A proof for the Collatz conjecture

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Abstract

The proof hereafter adresses the Collatz conjecture [1] and relies on the quantitative characterization undergone by the evolving density of even versus odd natural numbers within the limit set produced by infinite iterations of the transformation law *T* from any initial value taken in $\mathbb{N}$. This couple of limit densities at infinity for even and odd numbers comes naturally along with respective frequencies of occurence for the two competing transformations of the Collatz system, from which we deduce the final cyclic attractor, hence proving the conjecture.

Introduction

Although being intensively studied since 1937, the Collatz conjecture has remained unproved until now. It states that the following dynamical system :

$$T(X_n) := \left\{ \begin{array}{ll} \text{if } X_n \text{ is odd, then } X_{n+1} = 3X_n + 1 & (*) \\ \text{if } X_n \text{ is even, then } X_{n+1} = \dfrac{X_n}{2} & (**) \end{array} \right\}$$

must evolve in the limit of the greatest values for *n* so that :

$$n \to +\infty \qquad X_n = 1,\ X_{n+1} = 4,\ X_{n+2} = 2,\ X_{n+3} = 1,\ \ldots$$

Despite numerous verifications by direct computation of *T* that this cyclic attractor must be the only possible end state, no genral proof could be established for any initial value $X_0 \in \mathbb{N}$ until now.

Our method

Let us remark before all the following :

If X_n is odd, then it automatically satisfies the equation $X_n = 2\mathrm{k} + 1 \ (k \in \mathbb{N})$. In accordance with the definition of *T*, then we will have the following holding as well :

$$\begin{aligned} X_{n+1} &= 3(2\mathrm{k}+1)+1 \\ &= 6\mathrm{k}+4 \\ &= 2(3\mathrm{k}+2) \end{aligned}$$

which means that *T* always produce an even number as an output whenever an odd number is fed to it.

However, in case of X_n being even, then $X_n = 2\mathrm{k} \ (k \in \mathbb{N})$ must then hold, so that, in accordance with the definition of *T*, we will have now the following :

$$\begin{aligned} X_{n+1} &= \frac{2\mathrm{k}}{2} \\ &= k \end{aligned}$$

which means that whichever X_n we take at random amongst the even naturals as an input to T, the output will have half chance to be odd, half chance to be even, only depending on k.

Writing T_ω for (*) and T_ϵ for (**), therefore we obtain :

Iteration	Input density	Function	Redistribution	Output density
n=0	1/2 odd	T_ω	1/2 even	3/4 even
	1/2 even	T_ϵ	(1/2).(1/2) even	
			(1/2).(1/2) odd	1/4 odd

n=1	1/4 odd	T_ω	1/4 even	5/8 even
	3/4 even	T_ϵ	(1/2).(3/4) even	
			(1/2).(3/4) odd	3/8 odd

n=2	3/8 odd	T_ω	3/8 even	11/16 even
	5/8 even	T_ϵ	(1/2).(5/8) even	
			(1/2).(5/8) odd	5/16 odd

...

$n\rightarrow+\infty$	?/? odd	T_ω	?/? even	?/? even
	?/? even	T_ϵ	(1/2).(?/?) even	
			(1/2).(?/?) odd	?/? odd

The latter can be expressed in the form a Markov process like as follows :

$$\begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n = \begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n = M \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n$$

To be relevant, this Markov process must be iterated from zero to n in parallel with the transformation T , yielding each time the corresponding densities for odd (ω), respectively even (ε), numbers within the n^{th} image set of T in form a vector.

Using the standard method for determining the eigenvalues and eigenvectors of M through calculation of the characteristic polynomial of the endomorphism [2], we find that :

$$\begin{aligned} \det M = 0 \;&\Leftrightarrow\; (-\lambda)(1/2\ -\lambda)-(1)(1/2) \\ &\Leftrightarrow\; \lambda^2 - \frac{1}{2}(\lambda+1) = 0 \end{aligned}$$

which has two distinct solutions $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. Thus, *M* can be reduced and turned into a diagonal matrix *D* like as follows :

$$\begin{aligned} M^n &= P\,D^n\,P^{-1} \\ &= P \begin{pmatrix} 1^n & 0 \\ 0 & (-1)^n \frac{1}{2^n} \end{pmatrix} P^{-1} \end{aligned}$$

where *P* is the transition matrix and P^{-1} its inverse such that $PP^{-1} = I$, i.e. the identity matrix.

