bar q)^{-\frac{1}{2}}_{\tau_{1/\sqrt{q}}(\tau)} \dot{\bar q}_{\tau_{1/\sqrt{q}}(\tau)} \overbrace{|q_{\tau_{1/\sqrt{q}}(\tau)}| \cdot \|\tfrac{1}{q}\|_{L^1}}^{=\dot\tau_{1/\sqrt{q}}(\tau)}\Big) \\
&\stackrel{5}{=} \tfrac{1}{2}\|\tfrac{1}{q}\|^2_{L^1}\Big(\underbrace{\ddot q}_{(2.8)} \underbrace{|q|\,(\bar q)^{\frac{1}{2}}}_{=\bar q q^{\frac{1}{2}}} + \underbrace{\tfrac{1}{2}|\dot q|^2}_{e_q + \frac{1}{|q|}}\, q^{\frac{1}{2}}\Big)_{\tau_{1/\sqrt{q}}(\tau)} \\
&\stackrel{6}{=} \tfrac{1}{2}\|\tfrac{1}{q}\|^2_{L^1}\Big(- \underbrace{B|_q}_{\text{bd. } \mathbb{S}^1 \times \operatorname{im} q} \underbrace{(j_0 \dot q)\, \bar q q^{\frac{1}{2}}}_{(4.68)} - \underbrace{\dot{\boldsymbol{A}}|_q}_{\text{bd. } \mathbb{S}^1 \times \operatorname{im} q} \underbrace{\bar q q^{\frac{1}{2}}}_{\to 0} \\
&\qquad \underbrace{- \tfrac{q}{|q|^3}\, \bar q q^{\frac{1}{2}} + \tfrac{1}{|q|} q^{\frac{1}{2}}}_{=0} + \underbrace{e_q q^{\frac{1}{2}}}_{\stackrel{\tau \to \tau_*}{\longrightarrow} 0}\Big)_{\tau_{1/\sqrt{q}}(\tau)} \\
&\stackrel{\tau \to \tau_*}{\longrightarrow} 0.
\end{aligned} \tag{4.67}$$

Equality 2 uses (4.63) for $\dot\tau_{1/\sqrt{q}}$. Equality 3 replaces the complex absolute value $|q|$ by $q^{1/2}(\bar q)^{1/2}$, then we cancel factor one with the denominator. Equality 4 is by the chain rule and (4.63) for $\dot\tau_{1/\sqrt{q}}$. Equality 5 just summarized in simplified notation. Equality 6 is the key step, here we bring in the energy function e_q from (4.59) and we use that q solves the classical ODE (2.8) replacing $\ddot q$ accordingly. As a consequence a marvelous elimination of the two singular potential terms takes place.

Both maps B and $\dot{\boldsymbol{A}}$ are continuous on $\mathbb{S}^1 \times \operatorname{im} q$, hence bounded. The product $e_q q^{\frac{1}{2}}$ tends to zero, because energy $e_q \colon \mathbb{S}^1 \to \mathbb{R}$ is continuous, hence bounded, and q collides with the origin, hence is zero, at collision times.

Furthermore, we profit from the additional factor $|\bar q| = |q|$ in $(j_0 \dot q)\bar q q^{\frac{1}{2}}$, as compared to (4.65), to obtain zero in the limit

$$\begin{aligned}
\left|(j_0\dot q)\bar q q^{\frac12}\right|_{\tau_{1/\sqrt{q}}(\tau)} &\overset{1}{=} \left(\left|\sqrt{2e_q+\tfrac{2}{|q|}}\right| |q|^{\frac12}\,|q|\right)_{\tau_{1/\sqrt{q}}(\tau)} \\
&\overset{2}{=} \sqrt{2}\underbrace{\left(\sqrt{e_q\,|q|+1}\right)_{\tau_{1/\sqrt{q}}(\tau)}}_{\overset{\tau\to\tau_*}{\longrightarrow}1,\text{ by (4.66)}}\underbrace{|q_{\tau_{1/\sqrt{q}}(\tau)}|}_{\overset{\tau\to\tau_*}{\longrightarrow}0} \qquad (4.68)\\
&\overset{\tau\to\tau_*}{\longrightarrow} 0.
\end{aligned}$$

Equality 1 uses that the rotation $j_0 \simeq i$ is an isometry, that $|\bar q| = |q|$, that $|\sqrt{\cdot}| = \sqrt{|\cdot|}$ on $\mathbb{C}$. It also uses the energy identity (4.59) resolved for $\dot q$. In equality 2 we multiply out. This concludes the proof of Lemma 4.27. □

Lemma 4.28 (C^2)**.** *Let $q \in \operatorname{Coll}\mathcal{S}$. Then $z_q := q^{\frac12}\circ\tau_{1/\sqrt{q}}$ is of class C^2.*

Proof. By Lemma 4.27, it suffices to show that the first derivative of z_q is continuous at collisions.

The map $\tau \mapsto |z_q'(\tau)|$ is continuous, by (4.65), and at each collision time we have $|z_q'(\tau_*)| = \frac{1}{\sqrt2}\|q^{-1}\|_{L^1} =: c_q > 0$, by Lemma 4.26. Hence there exists $\varepsilon_0 > 0$ such that for each collision time τ_* there is the bound $|z_q'(\tau_*+\varepsilon)| > c_q/2 > 0$ whenever $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$. In particular, there exists a well defined angle function $\phi\colon (-\varepsilon_0,\varepsilon_0) \to \mathbb{S}^1_{\mathbb{C}}$ such that

$$z_q'(\tau_*+\varepsilon) = |z_q'(\tau_*+\varepsilon)|\cdot\phi(\varepsilon)$$

whenever $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ and it remains to check that ϕ is continuous at zero. Since $z_q(\tau_*) = 0$, by Taylor's theorem, there exists $r_\varepsilon \in [0,1]$ such that $z_q(\tau_*+\varepsilon) = z_q'(\tau_*+r_\varepsilon\varepsilon)$. This and the previous displayed equation yield equality one

$$\begin{aligned}
\phi(r_\varepsilon\cdot\varepsilon) &= \frac{z_q(\tau_*+\varepsilon)}{|z_q'(\tau_*+r_\varepsilon\cdot\varepsilon)|} \\
&= \frac{q^{\frac12}\circ\tau_{1/\sqrt{q}}(\tau_*+\varepsilon)}{|z_q'(\tau_*+r_\varepsilon\cdot\varepsilon)|} \\
&\overset{3}{=} {\alpha_q}^{\frac12}\circ\tau_{1/\sqrt{q}}(\tau_*+\varepsilon)\cdot\frac{|q^{\frac12}\circ\tau_{1/\sqrt{q}}(\tau_*+\varepsilon)|}{|z_q'(\tau_*+r_\varepsilon\cdot\varepsilon)|} \\
&\overset{4}{=} {\alpha_q}^{\frac12}\circ\tau_{1/\sqrt{q}}(\tau_*+\varepsilon).
\end{aligned}$$

Equality 3 holds by definition (4.59) of α. Equality 4 uses the fact that ϕ and α take values in $\mathbb{S}^1_{\mathbb{C}} \subset \mathbb{C}$, hence the real quotient factor must be 1. Taking the limit $\varepsilon \to 0$ in the above equation and using that $\tau_{1/\sqrt{q}}$ is continuous by Corollary 4.24 and α_q is continuous by assumption, we deduce $\phi(0) = {\alpha_q}^{\frac12}\circ\tau_{1/\sqrt{q}}(\tau_*)$. Hence at collision times ϕ is continuous since α is. This proves Lemma 4.28. □

Lemma 4.29. *Let $q \in \Lambda^\times_{\rm coll}\mathfrak{Q}$. Then Lemma 4.10 applies to z_q and*

$$t_{z_q} \overset{1}{=} \tau_{z_q}^{-1} : \mathbb{S}^1_\tau \to \mathbb{S}^1_t, \quad \tau_{1/\sqrt{q}} \overset{2}{=} t_{z_q} : \mathbb{S}^1_\tau \to \mathbb{S}^1_t, \quad \tau_z \overset{3}{=} t_{1/\sqrt{q_z}} : \mathbb{S}^1_t \to \mathbb{S}^1_\tau. \tag{4.69}$$

Proof. Identity 1 holds by Lemma 4.10. Identity 2 holds by the computation (3.28). Identity 3 holds by the computation (3.30) with (3.29) replaced by (4.61) which applies since $z \in \operatorname{Crit}\mathcal{B} \subset \bar{\mathcal{L}}^\times_{\rm coll}\mathfrak{Z}$ by Lemma 4.17. $\square$

Image of $\mathcal{Z}$ lies in Crit$\mathcal{B}$

Lemma 4.30 (energy identity)**.** *Any classical collisional $q \in \operatorname{Coll}\mathcal{S}$ satisfies*

$$e_q(t) = \tfrac{1}{2}\|\dot q\|^2 - \|\tfrac{1}{q}\|_{L^1} - \int_0^1 s\langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot q_s\rangle_0 \, ds + \int_{s=t}^1 \langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot q_s\rangle_0 \, ds$$

at every time t and where at non-collision times $e_q(t) = \frac{1}{2}|\dot q_t|^2 - \frac{1}{|q_t|}$ by (4.59).

Proof. At a non-collisional time t we differentiate the energy

$$\begin{aligned}
\tfrac{d}{dt} e_q(t) = \tfrac{d}{dt}\left(\tfrac{1}{2}|\dot q_t|^2 - \tfrac{1}{|q_t|}\right) &= \langle \dot q_t, \ddot q_t + \tfrac{q_t}{|q_t|}\rangle_0 \\
&= \langle \dot q_t, -B_t|_{q_t} i\dot q_t - \dot{\boldsymbol{A}}_t|_{q_t}\rangle_0 \\
&= -\langle \dot q_t, \dot{\boldsymbol{A}}_t|_{q_t}\rangle_0
\end{aligned}$$

where equality two is by the classical ODE (2.8) and equality three by orthogonality $\dot q_t \perp i\dot q_t$. By definition of the L^2 and L^1 norms we get equality one

$$\begin{aligned}
\tfrac{1}{2}\|\dot q\|^2 - \|\tfrac{1}{q}\|_{L^1} &= \int_0^1 \left(\tfrac{1}{2}|\dot q_s|^2 - \tfrac{1}{|q_s|}\right) ds = \int_0^1 e_q(s)\, ds \\
&\overset{2}{=} \int_{t-1}^t 1 \cdot e_q(s)\, ds \\
&\overset{3}{=} s \cdot e_q(s)\big|_{t-1}^t + \int_{t-1}^t s\, \langle \dot q_s, \dot{\boldsymbol{A}}_s|_{q_s}\rangle_0\, ds \\
&\overset{4}{=} e_q(t) + \left(\int_{t-1}^0 + \int_0^1 - \int_t^1\right) s\, \langle \dot q_s, \dot{\boldsymbol{A}}_s|_{q_s}\rangle_0\, ds \\
&\overset{5}{=} e_q(t) + \int_t^1 (s-1)\langle \dot q_{s-1}, \dot{\boldsymbol{A}}_{s-1}|_{q_{s-1}}\rangle_0\, ds + \int_0^1 s\, \langle \dot q_s, \dot{\boldsymbol{A}}_s|_{q_s}\rangle_0\, ds \\
&\quad - \int_t^1 s\, \langle \dot q_s, \dot{\boldsymbol{A}}_s|_{q_s}\rangle_0\, ds.
\end{aligned}$$

Equality 2 is by periodicity of e_q.[13] Equality 3 is by integration by parts and the previous displayed identity for the energy derivative. Equality 4 uses again periodicity $e_q(t-1) = e_q(t)$ and additivity of the integral. Equality 5 shifts the integration interval by 1, so s becomes $s-1$. Then we use periodicity $q_{s-1} = q_s$ and, by (2.5), also $\dot{\boldsymbol{A}}_{s-1} = \dot{\boldsymbol{A}}_s$. Then the two grayed out summands cancel and we get equality 6 which proves Lemma 4.30. $\square$

[13] schematically $\int_0^1 e_s = \int_0^t e_s + \int_t^1 e_s = \int_0^t e_s + \int_{t-1}^0 e_{s+1} = \int_{t-1}^t e_s$ as $e_{s+1} = e_s$ is periodic

Proposition 4.31. *Let $q \in \operatorname{Coll}\mathcal{S}$. Then $z_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}} \in \operatorname{Crit}\mathcal{B}$.*

Proof. Pick $q \in \operatorname{Coll}\mathcal{S}$ and abbreviate $z = z_q := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}} \colon \mathbb{S}^1 \to \mathfrak{Z}$ which is C^2 by Lemma 4.28. We represented z'' at time τ via equality $(4.67)_6$ in terms of q and $\dot q$ at time $\tau_{1/\sqrt{q}}(\tau)$. While $q_{\tau_{1/\sqrt{q}}(\tau)} = z_\tau^2$, in order to express $\dot q$ in terms of z and z' observe first that

$$|q_{\tau_{1/\sqrt{q}}(\tau)}| = \bar z_\tau z_\tau, \qquad \tau'_{1/\sqrt{q}}(\tau) \overset{(4.63)}{=} |q_{\tau_{1/\sqrt{q}}(\tau)}| \cdot \|\tfrac{1}{q}\|_{L^1} = \frac{z_\tau \bar z_\tau}{\|z\|^2}.$$

