# Exact solution to Einstein's field equation

Antonio Mejias[1]

[1]Territorial Polytechnic University of Mérida (UPTM). antoniojmejiasm@gmail.com

***Abstract***

In this work, Einstein's field equation will be solved by removing the framework of tensor analysis.

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## 1. Introduction

Einstein's field equation is difficult to solve, according to most scientists. Einstein himself thought it couldn't be solved; in fact, when Schwarzschild solved it, Einstein was surprised. Schwarzschild solved it for a mass surrounded by empty space, which implied that the energy-momentum tensor would be equal to zero, but in doing so, Schwarzschild removed the relativistic component from the equation.

By performing the relevant and simple calculations on that field equation, the tensor clothing can be removed, revealing a more physical and simple equation, which, when integrated, yields the effective potential that is the true solution.

## Tensor Reduction to Fundamental Dynamic Variables

Einstein's field equation is as follows:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \text{ [1]} \qquad (1)$$

By contracting it with the inverse metric tensor, we obtain the following:

$$R_{\mu\nu} g^{\mu\nu} - \frac{1}{2} R g_{\mu\nu} g^{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} g^{\mu\nu}$$

$$R - 2R = \frac{8\pi G}{c^4} T \text{ [2]} \qquad (2)$$

But T for a perfect fluid, as the universe can be assumed to be, is:

$$T = -2\rho c^2 \text{ [3]} \qquad (3)$$

Substituting (3) into (2):

$$R - 2R = -\frac{16\pi G \rho c^2}{c^4}$$

$$\frac{1}{4} R - \frac{1}{2} R = -\frac{4\pi G \rho}{c^2} \qquad (4)$$

But the curvature scalar R is:

$$R = \frac{2}{r^2} \text{ [4]} \qquad (5)$$

Substituting (5) into (4):

$$\frac{1}{2}\frac{1}{r^2} - \frac{1}{r^2} = -\frac{4\pi G \rho}{c^2} \qquad (6)$$

$$\rho = \frac{M}{V} = \frac{3M}{4\pi r^3} \qquad (7)$$

Replacing (7) in (6):

$$\frac{1}{2}\frac{1}{r^2} - \frac{1}{r^2} = -\frac{3GM}{c^2 r^3} \qquad (8)$$

$$v^2 = \frac{l^2}{m^2r^2} \implies \frac{1}{r^2} = \frac{v^2m^2}{l^2} \quad (9)$$

Substituting (9) into (8):

$$\frac{1}{2}\frac{v^2m^2}{l^2} - \frac{v^2m^2}{l^2} = -\frac{3GM}{c^2r^3}$$

$$\frac{1}{2}\frac{v^2}{r} - \frac{v^2}{r} = -\frac{3GMl^2}{c^2m^2r^4} \quad (10)$$

$$\frac{1}{2}\frac{v^2}{r} = \frac{1}{2}a \qquad \frac{v^2}{r} = \frac{GM}{r^2} \quad (11)$$

Then (10) becomes:

$$\frac{1}{2}a - \frac{GM}{r^2} + \frac{3GMl^2}{c^2m^2r^4} = 0 \quad (12)$$

From equation (12) we obtain Einstein's field equation with a more physical structure.

$$a = \frac{2GM}{r^2} - \frac{6GMv^2}{c^2r^2}$$

$$a = \frac{8\pi G\rho r}{3}\left(1 - \frac{3v^2}{c^2}\right) \quad (13)$$

$$As \quad \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \ \ So$$

$$L = \frac{1}{2}arm + \frac{GMm}{r} - \frac{GMl^2}{c^2m^2r^3}$$

$$L = \frac{1}{2}arm + \frac{GMm}{r}\left(1 - \frac{l^2}{c^2m^2r^2}\right) \quad (14)$$

$$U_v = -\frac{GMm}{r}\left(1 - \frac{v^2}{c^2}\right) \text{ Generalized potential } (15)$$

## 3. Geodesic of time

From the equation of time:

$$d\tau = \frac{dt}{\sqrt{1-\frac{v^2}{c^2}}} \quad (16)$$

Si $v = v_e = \frac{2GM}{r}$ =escape velocity. Then:

$$ds = cdt = \sqrt{c^2\frac{d\tau^2}{dt^2}\left(1 - \frac{2GM}{c^2r}\right)}dt \qquad \frac{d\tau^2}{dt^2} = \dot{\tau}^2$$

$$ds = cdt = \sqrt{c^2\dot{\tau}^2\left(1 - \frac{2GM}{c^2r}\right)}dt \quad (17)$$

$$\sqrt{c^2\dot{\tau}^2\left(1 - \frac{2GM}{c^2r}\right)} = f \quad (18)$$

If we compare the left side with the right side of the equation (17) we see that f is the speed of light c, therefore f is constant.