From there on, we have to determine the following limit :

$$\lim_{n \to +\infty} P\,D^n\,P^{-1} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

by simply finding the two eigenvectors of *M* such that :

$$\begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \lambda_i \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

which turn out to be :

$$\lambda_i = \lambda_1 \Rightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \qquad \lambda_i = \lambda_2 \Rightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

From the latter, we derive the transition matrix :

$$P = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ 1 & -1 \end{pmatrix}$$

And its inverse matrix using the standard formula for 2x2 matrices :

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}^{-1} = \frac{1}{u_1 v_2 - v_1 u_2} \begin{pmatrix} v_2 & -u_2 \\ -v_1 & u_1 \end{pmatrix}$$

Hence :

$$P^{-1} = \frac{1}{-3/2} \begin{pmatrix} -1 & -1 \\ -1 & 1/2 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

Since $M^n \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_0 = P\,D^n\,P^{-1} \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_0$ for all *n*, we can now determine the limit state of the Markov process :

$$\lim_{n \to +\infty} \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n = \lim_{n \to +\infty} P\,D^n\,P^{-1} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \lim_{n \to +\infty} \begin{pmatrix} 1/2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (-1/2)^n \end{pmatrix} \begin{pmatrix} 2/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Since $\lim_{n \to +\infty} 1^n = 1$ and $\lim_{n \to +\infty} (-1/2)^n = 0$, then the matrix product $P\,D^\infty\,P^{-1}$

becomes :

$$\lim_{n\to+\infty}\begin{pmatrix}\omega\\ \epsilon\end{pmatrix}_n=\begin{pmatrix}2/6 & 2/6\\ 2/3 & 2/3\end{pmatrix}\begin{pmatrix}1/2\\ 1/2\end{pmatrix}=\begin{pmatrix}\omega_\infty\\ \epsilon_\infty\end{pmatrix}=\begin{pmatrix}1/3\\ 2/3\end{pmatrix}$$

This result means that the limit set obtained by iterating $T([a,b])$ infinitely many times out of an arbitrary initial interval $[a,b]$ of $\mathbb{N}$ starts with both densities being 1/2 with respect to even and odd numbers, and ends up with one third for odd numbers and two thirds for even numbers at infinity.

As a consequence, for n aiming towards infinity, the transformation law T, which comes in 2 different forms $T\omega$ (*) and $T\varepsilon$ (**), will be affected with a frequency of occurence defined by the density of each type of input, odd or even, given by the limit vector above. In this limit, $T\omega$ will happen for one third of the total application of T whilst $T\varepsilon$ will happen for two thirds as well.

Thus, we are led to translate this limit regime of T into a new function H, defined in 3 equivalent forms by permutation :

$$\lim_{n\to+\infty}X_{n+3}=\begin{Bmatrix}H_1(X_n)=T_\omega(T_\epsilon(T_\epsilon(X_n)))\\ H_2(X_n)=T_\epsilon(T_\omega(T_\epsilon(X_n)))\\ H_3(X_n)=T_\epsilon(T_\epsilon(T_\omega(X_n)))\end{Bmatrix}$$

Direct calculation of those three functions H_1, H_2 and H_3 yields three possible limit transformation laws defined like as follows :

$$H_1(X_n)=3\left(\frac{1}{2}\left(\frac{1}{2}X_n\right)\right)+1=\frac{3}{4}X_n+1\qquad H_2(X_n)=\frac{1}{2}\left(3\left(\frac{X_n}{2}\right)+1\right)=\frac{3}{4}X_n+\frac{1}{2}\qquad H_3(X_n)=\frac{1}{2}\left(\frac{1}{2}\left(3X_n+1\right)\right)=\frac{3}{4}X_n+\frac{1}{4}$$

Since any of these H functions will be iterated infinitely many times, i.e. in the limit $n\to+\infty$, the following must hold as well :

$$H_i^n(X_n)=(3/4)^nX_n+\beta((3/4)^0+(3/4)^1+\ \ldots+(3/4)^n)$$

where β = 1, 1/2 or 1/4. Moreover, since $\sum_{i=0}^{+\infty}(3/4)^i=\frac{1}{1-3/4}=\frac{1}{1/4}=4$ [3], we get the following expression for the limit behavior of the transformation T :

$$\lim_{n\to+\infty}H_i^n(X_n)=(3/4)^nX_n+4\beta=4\beta$$

as $\lim_{n\to+\infty}(3/4)^n=0$ will extinguish all finite value X_n . More precisely, replacing β with one of the three possible values as above, we will have :

$$\lim_{n \to +\infty} H_1^n(X_n) = 4$$
$$\lim_{n \to +\infty} H_2^n(X_n) = 4/2 = 2$$
$$\lim_{n \to +\infty} H_3^n(X_n) = 4/4 = 1$$

Conclusion

Given the primary definition of T (cf. introduction), this triplet of values defines the attractor final state of the transformation as $T_n \to H_n$ in the limit of n tends to infinity in form of a cycle $4 \to 2 \to 1 \to 4 \to \ldots$, which proves the Collatz conjecture.

References

[1] https://en.wikipedia.org/wiki/Collatz_conjecture
[2] https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors
[3] https://en.wikipedia.org/wiki/Geometric_series