Use identity two and the definition of $z = z_q$ to get equality 2 of what follows

$$z'_\tau \overset{(4.65)}{=} \frac{\dot q_{\tau_{1/\sqrt{q}}(\tau)} \cdot \tau'_{1/\sqrt{q}}(\tau)}{2 q^{1/2}_{\tau_{1/\sqrt{q}}(\tau)}} \overset{2}{=} \frac{\dot q_{\tau_{1/\sqrt{q}}(\tau)}}{2 z_\tau} \cdot \frac{z_\tau \bar z_\tau}{\|z\|^2} = \frac{\dot q_{\tau_{1/\sqrt{q}}(\tau)} \bar z_\tau}{2\|z\|^2}.$$

Equivalently

$$\dot q_{\tau_{1/\sqrt{q}}(\tau)} = 2\|z\|^2 \frac{z'_\tau}{\bar z_\tau}. \qquad \text{Thus } \tfrac{1}{2}|\dot q_{\tau_{1/\sqrt{q}}(\tau)}|^2 = 2\|z\|^4 \frac{|z'_\tau|^2}{|z_\tau|^2}. \tag{4.70}$$

Use the previous identity at time $t := \tau_{1/\sqrt{q}}(\tau)$, in which case $\tau = (\tau_{1/\sqrt{q}})^{-1}(t) = (t_z)^{-1}(t) = \tau_z(t)$ by (4.69), in equality 2 of what follows

$$\begin{aligned}\tfrac{1}{2}\|\dot q\|^2 :&= \int_0^1 \tfrac{1}{2}|\dot q_t|^2\, dt \overset{2}{=} 2\|z\|^4 \int_0^1 \frac{|z'_{\tau_z(t)}|^2}{|z_{\tau_z(t)}|^2}\, dt \overset{3}{=} 2\|z\|^4 \int_0^1 \frac{|z'_\tau|^2}{|z_\tau|^2} \frac{|z_\tau|^2}{\|z\|^2}\, d\tau \\ &\overset{4}{=} 2\|z\|^2 \|z'\|^2.\end{aligned} \tag{4.71}$$

Equality 3 is by variable substitution $t = t_z(\tau)$, to $dt = t'_z(\tau)\, d\tau$, apply (4.52). By (4.67) we obtain equality one

$$\begin{aligned}
z''_\tau &\overset{1}{=} \tfrac{1}{2}\|\tfrac{1}{q}\|^2_{L^1} \left(-B|_q(j_0\dot q)\bar q q^{\frac{1}{2}} - \dot{\boldsymbol{A}}|_q \bar q q^{\frac{1}{2}} + e_q q^{\frac{1}{2}}\right)_{\tau_{1/\sqrt{q}}(\tau)} \\
&\overset{2}{=} \frac{z_\tau}{2\|z\|^4} \Bigg(-\operatorname{rot} \boldsymbol{A}|_q\, (j_0\dot q)\bar q - \dot{\boldsymbol{A}}|_q \bar q + \tfrac{1}{2}\|\dot q\|^2 - \|\tfrac{1}{q}\|_{L^1} \\
&\qquad - \int_0^1 s\langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot q_s\rangle_0\, ds + \int_{s=\tau_{1/\sqrt{q}}(\tau)}^1 \langle \dot{\boldsymbol{A}}_s|_{q_s}, \dot q_s\rangle_0\, ds\Bigg)_{\tau_{1/\sqrt{q}}(\tau) \overset{(4.69)}{=} t_z(\tau)} \\
&\overset{3}{=} \frac{-z_\tau \operatorname{rot} \boldsymbol{a}_{t_z(\tau)}|_{z_\tau}}{8\bar z_\tau z_\tau \|z\|^4} \left(2\|z\|^2 \frac{i z'_\tau}{\bar z_\tau}\right) \bar z_\tau^2 - \frac{z_\tau \bar z_\tau \bar z_\tau}{2\|z\|^4} \frac{\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}}{2\bar z_\tau} + z_\tau \frac{\|z'\|^2}{\|z\|^2} - \frac{z_\tau}{2\|z\|^6} \\
&\qquad - \frac{z_\tau}{2\|z\|^4} \int_0^1 \overbrace{\frac{\int_0^\sigma |z_\rho|^2\, d\rho}{\|z\|^2}}^{= t_z(\sigma) = s} \operatorname{Re}\left(\overline{\frac{1}{2\bar z_\sigma} \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}} \cdot 2\|z\|^2 \frac{z'_\sigma}{z_\sigma}\right) \overbrace{\frac{|z_\sigma|^2}{\|z\|^2}}^{t'_z(\sigma)}\, d\sigma \\
&\qquad + \frac{z_\tau}{2\|z\|^4} \int_{\sigma = t_z^{-1} \circ \tau_{1/\sqrt{q}}(\tau) = \tau}^1 \operatorname{Re}\left(\overline{\frac{1}{2\bar z_\sigma} \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}} \cdot 2\|z\|^2 \frac{z'_\sigma}{z_\sigma}\right) \frac{|z_\sigma|^2}{\|z\|^2}\, d\sigma
\end{aligned}$$

$$\overset{4}{=} -\frac{\operatorname{rot} \boldsymbol{a}_{t_z(\tau)}|_{z_\tau}}{4\|z\|^2}\, i z'_\tau - \frac{|z_\tau|^2}{4\|z\|^4} \cdot \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} + z_\tau \frac{\|z'\|^2}{\|z\|^2} - \frac{z_\tau}{2\|z\|^6}$$
$$- \frac{z_\tau}{2\|z\|^6} \int_0^1 \textstyle\int_0^\sigma |z_\rho|^2\, d\rho \cdot \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma$$
$$+ \frac{z_\tau}{2\|z\|^4} \int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma.$$

But this is the regularized critical point DDE (4.46) for z. To conclude the proof of Proposition 4.31 it remains to explain equalities 2, 3, and 4.

Equality 2 uses (4.64), that $B = \operatorname{rot} \boldsymbol{A}$ by definition (2.8), that $q^{\frac{1}{2}}$ rescaled is z at τ, then we substitute the energy e_q according to Lemma 4.30.
Equality 3 uses that $\tau_{1/\sqrt{q}}(\tau) = t_z(\tau)$ according to (4.69). Summand one: we replace $\operatorname{rot} \boldsymbol{A}$ according to (4.36) and $\dot{q}$ according to $(4.70)_1$, for j_0 we write i, and $\bar{q}$ rescaled is $\bar{z}_\tau^2$. Summand two: replace $\dot{\boldsymbol{A}}$ according to the dotted version of (4.38) and $\bar{q}$ rescaled by $\bar{z}^2$ at τ. Summand three: use (4.71). Summand four: use (4.64). Summands five and six: variable substitution $s = t_z(\sigma)$, replace $\dot{\boldsymbol{A}}$ using the dotted version of (4.38), rewrite the inner product in the $\mathbb{R}^2$ picture in terms of the real part (3.10) in the $\mathbb{C}$ picture, replace $\dot{q}_s$ for $s = t_z(\sigma) = \tau_{1/\sqrt{q}}(\sigma)$ according to $(4.70)_1$.
Now a lot of factors annulate in pairs and (in equality 4) we go back to the $\mathbb{R}^2$ picture and inner product. This proves the critical point equation (4.46) for z.,

Since $z \in C^2$, bootstrapping (4.46) we see that z is smooth. This proves Proposition 4.31. □

4.5.5 Proof of Main Theorem 4.15 (bijection)

Pick $z \in \operatorname{Crit} \mathcal{B}$ and $q \in \operatorname{Coll} \mathcal{S}$. Then with the definitions $q_z := \mathcal{Q}(z) := z^2 \circ \tau_z$ and $z_q := \mathcal{Z}(q) := q^{\frac{1}{2}} \circ \tau_{1/\sqrt{q}}$ we obtain

$$\mathcal{Q}(\mathcal{Z}(q)) = z_q^2 \circ \tau_{z_q} = q \circ \tau_{1/\sqrt{q}} \circ \tau_{z_q}$$

and

$$\mathcal{Z}(\mathcal{Q}(z)) = q_z^{\frac{1}{2}} \circ \tau_{1/\sqrt{q_z}} = z \circ \tau_z \circ \tau_{1/\sqrt{q_z}}.$$

This yields the following two equivalences

$$\mathcal{Q} \circ \mathcal{Z} = \mathrm{id} \quad \Leftrightarrow \quad \mathrm{id} = \tau_{1/\sqrt{q}} \circ \tau_{z_q} \overset{(4.69)_2}{=} t_{z_q} \circ \tau_{z_q} (\overset{3}{=} \mathrm{id}),$$
$$\mathcal{Z} \circ \mathcal{Q} = \mathrm{id} \quad \Leftrightarrow \quad \mathrm{id} = \tau_z \circ \tau_{1/\sqrt{q_z}} \overset{(4.69)_3}{=} t_{1/\sqrt{q_z}} \circ \tau_{1/\sqrt{q_z}} (\overset{3}{=} \mathrm{id}).$$

The right hand side of the first equivalence holds true (eq. 3) by $(4.69)_1$. The right hand side of the second equivalence holds true (eq. 3) by Corollary 4.24 for q_z which indeed lies in $\Lambda^\times_{\mathrm{coll}} \mathfrak{Q}$ by Proposition 4.18. This proves Theorem 4.15.

5 Non-local Hamiltonian mechanics – periodic

Throughout Section 5 we specialize to periodic 1-forms

$$\theta = \{\theta_t\}_{t \in \mathbb{S}^1}, \quad \theta_{t+1} = \theta_t, \quad f = 0.$$

The reason is that in this case there exists a non-local magnetic potential, more precisely a magnetic 1-form Θ on loop space, which induces the magnetic function $\mathcal{M}$, see (4.1), and whose exterior derivative, since it is automatically closed, induces a twisted symplectic form Ω_Θ on the cotangent bundle of loop space.

5.1 Cotangent bundle of twisted loop space

Considering twisted loop space allows for defining square roots, see Remark 3.2. The open subsets $\mathfrak{Z}^\times \subset \mathbb{C} \setminus \{0\}$ and $\mathfrak{Z} = \mathfrak{Z}^\times \cup \{0\}$ are defined by (3.11).

Definition 5.1 (cotangent bundle of regularization loop space $\bar{\mathcal{L}}^\times \mathfrak{Z}$**).**
Both cotangent bundles

$$T^*\mathcal{L}_\pm \mathfrak{Z} := \{(z, \eta) \in C^\infty(\mathbb{R}, \mathfrak{Z} \times \mathbb{C}) \mid \forall \tau \in \mathbb{R}\colon\ (z_{\tau+1}, \eta_{\tau+1}) = \pm (z_\tau, \eta_\tau)\},$$

are invariant under the I-induced, see (3.15), **sign involution**

$$T^*I = -\mathrm{Id}_2 \colon T^*\mathcal{L}_\pm \mathfrak{Z} \to T^*\mathcal{L}_\pm \mathfrak{Z}, \quad (z, \eta) \mapsto -(z, \eta), \qquad \mathrm{Id}_k := \mathrm{Id}_{\mathbb{C}^k},$$

because the target $\mathfrak{Z} \times \mathbb{C}$ is, see (3.12). The intersection of both cotangent bundles consists of the zero element $(0, 0)$, the unique fixed point of T^*I, in symbols $T^*\mathcal{L}_+ \mathfrak{Z} \cap T^*\mathcal{L}_- \mathfrak{Z} = \{(0, 0)\} = \operatorname{Fix} T^*I$. Taking away the fiber over the zero loop we get the two cotangent bundles

$$T^*\mathcal{L}^\times_\pm \mathfrak{Z} := \{(z, \eta) \in T^*\mathcal{L}_\pm \mathfrak{Z} \colon \|z\| \neq 0\}, \qquad T^*\mathcal{L}^\times_+ \mathfrak{Z} \cap T^*\mathcal{L}^\times_- \mathfrak{Z} = \emptyset.$$

To put it differently

$$T^*\mathcal{L}^\times_+ \mathfrak{Z} = \mathcal{L}^\times_+ \mathfrak{Z} \times \mathcal{L}_+ \mathbb{C}, \qquad T^*\mathcal{L}^\times_- \mathfrak{Z} = \mathcal{L}^\times_- \mathfrak{Z} \times \mathcal{L}_- \mathbb{C}.$$

These are disjoint and the sign involution T^*I acts freely on each one of them, so on their union. The **base point projection**, defined by

$$\pi \colon T^*\mathcal{L}^\times_\pm \mathfrak{Z} \to \mathcal{L}^\times_\pm \mathfrak{Z}, \quad (z, \eta) \mapsto z, \qquad \pi \circ T^*I = I \circ \pi,$$

is equivariant with respect to the sign involutions T^*I and I in (3.15). We introduce the two quotient spaces (indicated by a bar)

$$T^*\bar{\mathcal{L}}^\times_\pm \mathfrak{Z} := \frac{\mathcal{L}^\times_\pm \mathfrak{Z} \times \mathcal{L}_\pm \mathbb{C}}{T^*I}, \qquad T^*\bar{\mathcal{L}}^\times \mathfrak{Z} := T^*\bar{\mathcal{L}}^\times_+ \mathfrak{Z} \cup T^*\bar{\mathcal{L}}^\times_- \mathfrak{Z}. \tag{5.72}$$

For $\bar{\mathcal{L}}^\times \mathfrak{Z}$ see (3.16). The elements of quotient space $T^*\bar{\mathcal{L}}^\times \mathfrak{Z}$ are still denoted by (z, η) keeping in mind that each element has two representatives $\pm(z, \eta)$ and our constructions must be independent of choosing (z, η) or $-(z, \eta)$. The base point projection descends to quotient spaces, still denoted by

$$\pi \colon T^*\bar{\mathcal{L}}^\times \mathfrak{Z} \to \bar{\mathcal{L}}^\times \mathfrak{Z}, \quad (z, \eta) \mapsto z. \tag{5.73}$$

This concludes Definition 5.1.