Applying the principle of least action

$$\frac{\partial f}{\partial \tau} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\tau}}\right) = 0$$

$$\text{As } \frac{\partial f}{\partial \tau} = 0 \qquad \text{Then } \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\tau}}\right) = 0$$

$$\frac{\partial f}{\partial \dot{\tau}} = \frac{c^2\dot{\tau}\left(1-\frac{2GM}{c^2r}\right)}{f}$$

Since $f$ is constant, then

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\tau}}\right) = \frac{c^2}{f}\left(\ddot{\tau}\left(1 - \frac{2GM}{c^2r}\right) + \frac{2GM}{c^2r^2}\right) = 0$$

From here:

$$\ddot{\tau}\left(1 - \frac{2GM}{c^2r}\right) + \dot{\tau}\frac{2GM}{c^2r^2}\dot{r} = 0$$

$$\frac{\ddot{\tau}}{\dot{\tau}} = -\frac{2GM}{c^2r^2\left(1-\frac{2GM}{c^2r}\right)}\dot{r} \quad (19)$$

Integrating (18):

$$\int_\tau^t \frac{\ddot{\tau}}{\dot{\tau}}dt = \int_r^\infty \frac{2GM}{c^2r^2\left(1-\frac{2GM}{c^2r}\right)}dr$$

$$ln\frac{d\tau}{dt} - ln\frac{dt}{dt} = -ln\left(1 - \frac{2GM}{c^2r}\right) + ln\left(1 - \frac{2GM}{c^2\infty}\right)$$

$$ln\frac{d\tau}{dt} - ln1 = -ln\left(1 - \frac{2GM}{c^2r}\right) + ln(1-0)$$

$$ln\frac{d\tau}{dt} = -ln\left(1 - \frac{2GM}{c^2r}\right) \quad ln\frac{d\tau}{dt} = -ln\left(1 - \frac{2GM}{c^2r}\right)$$

$$ln\frac{d\tau}{dt} = ln\frac{1}{\left(1-\frac{2GM}{c^2r}\right)} \quad \implies \frac{d\tau}{dt} = \frac{1}{\left(1-\frac{2GM}{c^2r}\right)}$$

Therefore:

$$dt = d\tau\left(1 - \frac{2GM}{c^2r}\right) \quad (20)$$

## 4. Schwarzschild Metric

For a system where there is variation of r with respect to time, the Karajan is:

$$L = \frac{1}{2}\dot{r}^2 m + \frac{1}{2}\omega^2 r^2 m + \frac{GMm}{r}\left(1 - \frac{v^2}{c^2}\right) \quad (21)$$

$$l = vrm = cte.$$

$$\dot{v}rm + v\dot{r}m = 0$$

$$\dot{v} = \frac{v^2}{r}$$

$$\frac{v^2}{r}r + v\dot{r} = 0 \quad v + \dot{r} = 0 \quad v = -\dot{r} \Longrightarrow \quad v^2 = \dot{r}^2$$

Si v = c

$$L = \frac{1}{2}c^2 m + \frac{1}{2}c^2 m + \frac{GMm}{r}\left(1 - \frac{c^2}{c^2}\right)$$

$$L = c^2 m \quad (22)$$

Since $\omega r = v$ Substituting (21) into (20) then:

$$c^2 = \frac{1}{2}\dot{r}^2 + \frac{1}{2}v^2 + \frac{GM}{r}\left(1 - \frac{v^2}{c^2}\right)$$

$$0 = \dot{r}^2 + v^2 + \frac{2GM}{r} - \frac{2GM}{r}\frac{v^2}{c^2} - 2c^2$$

$$c^2 = -c^2 + \dot{r}^2 + v^2 + \frac{2GM}{r} - \frac{2GM}{r}\frac{v^2}{c^2}$$

$$c^2 = -c^2 + \frac{2GM}{r} + \dot{r}^2 + v^2 - \frac{2GM}{r}\frac{v^2}{c^2}$$

$$c^2 = -c^2\left(1 - \frac{2GM}{rc^2}\right) + \dot{r}^2 + v^2\left(1 - \frac{2GM}{rc^2}\right)$$

$$c^2\left(1 - \frac{2GM}{rc^2}\right) = -c^2 + \dot{r}^2 + v^2\left(1 - \frac{2GM}{rc^2}\right)$$

$$\dot{r}^2 = \frac{dr^2}{dt^2} \quad y \quad v^2 = \frac{d\Omega^2}{dt^2}r^2$$

$$c^2 = -\frac{c^2}{\left(1 - \frac{2GM}{rc^2}\right)} + \frac{dr^2}{dt^2\left(1 - \frac{2GM}{rc^2}\right)} + \frac{d\Omega^2 r^2}{dt^2}$$

$$c^2 dt^2 = -\frac{c^2 dt^2}{\left(1 - \frac{2GM}{rc^2}\right)} + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + d\Omega^2 r^2 \quad (23)$$

From (20):

$$dt^2 = d\tau^2\left(1 - \frac{2GM}{rc^2}\right)^2 \quad (24)$$

Replacing (24) in (23)

$$c^2 dt^2 = -c^2 d\tau^2\left(1 - \frac{2GM}{rc^2}\right) + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + d\Omega^2 r^2 \quad (25)$$

$$d\Omega^2 = d\theta^2 + d\varphi^2 sin^2\theta$$

Equation (25) is the true Schwarzschild metric.

## 5. Conclusions

Underlying Einstein's field equation is a simpler field equation than the one he formulated; one with a more physical meaning and without the practically unnecessary mathematical framework of tensor calculus. Perhaps this is why Einstein didn't fully understand his equation, attributing to it a one-dimensional space-time relationship and interpreting it as space being curved. Furthermore, he failed to realize that the formula he so desperately sought was embedded within that field equation, hidden beneath an unnecessary mathematical scaffold.

The Schwarzschild solution is simply the Lagrangian expressed in a different way.

## 6. References


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