Remark 5.2 (tangent spaces)**.** At "Upsilon" $\Upsilon = (z, \eta)$ the tangent spaces

$$T_\Upsilon T^* \mathcal{L}^\times_\pm \mathfrak{Z} = \mathcal{L}_\pm \mathbb{C}^2 := \{\hat{\Upsilon} = (\hat{z}, \hat{\eta}) \in C^\infty(\mathbb{R}, \mathbb{C}^2) \mid \forall \tau \in \mathbb{R} \colon \hat{\Upsilon}_{\tau+1} = \pm \hat{\Upsilon}_\tau\}$$

are disjoint and invariant under sign involution $dT^*I|_\Upsilon = -\mathrm{Id}$ which acts freely on each of them. We introduce the two quotient spaces (indicated by a bar)

$$T_\Upsilon T^* \bar{\mathcal{L}}^\times_\pm \mathfrak{Z} := \bar{\mathcal{L}}_\pm \mathbb{C}^2 = \frac{\mathcal{L}_\pm \mathbb{C}^2}{-\mathrm{Id}}, \qquad T_\Upsilon T^* \bar{\mathcal{L}}^\times \mathfrak{Z} := T_\Upsilon T^* \bar{\mathcal{L}}^\times_+ \mathfrak{Z} \cup T_\Upsilon T^* \bar{\mathcal{L}}^\times_- \mathfrak{Z}.$$

The elements of quotient space $T_\Upsilon T^* \bar{\mathcal{L}}^\times \mathfrak{Z} = \bar{\mathcal{L}}_+ \mathbb{C}^2 \cup \bar{\mathcal{L}}_- \mathbb{C}^2$ are still denoted by $\hat{\Upsilon} = (\hat{z}, \hat{\eta})$ keeping in mind that each element has two representatives $\pm\hat{\Upsilon}$ and our constructions must be independent of choosing $\hat{\Upsilon}$ or $-\hat{\Upsilon}$.

The linearization of the projection $\pi(z, \eta) = z$ in (5.73) acts by

$$d\pi|_{(z,\eta)} \colon T_{(z,\eta)} T^* \bar{\mathcal{L}}^\times \mathfrak{Z} \to T_z \bar{\mathcal{L}}^\times \mathfrak{Z}, \qquad (\hat{z}, \hat{\eta}) \mapsto \hat{z}. \tag{5.74}$$

Remark 5.3 (sign involution on tangent bundle)**.** The tangent map of $T^*I = -\mathrm{Id}_2 \colon T^* \bar{\mathcal{L}}^\times \mathfrak{Z} \to T^* \bar{\mathcal{L}}^\times \mathfrak{Z}$ acts on the tangent bundle $TT^* \bar{\mathcal{L}}^\times \mathfrak{Z}$ as $TT^*I = -\mathrm{Id}_4$.

5.1.1 Canonical vector field and induced flow

The **canonical vector field** is generated by time derivative

$$\mathcal{V} \colon T^* \bar{\mathcal{L}}^\times \mathfrak{Z} \to TT^* \bar{\mathcal{L}}^\times \mathfrak{Z}, \quad \Upsilon \mapsto (\Upsilon, \Upsilon'). \tag{5.75}$$

For the principal part we use the same notation $\mathcal{V}(\Upsilon) = \Upsilon'$. Note that $\mathcal{V}$ is well defined on quotients, indeed $\mathcal{V}(-\Upsilon) = (-\Upsilon, (-\Upsilon)') = -(\Upsilon, \Upsilon') = -\mathcal{V}(\Upsilon)$.

Remark 5.4 (induced flows and Lie derivative)**.** The flow induced by $\mathcal{V} = \partial_\tau$ on $T^* \bar{\mathcal{L}}^\times \mathfrak{Z} \ni \Upsilon$ is time shift $\Phi^r_\mathcal{V} \Upsilon = r_* \Upsilon := \Upsilon(\cdot + r)$ whenever $\tau \in \mathbb{R}$. Given $\hat{\Upsilon} \in T_\Upsilon T^* \bar{\mathcal{L}}^\times \mathfrak{Z}$, analogous as for the flow $\phi^r \nu$ on $\bar{\mathcal{L}}^\times \mathfrak{Z}$ in (3.20), one shows that $\left(d\Phi^r_\mathcal{V}|_z \hat{\Upsilon}\right)_\tau = \hat{\Upsilon}_{\tau+r}$. The base point projection (5.73) relates the two flows

$$\pi \circ \Phi^r_\mathcal{V} = \phi^r_\nu \circ \pi. \tag{5.76}$$

Indeed $\pi \circ \Phi^r_\mathcal{V}(z_\tau, \eta_\tau) = \pi(z_{\tau+r}, \eta_{\tau+r}) = z_{\tau+r} = \phi^r_\nu z_\tau = \phi^r_\nu \circ \pi(z_\tau, \eta_\tau)$.

Pull-back under base point projection intertwines the two **Lie derivatives**

$$\begin{aligned} L_\mathcal{V} \pi^* &:= \tfrac{d}{dr}\Big|_{r=0} (\Phi^r_\mathcal{V})^* \pi^* \\ &= \pi^* \tfrac{d}{dr}\Big|_{r=0} (\phi^r_\nu)^* \\ &=: \pi^* L_\nu. \end{aligned} \tag{5.77}$$

Here the first and the last equality are by definition of the Lie derivative. Equality two is by (5.76) and *-functoriality $(\Phi^r_\mathcal{V})^* \pi^* = (\pi \circ \Phi^r_\mathcal{V})^* = (\phi^r_\nu \circ \pi)^* = \pi^* (\phi^r_\mathcal{V})^*$ and since π^* does not depend on r. This concludes Remark 5.4.

5.1.2 Canonical 1-form Λ and symplectic form Ω

Let $\lambda = \eta\, dz$ be the canonical 1-form on the cotangent bundle of $\mathfrak{Z}^\times \subset \mathbb{C}$. The **canonical 1-form** on $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ is defined by assigning to an element $\hat\Upsilon = (\hat z, \hat\eta)$ of the tangent space to $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ at a point $\Upsilon = (z,\eta)$ the real value

$$\Lambda_\Upsilon \hat\Upsilon = \Lambda_{(z,\eta)}(\hat z,\hat\eta) := \int_0^1 \eta_\tau(\hat z_\tau)\, d\tau = \langle \eta, \hat z\rangle\,, \quad \Lambda = \int_0^1 \lambda\, d\tau,$$

cf Definition A.1. Note that Λ is invariant under sign involution. Indeed

$$(T^*I)^*\Lambda\,(\Upsilon,\hat\Upsilon) = \Lambda_{T^*I\,\Upsilon} TT^*I\,\hat\Upsilon = \Lambda_{-\Upsilon}(-\hat\Upsilon) = \langle -\eta, -\hat z\rangle = \langle \eta,\hat z\rangle = \Lambda(\Upsilon,\hat\Upsilon).$$

We evaluate Λ on the canonical vector field $\mathcal{V}$ along $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$, see (3.19). At any point $\Upsilon = (z,\eta) \in T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$, for later use in (5.84), we compute

$$(i_\mathcal{V}\Lambda)_\Upsilon = \Lambda_\Upsilon \Upsilon' = \int_0^1 \eta_\tau(z'_\tau)\, d\tau = \langle \eta, z'\rangle\,.$$

Definition 5.5 (canonical symplectic form)**.** At a point $\Upsilon = (z,\eta) \in T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ and for tangent vectors $\hat\Upsilon^j = (\hat z^j, \hat\eta^j) \in T_\Upsilon T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ for $j = 1,2$ we define and compute the **canonical symplectic form Ω** on $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ as follows

$$\begin{aligned}\Omega_\Upsilon(\hat\Upsilon^1,\hat\Upsilon^2) :&= d\Lambda_\Upsilon(\hat\Upsilon^1,\hat\Upsilon^2) \overset{2}{=} \int_0^1 d\lambda_{(z,\eta)}(\hat\Upsilon^1_\tau,\hat\Upsilon^2_\tau)\, d\tau \\ &= \int_0^1 \left(\langle \hat\eta^1_\tau, \hat z^2_\tau\rangle_0 - \langle \hat\eta^2_\tau, \hat z^1_\tau\rangle_0\right) d\tau = \langle \hat\eta^1, \hat z^2\rangle - \langle \hat\eta^2,\hat z^1\rangle.\end{aligned} \tag{5.78}$$

Here equality 2 uses (A.88), and $\langle\cdot,\cdot\rangle$ is the L^2-inner product, $d\lambda = d\eta\wedge dz$, and $\langle\cdot,\cdot\rangle_0$ is the Euclidean inner product (3.10) on $\mathbb{C}\simeq\mathbb{R}^2$. Similarly as for Λ one checks that $\Omega_{\rm can}$ is invariant under sign involution, namely $(T^*I)^*\Omega_{\rm can} = \Omega_{\rm can}$.

5.1.3 Exact twisted symplectic form Ω_Θ

Definition 5.6 (non-local magnetic 1-form)**.** Let $\vartheta = \varsigma^*\theta$ be a periodic 1-form on $\mathfrak{Z}$ as in (3.11). Define the reparametrization $t_z\colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$ by formula (4.52). We get a 1-form on loop space $\bar{\mathcal{L}}^\times\mathfrak{Z}$, cf (C.91), defined for $(z,\xi)\in T\bar{\mathcal{L}}^\times\mathfrak{Z}$ by

$$\Theta_z\xi = \Theta_z^{\vartheta,T}\xi := \int_{\mathbb{S}^1} \vartheta_{t_z}|_z\xi := \int_0^1 \vartheta_{t_z(\tau)}|_{z_\tau}\xi_\tau\, d\tau. \tag{5.79}$$

Pull-back Θ by the base point projection π in (5.73), then insert the canonical vector field $\mathcal{V}$ from (5.75), see also ν in (3.19), to recover the magnetic functional $\mathcal{M}$ in (4.33): At $\Upsilon = (z,\eta)\in T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$, for use in (5.84), we compute

$$(i_\mathcal{V}\pi^*\Theta)_\Upsilon = \Theta_{\pi(\Upsilon)} d\pi|_\Upsilon \Upsilon' \overset{(5.74)}{=} \Theta_z z' = i_\nu\Theta_z \overset{f=0}{=} \mathcal{M}(z).$$

Lemma 5.7 (twisted non-local symplectic form)**.** *For $d\Theta$ in (C.99) the 2-forms*

$$\Omega_\Theta := \Omega + \pi^* d\Theta = d\left(\Lambda + \pi^*\Theta\right), \qquad \Omega := d\Lambda$$

are weak[14] *symplectic forms on $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ where d is defined by (A.88).*

Proof. Since $dd = 0$, cf. Remark A.3, Corollary A.6 applies. ☐

5.2 Hamiltonian vector field $\mathcal{X}_{\mathcal{H}}^{\Omega_\Theta}$ and Lorentz force $\mathcal{Z}_\Theta$

Definition 5.8. On the cotangent bundle $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$, see (5.72), the **mechanic** (kinetic plus potential energy) **Hamiltonian** is defined by

$$\begin{aligned} \mathcal{H}\colon T^*\bar{\mathcal{L}}^\times\mathfrak{Z} &\to \mathbb{R} \qquad\qquad \langle\cdot,\cdot\rangle^z := \tfrac{1}{4\|z\|^2}\langle\cdot,\cdot\rangle \\ (z,\eta) &\mapsto \tfrac{1}{2}\langle\eta,\eta\rangle^z - \tfrac{1}{\|z\|^2} = \underbrace{\tfrac{\|\eta\|^2}{8\|z\|^2}}_{\mathcal{K}^*(z,\eta)} - \underbrace{\tfrac{1}{\|z\|^2}}_{\mathcal{U}^*(z)}. \end{aligned}$$

The non-local **Hamiltonian vector field** is determined and abbreviated by

$$d\mathcal{H} = \Omega_\Theta(\cdot,\mathcal{X}_\Theta) = \Omega(\cdot,\mathcal{X}_\Theta) + d\Theta(d\pi\cdot,\mathcal{X}_{\mathcal{H}}^{\Omega_\Theta}), \qquad \mathcal{X}_\Theta = \mathcal{X}_{\mathcal{H}}^{\Omega_\Theta}.$$

This guarantees uniqueness, but not necessarily existence since we only have a weak symplectic form; cf. Remark A.5. The next lemma guarantees existence.

Lemma 5.9 (Non-local Hamiltonian vector field)**.** *At $(z,\eta) \in T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ we have*

$$\mathcal{X}_\Theta(z,\eta) = \tfrac{1}{4\|z\|^2}\begin{pmatrix}\eta \\ \frac{\|\eta\|^2-8}{\|z\|^2}\,z\end{pmatrix} + \tfrac{1}{4\|z\|^2}\begin{pmatrix}0\\ \mathcal{Z}_z\eta\end{pmatrix}. \tag{5.80}$$

*The **non-local Lorentz force** $\mathcal{Z}_z \in \mathcal{L}(T_z\bar{\mathcal{L}}^\times\mathfrak{Z})$ is uniquely determined*[15] *by*

$$\langle\mathcal{Z}_z\eta,\cdot\rangle = -d\Theta_z(\eta,\cdot). \tag{5.81}$$

Proof. The Hamiltonian vector field of $\mathcal{H}$ for the untwisted symplectic form Ω is at a point $\Upsilon = (z,\eta)$ given by

$$\begin{cases} \mathcal{X}_0^1 = \partial_\eta\mathcal{H} &= \frac{1}{4\|z\|^2}\,\eta \\ \mathcal{X}_0^2 = -\partial_z\mathcal{H} &= \frac{\|\eta\|^2}{4\|z\|^4}\,z - \frac{2}{\|z\|^4}\,z \end{cases}, \qquad \mathcal{X}_0(z,\eta) = \tfrac{1}{4\|z\|^2}\begin{pmatrix}\eta\\ \frac{\|\eta\|^2}{\|z\|^2}\,z - \frac{8}{\|z\|^2}\,z\end{pmatrix}.$$

Pick $\mathcal{Y} \in T_\Upsilon T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$, by (5.78) and then by definition of $\mathcal{X}_0$ and of $\mathcal{X}_\Theta$, we get

$$\begin{aligned} \langle\mathcal{X}_0^2,\mathcal{Y}^1\rangle - \langle\mathcal{X}_0^1,\mathcal{Y}^2\rangle &= i_{\mathcal{Y}} i_{\mathcal{X}_0}\Omega \\ &= -i_{\mathcal{Y}} d\mathcal{H} \\ &= i_{\mathcal{Y}} i_{\mathcal{X}_\Theta}\Omega_\Theta \\ &= \Omega(\mathcal{X}_\Theta,\mathcal{Y}) + d\Theta(\pi_*\mathcal{X}_\Theta,\pi_*\mathcal{Y}) \\ &= \langle\mathcal{X}_\Theta^2,\mathcal{Y}^1\rangle - \langle\mathcal{X}_\Theta^1,\mathcal{Y}^2\rangle + d\Theta(\mathcal{X}_\Theta^1,\mathcal{Y}^1). \end{aligned} \tag{5.82}$$

[14] see Remark A.5

[15] Formula (C.99) shows that $d\Theta$ is a continuous bilinear form on L^2, hence by the Riesz representation theorem it determines an isomorphism.

The last but one identity uses $\Omega_\Theta := \Omega + \pi^* d\Theta$, the last one (5.78) and the linearized projection $\pi_* = d\pi$ in (5.74). For vectors of the form $\mathcal{Y} = (0, \mathcal{Y}^2)$ the displayed formula yields that $-\langle \mathcal{X}^1_0, \mathcal{Y}^2\rangle = -\langle \mathcal{X}^1_\Theta, \mathcal{Y}^2\rangle$ for every $\mathcal{Y}^2$. Thus $\mathcal{X}^1_\Theta = \mathcal{X}^1_0 = \frac{\eta}{4\|z\|^2}$, so in the displayed formula two terms cancel and it reduces to

$$\underline{\langle \mathcal{X}^2_\Theta - \mathcal{X}^2_0, \mathcal{Y}^1\rangle = -d\Theta(\mathcal{X}^1_\Theta, \mathcal{Y}^1)} = -d\Theta(\tfrac{\eta}{4\|z\|^2}, \mathcal{Y}^1) \overset{(5.81)}{=} \langle \mathcal{Z}_z \tfrac{\eta}{4\|z\|^2}, \mathcal{Y}^1\rangle$$

for every $\mathcal{Y}^1$. Thus the difference is a vertical vector field, the Lorentz force

$$(\mathcal{X}_\Theta - \mathcal{X}_0)|_{(z,\eta)} = \begin{pmatrix} \mathcal{X}^1_\Theta - \mathcal{X}^1_0 \\ \mathcal{X}^2_\Theta - \mathcal{X}^2_0 \end{pmatrix} |_{(z,\eta)} = \tfrac{1}{4\|z\|^2} \begin{pmatrix} 0 \\ \mathcal{Z}_z \eta \end{pmatrix}.$$

In (C.99) we calculate $d\Theta$. The proof of Lemma 5.9 is complete. □

5.3 Euler vector field $\mathcal{Y}^{\Omega_\Theta}_{\mathcal{V}}$

Let Θ be defined by (5.79). The base-point projection π intertwines the upstair and the downstair Lie derivative by (5.77).

Definition 5.10. The **non-local Euler vector field** $\mathcal{Y}_\Theta$ is determined by

$$i_{\mathcal{Y}_\Theta} \Omega_\Theta = \pi^* L_\nu \Theta \quad \big(= L_{\mathcal{V}} \pi^* \Theta\big).$$

Lemma 5.11. *At $(z, \eta) \in T^* \bar{\mathcal{L}}^\times \mathfrak{Z}$ the Euler vector field is time-wise given by*

$$\mathcal{Y}_\Theta|_z(\tau) = \tfrac{1}{4\|z\|^2} \begin{pmatrix} 0 \\ -4\,|z_0|^2\,\dot\vartheta_{t_z(\tau)}|_{z_\tau} \end{pmatrix}. \tag{5.83}$$

Proof of Lemma 5.11. Pick $\mathcal{X} = (\mathcal{X}^1, \mathcal{X}^2) \in T_{(z,\eta)} T^* \bar{\mathcal{L}}^\times \mathfrak{Z}$, then by definition of $\mathcal{Y}_\Theta$ and similar to the last two lines of (5.82), we obtain

$$\begin{aligned} i_{\mathcal{X}} L_{\mathcal{V}} \pi^* \Theta &= i_{\mathcal{X}} i_{\mathcal{Y}_\Theta} \Omega_\Theta \\ &= \Omega(\mathcal{Y}_\Theta, \mathcal{X}) + d\Theta(\pi_* \mathcal{Y}_\Theta, \pi_* \mathcal{X}) \\ &= \langle \mathcal{Y}^2_\Theta, \mathcal{X}^1\rangle - \langle \mathcal{Y}^1_\Theta, \mathcal{X}^2\rangle + d\Theta(\mathcal{Y}^1_\Theta, \mathcal{X}^1). \end{aligned}$$

Now at a point $\Upsilon = (z, \eta)$ the left hand side evaluates to

$$\big(i_{\mathcal{X}} \underline{L_{\mathcal{V}} \pi^*} \Theta\big)_\Upsilon \overset{(5.77)}{=} \big(i_{\mathcal{X}} \underline{\pi^* L_\nu} \Theta\big)_\Upsilon \overset{3}{=} (L_\nu \Theta)_{\pi(\Upsilon)}\, d\pi|_\Upsilon \mathcal{X} \overset{(5.74)}{=} (L_\nu \Theta)_z \mathcal{X}^1$$

where equality 3 is by definition of pull-back. So the earlier computation yields

$$\langle \mathcal{Y}^2_\Theta, \mathcal{X}^1\rangle - \langle \mathcal{Y}^1_\Theta, \mathcal{X}^2\rangle + d\Theta(\mathcal{Y}^1_\Theta, \mathcal{X}^1) = (L_\nu \Theta|_z) \mathcal{X}^1.$$

For vectors of the form $\mathcal{X} = (0, \mathcal{X}^2)$ this implies $-\langle \mathcal{Y}^1_\Theta, \mathcal{X}^2\rangle = 0$ for every $\mathcal{X}^2$, thus $\mathcal{Y}^1_\Theta = 0$. Hence $\langle \mathcal{Y}^2_\Theta, \mathcal{X}^1\rangle = (L_\nu \Theta|_z) \mathcal{X}^1$, equivalently

$$\int_0^1 \langle \mathcal{Y}^2_\Theta|_{(z,\eta)}(\tau), \mathcal{X}^1|_{(z,\eta)}(\tau)\rangle_0 \, d\tau \overset{(C.98)}{=} -\int_0^1 \tfrac{|z_0|^2}{\|z\|^2} \dot\vartheta_{t_z(\tau)}|_{z_\tau} \mathcal{X}^1|_{(z,\eta)}(\tau)\, d\tau.$$

This proves Lemma 5.11. □

5.4 Action $\mathcal{A}_{\mathcal{H}}^{\Lambda_\Theta}$ and Euler-Hamilton equation

Definition 5.12. Motivated by [FW26b, §4], we define the **perturbed symplectic action functional** on $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ as follows[16] $\mathcal{A}=\mathcal{A}_{\mathcal{H}}^{\Lambda_\Theta}:=i_{\mathcal{V}}\left(\Lambda+\pi^*\Theta\right)-\mathcal{H}$. More precisely, the action is of the form

$$\begin{aligned}
&\mathcal{A}\colon T^*\bar{\mathcal{L}}^\times\mathfrak{Z}\to\mathbb{R}\\
&\Upsilon=(z,\eta)\mapsto\Lambda_\Upsilon\Upsilon'+\Theta_z z'-\mathcal{H}(z,\eta)\\
&\qquad\qquad=\langle\eta+\Theta_z,z'\rangle-\tfrac{\|\eta\|^2}{8\|z\|^2}+\tfrac{1}{\|z\|^2}=\mathcal{A}(z,\eta).
\end{aligned}\tag{5.84}$$

We compute $d\mathcal{A}$ using Cartan's formula in equation two

$$\begin{aligned}
d\mathcal{A}&=di_{\mathcal{V}}\left(\Lambda+\pi^*\Theta\right)-d\mathcal{H}\\
&=L_{\mathcal{V}}\left(\Lambda+\pi^*\Theta\right)-i_{\mathcal{V}}d\left(\Lambda+\pi^*\Theta\right)+i_{\mathcal{X}_\Theta}\Omega_\Theta\\
&=L_{\mathcal{V}}\Lambda+L_{\mathcal{V}}\pi^*\Theta-i_{\mathcal{V}}\Omega_\Theta+i_{\mathcal{X}_\Theta}\Omega_\Theta\\
&=L_{\mathcal{V}}\pi^*\Theta-i_{\mathcal{V}}\Omega_\Theta+i_{\mathcal{X}_\Theta}\Omega_\Theta\\
&=i_{\mathcal{Y}_\Theta}\Omega_\Theta-i_{\mathcal{V}}\Omega_\Theta+i_{\mathcal{X}_\Theta}\Omega_\Theta\\
&=i_{(\mathcal{X}_\Theta+\mathcal{Y}_\Theta-\mathcal{V})}\,\Omega_\Theta.
\end{aligned}$$

Equation two is by definition of $\mathcal{X}_\Theta$. Equation four uses that $L_{\mathcal{V}}\Lambda=0$ by rotation invariance of Λ. Equation five is Definition 5.10 of the Euler vector field. This, together with the formulas (5.80) for $\mathcal{X}_\Theta$ and (5.83) for $\mathcal{Y}_\Theta$, proves the first two displayed formulas in the next lemma.

Theorem 5.13 (critical points of $\mathcal{A}$)**.** *The critical points of the action functional $\mathcal{A}$ are the solutions of the* ***delay Euler-Hamilton equation***

$$\mathcal{V}=\mathcal{X}_\Theta+\mathcal{Y}_\Theta.$$

The delay Euler-Hamilton equation for $(z,\eta)\in T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$ is of the form

$$\begin{pmatrix}z'\\ \eta'\end{pmatrix}=\tfrac{1}{4\|z\|^2}\begin{pmatrix}\eta\\ \frac{\|\eta\|^2-8}{\|z\|^2}\,z\end{pmatrix}+\tfrac{1}{4\|z\|^2}\begin{pmatrix}0\\ \mathcal{Z}_z\eta-4\left|z_0\right|^2\dot\vartheta_{t_z(\tau)}|_{z_\tau}\end{pmatrix}.\tag{5.85}$$

Along the image of the canonical injection (1.1), i.e. $\eta=4\|z\|^2z'$, it holds that

$$\tfrac{1}{4\|z\|^2}\left(\mathcal{Z}_z\eta-4\left|z_0\right|^2\dot\vartheta_{t_z(\tau)}|_{z_\tau}\right)\overset{(5.83)}{=}\mathcal{Z}_z z'+\mathcal{Y}_\Theta^2|_z\overset{(C.100)}{=}\operatorname{grad}\mathcal{M}|_z.$$

Equivalently to (5.85) the delay Euler-Hamilton equation is the first order DDE

$$\begin{aligned}
z'&=\tfrac{1}{4\|z\|^2}\,\eta\\
\eta'&=\tfrac{\|\eta\|^2-8}{4\|z\|^4}z-\tfrac{2z}{\|z\|^4}\int_0^1\textstyle\int_0^s\left|z_\sigma\right|^2d\sigma\cdot\langle\dot{\boldsymbol{a}}_{t_z(s)}|_{z_s},z_s'\rangle_0\,ds\\
&\quad+\tfrac{2z}{\|z\|^2}\int_{\sigma=\tau}^1\langle\dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma},z_\sigma'\rangle_0\,d\sigma-\tfrac{|z|^2}{\|z\|^2}\dot{\boldsymbol{a}}_{t_z}|_z-\left(\operatorname{rot}\boldsymbol{a}_{t_z}|_z\right)j_0z'.
\end{aligned}\tag{5.86}$$

Proof. The formula for $\operatorname{grad}\mathcal{M}$ in Lemma 4.3 proves (5.86), so Theorem 5.13. ∎

[16] $\mathcal{A}(\Upsilon)=\Lambda|_\Upsilon\mathcal{V}(\Upsilon)+\Theta_{\pi(\Upsilon)}d\pi|_\Upsilon\mathcal{V}(\Upsilon)-\mathcal{H}(\Upsilon)$

5.5 Euler-Hamilton and Lagrange solutions correspond

In this section we prove Theorem B from the introduction. We need to show that there is a 1-to-1 correspondence between the critical point equation (5.85) of $\mathcal{A}$ in Theorem 5.13 and the critical point equation (4.46) of $\mathcal{B}$ in Theorem 4.7.

Euler-Hamilton implies Lagrange

Lemma 5.14. *If (z, η) solves the first order Euler-Hamilton DDE (5.85), then the base part z solves the regularized second order Lagrangian DDE (4.46).*

Proof. Suppose (z, η) solves (5.85). Differentiate component one with respect to time τ and then, in equality 2, substitute η' according to component two, use in addition that $\eta = 4\|z\|^2 z'$ by component one

$$z'' = \frac{1}{4\|z\|^2}\eta' \stackrel{2}{=} \frac{\|z'\|^2 z}{\|z\|^2} - \frac{z}{2\|z\|^6} + \frac{\operatorname{grad}\mathcal{M}|_z}{4\|z\|^2}.$$

This proves (4.46), hence Lemma 5.14. □

Lagrange implies Euler-Hamilton

Here we need to inject appropriately a Lagrange solution z to the Hamiltonian side which requires a pair. We define the injection by

$$\iota(z) := (z, \eta_z), \quad \eta_z := 4\|z\|^2 z'.$$

Lemma 5.15. *If z solves the regularized second order Lagrangian DDE (4.46), then (z, η_z) solves the first order Euler-Hamilton DDE (5.85).*

Proof. Suppose z solves (4.46). First component of (5.85): Indeed $z' = \frac{1}{4\|z\|^2}\eta_z$ by definition of η_z. Second component: Differentiate η_z and then, in equality 2, replace z'' according to (4.46), using in addition that $z' = \frac{1}{4\|z\|^2}\eta_z$ to get

$$\eta_z' = 4\|z\|^2 z'' \stackrel{2}{=} 4\|z\|^2 \left(\frac{\|\frac{1}{4\|z\|^2}\eta_z\|^2 z}{\|z\|^2} - \frac{z}{2\|z\|^6} + \frac{\operatorname{grad}\mathcal{M}|_z}{4\|z\|^2} \right)$$

which is exactly the second component of (5.85). This proves Lemma 5.15. □

5.6 Lagrangian action dominates Hamiltonian action

Lemma 5.16 (Lagrangian domination)**.** *There are the identities*

$$\begin{aligned}\mathcal{B}(z) &= \mathcal{A}(z, \eta) + \frac{1}{2}\left\| 2\|z\|\, z' - \frac{\eta}{2\|z\|} \right\|^2 \\ &= \mathcal{A}(z, \eta) + \frac{1}{2}\left\| 4\|z\|^2\, z' - \eta \right\|^2_{\frac{1}{4\|z\|^2}}\end{aligned}$$

for every pair of loops $(z, \eta) \in T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$.

Proof. By definition (5.84) of $\mathcal{A}$, just multiplying out the inner product we get

$$\begin{aligned}
&\mathcal{A}(z,\eta)+\frac{1}{2}\left\|2\|z\|z'-\frac{\eta}{2\|z\|}\right\|^2\\
&=\langle\eta,z'\rangle-\frac{\|\eta\|^2}{8\|z\|^2}+\frac{1}{\|z\|^2}+\Theta_z z'+\frac{1}{2}\left\langle 2\|z\|z'-\frac{\eta}{2\|z\|},2\|z\|z'-\frac{\eta}{2\|z\|}\right\rangle\\
&=2\|z\|^2\|z'\|^2+\frac{1}{\|z\|^2}+\Theta_z z'\\
&=\mathcal{B}(z)
\end{aligned}$$

where the last step is by definition (4.42) of $\mathcal{B}$. This proves Lemma 5.16. □

Corollary 5.17 (equal values on critical points)**.** *Both functionals coincide*

$$\mathcal{B}(z)=\mathcal{A}(z,\eta_z),\qquad \eta_z:=4\|z\|^2\,z',$$

on critical points: solutions z of (4.46), equivalently, zeroes (z,η_z) of (5.85).

Proof. Lemma 5.16 and Section 5.5. □

In terms of the projection π and injection ι in (1.1) the corollary tells that

$$\mathcal{B}=\mathcal{A}\circ\iota,\qquad \mathcal{B}\circ\pi=\mathcal{A},$$

along critical points of $\mathcal{B}$, respectively of $\mathcal{A}$.

5.6.1 The diffeomorphism $\mathbb{L}$

By (4.42) with $\mathcal{M}(z)=\Theta_z z'$, see (5.1.3), and $\langle\cdot,\cdot\rangle_z:=4\|z\|^2\langle\cdot,\cdot\rangle$ we have

$$\mathcal{B}=\mathcal{K}-\mathcal{U}+\mathcal{M}\colon \bar{\mathcal{L}}^\times\mathfrak{z}\to\mathbb{R},\quad z\mapsto\tfrac{1}{2}\langle z',z'\rangle_z+\frac{1}{\|z\|^2}+\Theta_z z'.$$

The functional defined by

$$\mathcal{L}=\mathcal{L}^\Theta\colon T\bar{\mathcal{L}}^\times\mathfrak{z}\to\mathbb{R},\quad (z,\xi)\mapsto\tfrac{1}{2}\langle\xi,\xi\rangle_z+\frac{1}{\|z\|^2}+\Theta_z z' \tag{5.87}$$

naturally extends $\mathcal{B}(z)=\mathcal{L}(z,z')$ in the same way as the classical action $\mathcal{S}_L(q)=\mathcal{L}_L(q,\dot q)$ is extended by a corresponding functional $\mathcal{L}_L(q,v)$.

In the classical case a fiberwise strictly convex Lagrange function L on the tangent bundle determines a function H on the cotangent bundle: resolve $\mathfrak{p}:=d_\mathfrak{v}L(r,\mathfrak{v})$ for $\mathfrak{v}$ and substitute the obtained $\mathfrak{v}=\mathfrak{v}(\mathfrak{p})$ in the Legendre identity

$$\langle\mathfrak{p},\mathfrak{v}\rangle=L(r,\mathfrak{v})+H(r,\mathfrak{p}).$$

Returning to the non-local situation where the manifold is loop space and (z,ξ) and (z,η) are pairs of loops, an analogous approach yields

$$\eta:=d_\xi\mathcal{L}(z,\xi)=4\|z\|^2\xi,\qquad \xi=\tfrac{1}{4\|z\|^2}\eta,\qquad d_{\xi\xi}\mathcal{L}(z,\xi)=4\|z\|^2>0.$$

The **non-local Hamiltonian function** is then defined by

$$\mathcal{H}^{\Theta}(z,\eta) := \langle \eta, \xi\rangle - \mathcal{L}^{\Theta}(z,\xi)$$

and with $\langle\cdot,\cdot\rangle^z := \frac{1}{4\|z\|^2}\langle\cdot,\cdot\rangle$ given by the formula

$$\begin{aligned}\mathcal{H}^{\Theta}\colon T^*\bar{\mathcal{L}}^{\times}\mathfrak{z} &\to \mathbb{R}\\ (z,\eta) &\mapsto \frac{1}{2}\langle\eta,\eta\rangle^z - \frac{1}{\|z\|^2} - \Theta_z z' = \tfrac{1}{4\|z\|^2}\left(\tfrac{1}{2}\|\eta\|^2 - 4\right) - \Theta_z z'.\end{aligned}$$

Given the functional $\mathcal{L}$ in (5.87), define the non-local analogue of the diffeomorphism introduced in [AS15, p. 1891], in the local context, by the formula

$$\begin{aligned}\mathbb{L}\colon T\bar{\mathcal{L}}^{\times}\mathfrak{z} &\to T^*\bar{\mathcal{L}}^{\times}\mathfrak{z}\\ (z,\xi) &\mapsto (z, d_\xi\mathcal{L}(z,z'+\xi)) =: (z,\eta)\end{aligned}$$

where

$$\eta := d_\xi\mathcal{L}(z,z'+\xi) = 4\|z\|^2\,(z'+\xi)$$

Note that since the inverse is given by

$$\mathbb{L}^{-1}(z,\eta) = \left(z, \frac{\eta}{4\|z\|^2} - z'\right) =: (z,\xi),$$

the solutions (z,η) of the Euler-Hamilton equations (5.85) are zeroes of $\mathbb{L}^{-1}$.

As in the ODE case [AS15], also in the present delay equation situation both functionals are related through the maps $\iota(z) = (z, 4\|z\|^2 z')$ and $\pi(z,\eta) = z$ in (1.1), Lemma 5.16, in the form

$$\mathcal{B}\circ\pi(z,\eta) = \mathcal{A}(z,\eta) + \mathcal{U}^*(z,\eta), \qquad \mathcal{U}^*(z,\eta) := \tfrac{1}{2}\,\|\iota(z)-\eta\|^2_{\frac{1}{4\|z\|^2}}$$

for every $(z,\eta)\in T^*\bar{\mathcal{L}}^{\times}\mathfrak{z}$. Observe that the map $\mathcal{U}^*\geq 0$ vanishes precisely along the critical points.

With the non-negative functional $\mathcal{U}$ defined and given by

$$\mathcal{U}(z,\xi) := \mathcal{U}^*\circ\mathbb{L}(z,\xi) = \tfrac{1}{2}\langle\xi,\xi\rangle_z \geq 0$$

the functionals $\mathcal{A}$ and $\mathcal{B}$ are related by the formula

$$\begin{aligned}\mathcal{A}\circ\mathbb{L}(z,\xi) &= \mathcal{B}(z) - \mathcal{U}(z,\xi)\\ &= \tfrac{1}{2}\langle z',z'\rangle_z + \frac{1}{\|z\|^2} + \Theta_z z' - \tfrac{1}{2}\langle\xi,\xi\rangle_z.\end{aligned}$$

A Twisted symplectic forms

Let N be a manifold, either of finite dimension n or a Banach manifold, see e.g. [Lan01]. The elements of the cotangent bundle T^*N are the pairs $z = (q,p)$ where $q \in N$ and $p \in T_q^*N$. The foot-point projection to the base is the map $\pi\colon T^*N \to N$, $z = (q,p) \mapsto q$. The derivative is a linear map $d\pi|_z\colon T_zT^*N \to T_qN$. The kernel $V_z := \ker d\pi|_z$ is called the vertical subspace of T_zT^*N. There is the canonical identification

$$T_q^*N \xrightarrow{\simeq} V_{(q,p)} \subset T_{(q,p)}T^*N, \quad \nu \mapsto \tfrac{d}{dt}\big|_{t=0}(q, p+t\nu) = (0,\nu).$$

Definition A.1. The **canonical 1-form** at a point $z = (q,p) \in T^*N$ applied to tangent vector $\zeta \in T_zT^*N$ is defined by foot-point evaluation $\lambda_z\zeta := p(d\pi|_z\zeta)$.

Definition A.2 (exterior derivative)**.** For a 1-form λ and a 2-form ω one defines

$$\begin{aligned} d\lambda(X,Y) &:= d(\lambda Y)X - d(\lambda X)Y - \lambda[X,Y] \\ d\omega(X,Y,Z) &:= d(\omega(Y,Z))X - d(\omega(X,Z))Y + d(\omega(X,Y))Z \\ &\quad - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) \end{aligned} \tag{A.88}$$

for vector fields X, Y, Z along N; see e.g. [Lan01, V Prop. 3.2]. The d's on the right hand side denote the differential of a function, evaluated on a vector field. A form is called **closed** if its exterior derivative vanishes.

Remark A.3. Pull-back commutes with d and $dd = 0$; see e.g. [Lan01, V §3].

Lemma A.4. *For $z = (q,p) \in T^*N$ let $\nu \in V_z \simeq T_q^*N$ and $\zeta \in T_zT^*N$, then*

$$d\lambda(\nu,\zeta) = \nu\left(d\pi|_z\zeta\right).$$

Proof. Since the question is local we can assume without loss of generality that N is an open subset of a vector space W. Then the cotangent bundle $T^*N \to N$ can be trivialized so that $T^*N = N \times W^*$ and $TT^*N = N \times W^* \times W \times W^*$.

Then we have $z = (q,p)$ and $\nu = (q,p;0,v)$ and $\zeta = (q,p;\xi,\eta)$. Now we extend these vector fields constantly as follows $\tilde\nu = (\tilde q, \tilde p; 0, v)$ and $\tilde\zeta = (\tilde q, \tilde p; \xi, \eta)$. Set $\tilde z = (\tilde q, \tilde p)$. The local flows of the vector fields $\tilde\nu$ and $\tilde\zeta$ are then given by $\phi^t_{\tilde\nu}(\tilde q, \tilde p) = (\tilde q, \tilde p + tv)$ and $\phi^s_{\tilde\zeta}(\tilde q, \tilde p) = (\tilde q + s\xi, \tilde p + s\eta)$.
In particular, the flows of $\tilde\zeta$ and $\tilde\nu$ commute, thus $[\tilde\zeta, \tilde\nu] = 0$. By Definition A.2 $d\lambda(\nu,\zeta)|_z = d(\lambda\tilde\zeta)|_z\nu - d(\lambda\tilde\nu)|_z\zeta - (\lambda[\tilde\nu,\tilde\zeta])|_z$. We calculate each summand. Summand one already represents the desired result

$$d(\lambda\tilde\zeta)|_z\nu = \tfrac{d}{dr}\big|_{r=0}\lambda_{\phi^r_{\tilde\nu}(z)}\tilde\zeta_{\phi^r_{\tilde\nu}(z)} \stackrel{2}{=} \tfrac{d}{dr}\big|_{r=0}(p+rv)d\pi|_{(q,p+rv)}\zeta \stackrel{3}{=} v\xi \stackrel{4}{=} \nu(d\pi|_z\zeta).$$

Equality 2 is by Definition A.1 of the canonical 1-form λ. It also uses that the extension vector field $\tilde\zeta \equiv (\xi,\eta) = \zeta$ is constant, hence so is the result under the linearized projection, namely $d\pi|_{(q,p+rv)}\zeta \equiv \xi$. Thus the r-derivative only hits the first map and we get $v\xi$ which is equality 3. But v is the local representative of ν and ξ the one of $d\pi|_z\zeta$. This is equality 4.
Summand two is zero, because $\lambda_{\tilde z}\nu = \tilde p(d\pi|_{\tilde z}(0,v)) = \tilde p(0) = 0$. Summand three is zero since $[\tilde\nu,\tilde\zeta] = 0$. This proves Lemma A.4. ☐

Remark A.5 (weak symplectic form)**.** A **weak symplectic form** is a closed 2-form which is non-degenerate in the sense that for any tangent vector there exists another vector in the same tangent space such that plugging in these two vectors in the 2-form has a non-zero value. In infinite dimension this does *not* imply that a weak symplectic form gives rise to an *isomorphism* between tangent and cotangent space. In finite dimension a weak symplectic form is a symplectic form.

Corollary A.6. *Let σ be a closed 2-form on N. Then $\omega_\sigma := d\lambda + \pi^*\sigma$ is a weak symplectic form on T^*N.*

Proof. Given that $d\omega_\sigma = dd\lambda + \pi^* d\sigma = 0$, by Remark A.3, it remains to show non-degeneracy. Let $z = (q,p) \in T^*N$. There are two cases.

Case 1. Suppose $\zeta \in T_z T^*N$ is such that $d\pi|_z \zeta \neq 0$ is non-zero.

To $0 \neq d\pi|_z\zeta \in T_qN$ there exists, by the Hahn-Banach Theorem, see e.g. [Lan01, I Prop. 2.3], a dual vector $\nu \in T_q^*N \simeq V_z := \ker d\pi|_z$ with non-zero pairing

$$0 \neq \nu(d\pi|_z\zeta) = d\lambda(\nu,\zeta) = d\lambda(\nu,\zeta) + \sigma(d\pi|_z\nu, d\pi|_z\zeta) =: \omega_\sigma(\nu,\zeta)$$

where equality one is Lemma A.4 and equality two adds zero since $\nu \in \ker d\pi|_z$.

Case 2. Suppose $\nu \in T_zT^*N$ is non-zero and $d\pi|_z\nu = 0$.

Since $\nu \in V_z \simeq T_q^*N$ is non-zero there exists a dual vector $\xi_0 \in T_qN$ with non-zero pairing $\nu(\xi_0) \neq 0$. Since $d\pi|_z$ is surjective there exists $\zeta \in T_zT^*N$ such that $d\pi|_z\zeta = \xi_0$. By Lemma A.4 and since $d\pi|_z\nu = 0$ we obtain

$$0 \neq \nu(\xi_0) = \nu(d\pi|_z\zeta) = d\lambda(\nu,\zeta) = d\lambda(\nu,\zeta) + \sigma(d\pi|_z\nu, d\pi|_z\zeta) =: \omega_\sigma(\nu,\zeta).$$

This proves Corollary A.6. □

B Non-exact magnetic 2-form Σ

In the case where a 1-form family, say $\vartheta_t = \varsigma^*\theta_t$, is *twisted-periodic*, and *not periodic*, we do not have a 1-form Θ on loop space $\bar{\mathcal{L}}^\times\mathfrak{Z}$, so $d\Theta$ as in (C.99) is not available to describe a magnetic 2-form. However, the key property of a magnetic 2-form is closedness. Therefore we use the right hand side of (C.99) as definition of a 2-form Σ and then show $d\Sigma = 0$ by hand.

Definition B.1 (Magnetic 2-form of ϑ)**.** For $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$ and $X, Y \in T_z\bar{\mathcal{L}}^\times\mathfrak{Z}$ set

$$\begin{aligned}&\Sigma_z(X,Y)\\ &:= \int_0^1 \Big(d\vartheta_{t_z}|_z\,(X,Y) + \big(dt|_zX\big)\dot\vartheta_{t_z}|_zY - \big(dt|_zY\big)\dot\vartheta_{t_z}|_zX\Big)_\tau d\tau.\end{aligned} \tag{B.89}$$

Note that $d\vartheta_t$ and $\dot\vartheta_t$ are both periodic in t by (2.4). Hence the $\mathbb{S}^1$-output of $t_z\colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$ is well received and, furthermore, integration $\int_r^{r+1}$ leads to the same value independent of $r \in \mathbb{R}$.

Theorem B.2. *It holds $d\Sigma = 0$ and $\Omega_\Sigma := \Omega_{\rm can} + P^*\Sigma$ is a weak symplectic form on the cotangent bundle $T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$, said **twisted weak symplectic**.*

Proof. By Corollary A.6 it suffices to prove $d\Sigma = 0$. To ease presentation we change our usual loop notation z to $\mathfrak{z}$. Let $\mathfrak{z} \in \bar{\mathcal{L}}^\times\mathfrak{Z}$ and $X, Y, Z \in T_\mathfrak{z}\bar{\mathcal{L}}^\times\mathfrak{Z}$. Since our manifold is an open subset of a vector space, namely $\mathfrak{Z} \subset \mathbb{C}$, therefore we constantly extend the tangent vectors at $\mathfrak{z}$ to tangent vectors at any $\tilde{\mathfrak{z}} \in \bar{\mathcal{L}}^\times\mathfrak{Z}$, namely we set time-wise $X_{\tilde{\mathfrak{z}}}(\tau) := X(\tau)$. In particular, commutators vanish and the two formulas (A.88) simplify to

$$\begin{aligned} d\lambda(X,Y) &:= d(\lambda Y)X - d(\lambda X)Y \\ d\omega(X,Y,Z) &:= d(\omega(Y,Z))X - d(\omega(X,Z))Y + d(\omega(X,Y))Z. \end{aligned} \tag{B.90}$$

To shorten formulas we abbreviate $x := (X(\mathfrak{z}))_\tau$, $y := Y_\tau$, and $z := Z_\tau$. To avoid parentheses we shall write $d\vartheta_t(x,y)$ as $d\vartheta_t xy$. We proceed similar to (C.99), but during equality two below we omit the step using the letter D. Notice that Σ defined by (B.89) has three summands which in equality two are treated one after the other, visualized by three integrals. Using repeatedly (B.90) we get

$$\begin{aligned}
d\Sigma\,(X,Y,Z) &\overset{\text{(B.90)}}{=} d(\Sigma(Y,Z))X - d(\Sigma(X,Z))Y + d(\Sigma(X,Y))Z \\
&\overset{\text{(B.89)}}{=} \int_0^1 \Big(d_x\left(d\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} yz\right) + d\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} yz(dt_\mathfrak{z}X) \\
&\qquad - d_y\left(d\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} xz\right) - d\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} xz(dt_\mathfrak{z}Y) \\
&\qquad + d_z\left(d\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} xy\right) + d\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} xy(dt_\mathfrak{z}Z)\Big) d\tau \\
&+ \int_0^1 \Big(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} z \cdot d_X(dt|_\mathfrak{z} Y) + (dt|_\mathfrak{z}Y)\cdot d_x(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} z) + (dt|_\mathfrak{z}Y)\cdot \ddot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} z\cdot(dt|_\mathfrak{z}X) \\
&\qquad - \dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} z \cdot d_Y(dt|_\mathfrak{z} X) - (dt|_\mathfrak{z}X)\cdot d_y(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} z) - (dt|_\mathfrak{z}X)\cdot \ddot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} z\cdot(dt|_\mathfrak{z}Y) \\
&\qquad + \dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} y \cdot d_Z(dt|_\mathfrak{z} X) + (dt|_\mathfrak{z}X)\cdot d_z(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} y) + \underline{(dt|_\mathfrak{z}X)\cdot \ddot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} y\cdot(dt|_\mathfrak{z}Z)}\Big) d\tau \\
&+ \int_0^1 \Big(- \dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} y \cdot d_X(dt|_\mathfrak{z} Z) - (dt|_\mathfrak{z}Z)\cdot d_x(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} y) - \underline{(dt|_\mathfrak{z}Z)\cdot \ddot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} y\cdot(dt|_\mathfrak{z}X)} \\
&\qquad + \dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} x \cdot d_Y(dt|_\mathfrak{z} Z) + (dt|_\mathfrak{z}Z)\cdot d_y(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} x) + (dt|_\mathfrak{z}Z)\cdot \ddot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} x\cdot(dt|_\mathfrak{z}Y) \\
&\qquad - \dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} x \cdot d_Z(dt|_\mathfrak{z} Y) - (dt|_\mathfrak{z}Y)\cdot d_z(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} x) - (dt|_\mathfrak{z}Y)\cdot \ddot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} x\cdot(dt|_\mathfrak{z}Z)\Big) d\tau \\
&= 0.
\end{aligned}$$

Equality three: The sum of the cyan terms is $dd\theta_{t_\mathfrak{z}}|_\mathfrak{z}(x,y,z)$ by (B.90), hence it vanishes since for the finite dimensional exterior derivative $dd = 0$. The gray terms cancel pairwise. Since the loop space exterior derivative satisfies $dd = 0$ by Remark A.4, the six red terms cancel pair-wise. For instance, the two terms with common factor $\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} z$ combine, by (B.90), as follows $(d_X(dt|_\mathfrak{z}Y) - (d_Y(dt|_\mathfrak{z}X) = ddt|_\mathfrak{z}(X,Y) = 0$. Again by (B.90) the remaining 9 black terms cancel triple-wise $d\vartheta_{t_\mathfrak{z}}|_\mathfrak{z}(x,y) - d_x\left(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} y\right) + d_y\left(\dot\vartheta_{t_\mathfrak{z}}|_\mathfrak{z} x\right) = 0$. This proves Theorem B.2. $\square$

C Periodic magnetic 1-forms on loop spaces

This section lives in an abstract setting were M is a manifold of finite dimension, not necessarily a subset $\mathfrak{Z}$ of the plane. Furthermore, we consider general circle reparametrizations T_z and $\mathcal{T}_z$, not necessarily the Barutello-Ortega-Verzini reparametrizations t_z and τ_z.

Note. Outside of Appendix C the letters T and $\mathcal{T}$ have a different meaning.

Definition C.1 (time reparametrization)**.** A smooth map from **loop space** $\mathcal{L}M := \{z \in C^\infty(\mathbb{R}, M) \mid \forall t \in \mathbb{R}\colon z(t+1) = z(t)\}$ to the group of orientation preserving circle diffeomorphisms

$$T\colon \mathcal{L}M \to \mathrm{Diff}_+\mathbb{S}^1, \quad z \mapsto T(z) =: T_z$$

is called a **time reparametrization**. The **inverse** time reparametrization is the map $\mathcal{T}\colon \mathcal{L}M \to \mathrm{Diff}_+\mathbb{S}^1$, $z \mapsto (T_z)^{-1} =: \mathcal{T}_z$.

A **periodic** 1-form on a manifold M, notation $\vartheta = \{\vartheta_t\}_{t\in\mathbb{S}^1}$, is a one-parameter-family of 1-forms on M such that $\vartheta_{t+1} = \vartheta_t$ for every $t \in \mathbb{R}$.

Definition C.2 (magnetic 1-form on loop space)**.** A periodic 1-form ϑ on a manifold M and a time reparametrization T of loop space $\mathcal{L}M$ induce on $\mathcal{L}M$ a 1-form Θ as follows. At a loop z the **magnetic 1-form** is on a tangent vector $\xi \in T_z\mathcal{L}M$, i.e. a vector field along z, defined by

$$\Theta_z\xi = \Theta_z^{\vartheta,T}\xi := \int_{\mathbb{S}^1} \vartheta_{T_z}|_z\xi := \int_0^1 \vartheta_{T_z(\tau)}|_{z_\tau}\xi_\tau \, d\tau. \tag{C.91}$$

To write the second integral we lift $T_z\colon \mathbb{S}^1_\tau \to \mathbb{S}^1_t$ to a map, still denoted by $T_z\colon \mathbb{R} \to \mathbb{R}$, which is equivariant with respect to the $\mathbb{Z}$-action on $\mathbb{R}$ given by $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $(k, \tau) \mapsto \tau + k$ and such that $T_z(0) = 0$, in symbols

$$T_z(\tau + k) = T_z(\tau) + k, \qquad T_z(0) = 0. \tag{C.92}$$

Since ϑ is periodic Θ is well defined, independent of the choice of integration interval as long as it covers one period; cf. Remark 2.4.

Remark C.3 (canonical vector field on loop space)**.** Along loop space there is a **canonical vector field** defined by time derivative $\nu(z) := z'$, cf. (3.19). The flow on $\mathcal{L}M$ of the canonical vector field $\nu = \partial_\tau$ is time shift, in symbols

$$(\phi_\nu^r z)_\tau = z_{\tau+r}, \qquad (d\phi_\nu^r|_z\xi)_\tau = \xi_{\tau+r}, \tag{C.93}$$

for every time $\tau \in \mathbb{R}$ and where $z \in \mathcal{L}M$ and $\xi \in T_z\mathcal{L}M$; cf. (3.20).

Definition C.4 (magnetic functional)**.** To a periodic 1-form ϑ on M and a time reparametrization T of loop space $\mathcal{L}M$ we associate a **magnetic functional** by evaluating the associated magnetic 1-form $\Theta^{\vartheta,T}$ along the canonical vector field

$$\mathcal{M} := i_\nu\Theta^{\vartheta,T}\colon \mathcal{L}M \to \mathbb{R}, \quad z \mapsto \int_{\mathbb{S}^1} \vartheta_{T_z}|_z z' := \int_0^1 \vartheta_{T_z(\tau)}|_{z_\tau} z'_\tau \, d\tau. \tag{C.94}$$

The definition is (C.91) and it is well defined by Lemma **??**.

Differential of magnetic functional

Motivated by Cartan's formula in finite dimension the differential of the magnetic functional $\mathcal{M}$ at $(z,\xi) \in T\mathcal{L}M$ should be given by the difference

$$(d\mathcal{M})_z\xi = (di_\nu\Theta)_z\xi = (L_\nu\Theta)_z\xi - (i_\nu d\Theta)_z\xi.$$

We prove this identity in case of the main player in this article, the magnetic functional $\mathcal{M}$ defined in (4.33) via the Barutello-Ortega-Verzini reparametrization $T_z = t_z$ in (3.21) and the pull-back $\vartheta = \varsigma^*\theta$. To get there we calculate the Lie and exterior derivatives of the more general magnetic 1-form Θ in (C.91).

Lie derivative – Euler force

Lemma C.5. *The Lie, or fisherman, derivative is defined and given by*

$$L_\nu\Theta|_z := \tfrac{d}{dr}\big|_{r=0}(\phi_\nu^r)^*\Theta|_z = \int_0^1 \underbrace{\Big(\big(\underline{dT|_z z'}\big)_\tau - \tfrac{d}{d\tau}T_z(\tau)\Big)}_{\stackrel{(C.96)}{=} \frac{d}{dr}\big|_{r=0} T_{\phi_\nu^r(z)}(\tau-r)\ \in\mathbb{R}} \dot\vartheta_{T_z(\tau)}|_{z_\tau}\, d\tau$$

at any loop $z \in \mathcal{L}M$.

Proof. Let $z \in \mathcal{L}M$ and $\xi \in T_z\mathcal{L}M$. As a preparation we compute[17]

$$\begin{aligned}\left(\phi_\nu^{r\,*}\Theta\right)_z\xi := \Theta_{\phi_\nu^r(z)}\, d\phi_\nu^r|_z\xi &\stackrel{1}{=} \int_0^1 \vartheta_{T_{\phi_\nu^r(z)}(\sigma)}|_{z_{\sigma+r}}\xi_{\sigma+r}\,d\sigma \\ &\stackrel{2}{=} \int_r^{r+1} \vartheta_{T_{\phi_\nu^r(z)}(\tau-r)}|_{z_\tau}\xi_\tau\,d\tau.\end{aligned} \tag{C.95}$$

Equality 1 uses (C.93) in definition (C.91). Equality 2 is by change of variables $\tau(\sigma) := \sigma + r$.

Since the expression $\frac{d}{dr}\big|_{r=0} T_{\phi_\nu^r(z)}$ is an element of $T_{T_z}\mathrm{Diff}_+\mathbb{S}^1 = C^\infty(\mathbb{S}^1,\mathbb{R})$, evaluated at $\tau \in \mathbb{S}^1$ it becomes a real number. For $r \in \mathbb{R}$ Leibniz yields

$$\begin{aligned}\tfrac{d}{dr}\big|_{r=0} T_{\phi_\nu^r(z)}(\tau - r) &= \left(\tfrac{d}{dr}\big|_{r=0} T_{\phi_\nu^r(z)}\right)_\tau + \tfrac{d}{dr}\big|_{r=0} T_z(\tau - r) \\ &= (dT|_z z')_\tau - \tfrac{d}{d\tau}T_z(\tau) \in \mathbb{R}.\end{aligned} \tag{C.96}$$

Differentiating equality 2 in (C.95) we obtain equality 1 in what follows

$$\begin{aligned}&\tfrac{d}{dr}\big|_{r=0}\left(\phi_\nu^{r\,*}\Theta\right)_z\xi \\ &\stackrel{1}{=} \tfrac{d}{dr}\big|_{r=0}\int_r^{r+1} \vartheta_{T_{\phi_\nu^r(z)}(\tau-r)}|_{z(\tau)}\xi_\tau\,d\tau \\ &\stackrel{2}{=} \vartheta_{T_z(1)}|_{z_1}\xi_1 - \vartheta_{T_z(0)}|_{z_0}\xi_0 + \int_0^1 \tfrac{d}{dr}\big|_{r=0}\vartheta_{T_{\phi_\nu^r(z)}(\tau-r)}|_{z(\tau)}\xi_\tau\,d\tau \\ &\stackrel{3}{=} \vartheta_1|_{z_0}\xi_0 - \vartheta_0|_{z_0}\xi_0 + \int_0^1 \dot\vartheta_{T_z(\tau)}|_{z(\tau)}\left((dT|_z z')_\tau - \tfrac{d}{d\tau}T_z(\tau)\right)\xi_\tau\,d\tau.\end{aligned} \tag{C.97}$$

[17] Actually, by the footnote to Remark 2.4 (iii) we'd get $\stackrel{3}{=} \int_0^1 \vartheta_{T_{\phi_\nu^r(z)}(\tau-r)}|_{z_\tau}\xi_\tau\,d\tau$ in the periodic case. But formula (C.97) arising from $\stackrel{2}{=}$ could be useful in the twisted-periodic case.

Equality 2 is by the Leibniz integral rule and (C.96). Equality 3 is by (C.92), by periodicity $z_1 = z_0$ and $\xi_1 = \xi_0$, and by the chain rule. Now use periodicity $\vartheta_1 - \vartheta_0 = 0$. This proves Lemma C.5. $\square$

Example C.6 (identity reparametrization)**.** For the constant reparametrization $T \equiv \mathrm{id}_{\mathbb{S}^1} \in \mathrm{Diff}_+\mathbb{S}^1$ the value of (C.96) is $0 - \frac{d}{d\tau}\tau = -1$. So by Lemma C.5

$$L_\nu \Theta|_z = -\int_0^1 \dot\vartheta_{T_z(\tau)}|_{z_\tau}\, d\tau.$$

Example C.7 (BOV-reparametrization – Euler force)**.** Let $\vartheta_t = \varsigma^*\theta_t$. Consider the Barutello-Ortega-Verzini reparametrization $T_z = t_z$ in (4.52) where $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$. Plug the formula (4.40) for $(dt|_z\xi)_\tau$ and (4.52) for t_z' into (C.96) to obtain

$$\begin{aligned}
\tfrac{d}{dr}\big|_{r=0}\, t_{\phi_\nu^r(z)}(\tau - r) &= \frac{1}{\|z\|^2}\int_0^\tau 2\left\langle z_\sigma, z_\sigma'\right\rangle_0 d\sigma - \frac{2\left\langle z, z'\right\rangle}{\|z\|^4}\int_0^\tau \left|z_\sigma\right|^2 d\sigma - \frac{\left|z_\tau\right|^2}{\|z\|^2}\\
&\stackrel{2}{=} \frac{\left|z_\tau\right|^2 - \left|z_0\right|^2}{\|z\|^2} - \frac{\left|z_1\right|^2 - \left|z_0\right|^2}{\|z\|^4}\int_0^\tau \left|z_\sigma\right|^2 d\sigma - \frac{\left|z_\tau\right|^2}{\|z\|^2}\\
&= -\frac{\left|z_0\right|^2}{\|z\|^2}.
\end{aligned}$$

Equality 2 uses that $\int_0^s 2\left\langle z_\sigma, z_\sigma'\right\rangle ds = \int_0^s \frac{d}{d\sigma}\left|z_\sigma\right|^2 ds = \left|z_s\right|^2 - \left|z_0\right|^2$. For $s = 1$ the difference vanishes since $z_1 = \pm z_0$. Twisted-periodicity of θ_t with function f implies such for ϑ_t with function $F := \varsigma^* f = f \circ \varsigma$. Hence Lemma C.5 yields

$$L_\nu\Theta|_z = -\tfrac{|z_0|^2}{\|z\|^2}\int_0^1 \dot\vartheta_{t_z(\tau)}|_{z_\tau}\, d\tau \stackrel{2}{=} \tfrac{1}{4\|z\|^2}\left\langle -4|z_0|^2\dot\vartheta_{t_z(\tau)}|_{z_\tau}, \cdot\right\rangle \stackrel{3}{=} \left\langle \mathcal{Y}^2_\Theta|_{(z,\eta)}, \cdot\right\rangle \qquad \text{(C.98)}$$

$\forall (z,\eta) \in T^*\bar{\mathcal{L}}^\times\mathfrak{Z}$. Step 2 uses that $\mathbb{R}^2 \simeq (\mathbb{R}^2)^*$, step 3 is the Euler force (5.83).

Exterior derivative – Lorentz force

Let $\xi, \eta \in T_z\mathcal{L}M$. In order to apply formula (A.88) involving commutators, we extend the tangent vectors ξ and η to vector fields defined in a neighborhood of z in $\mathcal{L}M$ which we denote by the same letters ξ and η. By definition (A.88) we get equality one

$$\begin{aligned}
&(d\Theta)_z(\xi,\eta)\\
&= \int_0^1 \Big(D_\xi\left(\vartheta_{T_z(\cdot)}|_{z\cdot}\eta_\cdot\right)_\tau - D_\eta\left(\vartheta_{T_z(\cdot)}|_{z\cdot}\xi_\cdot\right)_\tau - \vartheta_{T_z(\tau)}|_{z_\tau}[\xi,\eta]_\tau\Big)d\tau\\
&\stackrel{2}{=} \int_0^1 \Big(d_\xi\left(\vartheta_{T_z(\cdot)}|_{z\cdot}\eta_\cdot\right)_\tau - d_\eta\left(\vartheta_{T_z(\cdot)}|_{z\cdot}\xi_\cdot\right)_\tau - \vartheta_{T_z(\tau)}|_{z_\tau}[\xi,\eta]_\tau \qquad \text{(C.99)}\\
&\qquad + \dot\vartheta_{T_z(\tau)}|_{z_\tau}\left(dT|_z\xi\right)_\tau\eta_\tau - \dot\vartheta_{T_z(\tau)}|_{z_\tau}\left(dT|_z\eta\right)_\tau\xi_\tau\Big)d\tau\\
&\stackrel{3}{=} \int_0^1 \Big(d\vartheta_{T_z(\tau)}|_{z_\tau}(\xi_\tau,\eta_\tau) + \dot\vartheta_{T_z(\tau)}|_{z_\tau}\big(\underline{\left(dT|_z\xi\right)}_\tau\eta_\tau - \left(dT|_z\eta\right)_\tau\xi_\tau\big)\Big)d\tau.
\end{aligned}$$

The letter D indicates differentiation with respect to all variables, spatial d and time ∂_t. Equality 2 unpacks D. Equality 3 is by (A.88) for d and $\lambda = \vartheta_T$.

Remark C.8 (Lorentz force)**.** On a finite dimensional manifold M let g be a Riemannian metric and σ_t a family of closed 2-forms depending on a parameter $t \in \mathbb{R}$, e.g. a family of exact forms $\sigma_t = d\vartheta_t$. Then at any point $q \in M$ a g-anti-symmetric linear map $Z_t|_q \colon T_qM \to T_qM$ is defined by the 1-form identity

$$g_q(Z_t|_q v, \cdot) = -\sigma_t|_q(v, \cdot)$$

equivalently $(Z_t|_q v)^\flat = -i_v \sigma_t|_q$ where $\flat \colon T_qM \to T_q^*M$ is the metric isomorphism $\xi \mapsto g_q(\xi, \cdot)$ and $\sharp$ the inverse. This is the **Lorentz force**, it is given by

$$Z_t|_q v = -(i_v \sigma_t|_q)^\sharp.$$

Euclidean space. On $M = \mathbb{R}^3$, given a 1-form θ, let $\boldsymbol{A}$ be the dual vector field, cf. (2.3), then the magnetic vector field is given by $\boldsymbol{B} = \operatorname{rot} \boldsymbol{A}$. In terms of differential forms this corresponds to $\sigma = d\theta$ and σ and $\boldsymbol{B}$ are related via the Hodge $*$-operator by $\sigma = *\boldsymbol{B}^\sharp$; see e.g. [Web17]. In this context the Lorentz force on a particle of electric charge $c \in \mathbb{R}$ at a point $\boldsymbol{r}(t)$ at time t, namely $cZ|_{r(t)}\dot{r}(t)$, is of the familiar cross product form $c\dot{r}(t) \times \boldsymbol{B}|_{r(t)}$. This ends Remark C.8.

Example C.9 (identity reparametrization)**.** For the constant reparametrization $T \equiv \mathrm{id}_{\mathbb{S}^1} \in \mathrm{Diff}_+\mathbb{S}^1$ term $dT|_z = 0$ vanishes in (C.99). The identity

$$\int_0^1 \langle (\mathcal{Z}_z\xi)_\tau, \eta_\tau \rangle_0 \, d\tau = \langle \mathcal{Z}_z\xi, \eta \rangle \overset{(5.81))}{=} -(d\Theta)_z(\xi, \eta) \overset{(C.99))}{=} \int_0^1 -d\vartheta_\tau|_{z_\tau}(\xi_\tau, \eta_\tau) \, d\tau$$

$\forall \xi, \eta$ determines the non-local Lorentz force $\mathcal{Z}_z$. The equality of Integrands $\langle (\mathcal{Z}_z\xi)_\tau, \cdot \rangle_0 = -d\vartheta_\tau|_{z_\tau}(\xi_\tau, \cdot)$, the latter is $\langle Z_\tau|_{z_\tau}\xi_\tau, \cdot \rangle_0$, tells $(\mathcal{Z}_z\xi)_\tau = Z_\tau|_{z_\tau}\xi_\tau$.

Example C.10 (BOV-reparametrization – Lorentz force)**.** Let $\vartheta_t = \varsigma^*\theta_t$. For the Barutello-Ortega-Verzini reparametrization $T_z = t_z$ in (4.52) the **Lorentz force** at $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$ is the linear map $\mathcal{Z}_z \colon T_z\bar{\mathcal{L}}^\times\mathfrak{Z} \to T_z\bar{\mathcal{L}}^\times\mathfrak{Z}$ determined by

$$\langle \mathcal{Z}_z\zeta, \xi \rangle = -d\Theta_z(\zeta, \xi)$$

for all $\xi, \zeta \in T_z\bar{\mathcal{L}}^\times\mathfrak{Z}$ where $d\Theta$ is given by (C.99).[18] In particular for $\zeta = z'$ computation (C.101) below, read backwards, yields equality one in what follows

$$\begin{aligned} \langle \operatorname{grad} \mathcal{M}|_z, \cdot \rangle &= (L_\nu\Theta)_z \cdot -(d\Theta)_z(z', \cdot) \\ &\overset{2}{=} \left\langle \mathcal{Y}_\Theta^2 + \mathcal{Z}_z z', \cdot \right\rangle \\ &\overset{3}{=} \left\langle \tfrac{1}{4\|z\|^2}(-4|z_0|^2 \dot{\vartheta}_{t_z(\tau)}|_{z_\tau}) + \mathcal{Z}_z z', \cdot \right\rangle. \end{aligned} \tag{C.100}$$

Equality 2 uses the relation (C.98) between Lie derivative and Euler force $\mathcal{Y}_\Theta$ as well as the above definition of the Lorentz force $\mathcal{Z}$. Equality 3 is by (C.98) again. This concludes Example C.10.

[18] **Note.** In the twisted-periodic, and not periodic, case let's take the same formula (B.89) to define a 2-form, say Σ. In doing so all we loose is exactness. Define $\mathcal{Z}$ as above.

Difference of Lie and exterior derivative – $d\mathcal{M}$

Proposition C.11. *For $z \in \mathcal{L}M$ and $\xi \in T_z\mathcal{L}M$ there is the identity*

$$
\begin{aligned}
&(L_\nu\Theta)_z\xi - (d\Theta)_z(z',\xi) = (L_\nu\Theta)_z\xi - (d\Theta)_z(z',\xi) \\
&= \int_0^1 \dot\vartheta_{T_z(\tau)}|_{z_\tau} \left((dT|_z\xi)_\tau\, z'_\tau - \left(\tfrac{d}{d\tau}T_z(\tau)\right)\xi_\tau \right) - d\vartheta_{T_z(\tau)}|_{z_\tau}(z'_\tau,\xi_\tau)\, d\tau.
\end{aligned}
$$

Proof. Lemma C.5 and (C.99); the two underlined terms add to zero. ☐

Corollary C.12. *For the BOV-reparametrization t_z in (3.21) and $\vartheta_t = \varsigma^*\theta_t$ Cartan's formula $L_\nu = i_\nu d + di_\nu$ holds for Θ along loop space $\bar{\mathcal{L}}_+^\times\mathfrak{Z}$; see (3.16).*

Example C.13 (BOV-reparametrization and $\vartheta_t = \varsigma^*\theta_t$)**.** In the setting of the Barutello-Ortega-Verzini reparametrization in Section 3 replace $\mathcal{L}M$ by $\bar{\mathcal{L}}^\times\mathfrak{Z}$ and T_z by t_z from (3.21). In this case the magnetic functional $\mathcal{M} = i_\nu\Theta$ is given by (4.33). For loops $z \in \bar{\mathcal{L}}^\times\mathfrak{Z}$ and tangent vector fields $\xi \in T_z\bar{\mathcal{L}}^\times\mathfrak{Z}$, we compute by Proposition (C.11) the difference in equality one

$$
\begin{aligned}
&(L_\nu\Theta)_z\xi - (d\Theta)_z(z',\xi) \qquad = \left\langle \tfrac{1}{4\|z\|^2}\left(-4|z_0|^2\dot\vartheta_{t_z(\tau)}|_{z_\tau}\right) + \mathcal{Z}_z z', \xi \right\rangle \\
&\stackrel{1}{=} \int_0^1 (dt|_z\xi)_\tau\, \dot\vartheta_{t_z(\tau)}|_{z_\tau} z'_\tau - \left(\tfrac{d}{d\tau}t_z(\tau)\right)\dot\vartheta_{t_z(\tau)}|_{z_\tau}\xi_\tau - d\vartheta_{t_z(\tau)}|_{z_\tau}(z'_\tau,\xi_\tau)\, d\tau \\
&\stackrel{2}{=} \frac{2}{\|z\|^2}\int_0^1 \underline{\textstyle\int_{\sigma=0}^\tau \langle z_\sigma,\xi_\sigma\rangle\, d\sigma} \cdot \dot\vartheta_{t_z(\tau)}|_{z_\tau} z'_\tau\, d\tau \\
&\quad - \frac{2\langle z,\xi\rangle}{\|z\|^4}\int_0^1 \underline{\textstyle\int_0^\tau |z_\sigma|^2\, d\sigma} \cdot \dot\vartheta_{t_z(\tau)}|_{z_\tau} z'_\tau\, d\tau \\
&\quad - \frac{1}{\|z\|^2}\int_0^1 |z_\tau|^2\,\dot\vartheta_{t_z(\tau)}|_{z_\tau}\xi_\tau\, d\tau - \int_0^1 d\vartheta_{t_z(\tau)}|_{z_\tau}(z'_\tau,\xi_\tau)\, d\tau.
\end{aligned}
$$

Equality 2 inserts the formulas for $(dt|_z\xi)_\tau$ and $\frac{d}{d\tau}t_z(\tau)$. We grayed out the terms which will not be modified in equality 3 below in which we actually only modify summand one: We change the order of integration and then interchange the names of the variables τ and σ in order to obtain

$$
\begin{aligned}
&\stackrel{3}{=} \frac{2}{\|z\|^2}\int_0^1 \left\langle \underline{\textstyle\int_{\sigma=\tau}^1 \dot\vartheta_{t_z(\sigma)}|_{z_\sigma} z'_\sigma\, d\sigma} \cdot z_\tau, \xi_\tau \right\rangle d\tau \\
&\quad - \frac{2\langle z,\xi\rangle}{\|z\|^4}\int_0^1 \underline{\textstyle\int_0^\tau |z_\sigma|^2\, d\sigma} \cdot \dot\vartheta_{t_z(\tau)}|_{z_\tau} z'_\tau\, d\tau \\
&\quad - \frac{1}{\|z\|^2}\int_0^1 |z_\tau|^2\,\dot\vartheta_{t_z(\tau)}|_{z_\tau}\xi_\tau\, d\tau - \int_0^1 d\vartheta_{t_z(\tau)}|_{z_\tau}(z'_\tau,\xi_\tau)\, d\tau \\
&\stackrel{4}{=} (d\mathcal{M})_z\xi \qquad = \langle \operatorname{grad}\mathcal{M}|_z, \xi\rangle.
\end{aligned}
\tag{C.101}
$$

Equality 4 is by Lemma 4.3. This proves Cartan's formula in the setting of the present article. This concludes Example C.13 and proves Corollary C.12.

References

[AS15] Alberto Abbondandolo and Matthias Schwarz. The role of the Legendre transform in the study of the Floer complex of cotangent bundles. *Comm. Pure Appl. Math.*, 68(11):1885–1945, 2015.

[BOV21] Vivina Barutello, Rafael Ortega, and Gianmaria Verzini. Regularized variational principles for the perturbed Kepler problem. *Adv. Math.*, 383:Paper No. 107694, 64, 2021. arXiv:2003.09383.

[CFV23] Kai Cieliebak, Urs Frauenfelder, and Evgeny Volkov. A variational approach to frozen planet orbits in helium. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 40(2):379–455, 2023.

[CFvK17] Kai Cieliebak, Urs Frauenfelder, and Otto van Koert. Periodic orbits in the restricted three-body problem and Arnold's J^+-invariant. *Regul. Chaotic Dyn.*, 22(4):408–434, 2017.

[For11] Otto Forster. *Analysis. 1.* Grundkurs Mathematik. Vieweg + Teubner, Wiesbaden, expanded edition, 2011. Differential- und Integralrechnung einer Veränderlichen.

[Fra25] Urs Frauenfelder. Periodic orbits in time-dependent planar Stark-Zeeman systems. *arXiv e-prints*, page arXiv:2503.09209, March 2025. Accepted for publication in Kyoto J. Math.

[FW21] Urs Frauenfelder and Joa Weber. The regularized free fall I – Index computations. *Russian Journal of Mathematical Physics*, 28(4):464–487, 2021. SharedIt.

[FW26a] Urs Frauenfelder and Joa Weber. Loop space blow-up and scale calculus. *Arch. Math. (Basel)*, 126:335–342, 2026. Open access.

[FW26b] Urs Frauenfelder and Joa Weber. Merry-go-round and time-dependent symplectic forms. *viXra e-prints science, freedom, dignity*, pages 1–18, January 2026. viXra: 2601.0019.

[FW26c] Urs Frauenfelder and Joa Weber. The linearized Floer equation in a chart. *SIGMA*, 22(032):38 pages, 2026. Special Issue on Geometry and Dynamics in memory of Will Merry. Open access.

[FW26d] Urs Frauenfelder and Joa Weber. Towards a Floer theory for Mars II – Floer Hessian field almost extends. *viXra e-prints science, freedom, dignity*, 2026. In preparation

[Kep09] Johannes Kepler. *Astronomia Nova*. Heidelberg: G. Voegelinus, 1609. Online: archive or ETH Zürich.

[Kep37] Johannes Kepler. *Gesammelte Werke. Astronomia nova*, volume 3 of *Kepler. Gesammelte Werke.* Max Caspar [Hg./Red.], Walther von Dyck [Hg./Red.], München, 1937. Online: vol 3 and BAdW.

[Kep92] Johannes Kepler. *New Astronomy,* translated by William H. Donahue. Cambridge Univ. Press, Cambridge, 1992. New Revised Edition, 2015.

[Lan01] Serge Lang. *Fundamentals of differential geometry.* Springer-Verlag, New York, corr. printing 2nd edition, 2001.

[Tes12] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics*. Online edition, authorized by American Mathematical Society, Providence, RI, 2012.

[Web17] Joa Weber. *Topological methods in the quest for periodic orbits*. Publicações Matemáticas. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2017. 31° Colóquio Brasileiro de Matemática. Access book.