Horizons in both original and symmetric Natario warp drives using two parallel and alternative ADM-MTW-Alcubierre formalisms

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Abstract

In 1994 Alcubierre developed the first warp drive theory using the original $3+1$ ADM-MTW formalism.The original ADM-MTW formalism(Arnowitt-Dresner-Misner)(Misner-Thorne-Wheeler) uses both contravariant and covariant shift vector components in its mathematical structure.It possesses mixed shift vector components. Seven years later in 2001 the same original $3+1$ ADM-MTW formalism appeared in the first part of the second warp drive theory developed by Natario.(The second part of Natario theory uses the Hodge Star). In this work we present two new $3+1$ ADM-MTW formalisms:One is the parallel contravariant in which all the shift vector components in its mathematical structure are completely contravariant and the other one is the parallel covariant in which all the shift vector components in its mathematical structure are completely covariant. We describe both the original and symmetric Natario warp drive vectors using the mathematical techniques of these parallel formalisms.We focused ourselves in the $3D$ spherical coordinates for variable speeds. Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.Also a real warp drive must accelerate and de-accelerate. One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble. The behavior of a photon sent to the front of the warp bubble in the case of the original and symmetric Natario warp drive vectors with variable speeds and a lapse function was also one of the main purposes of this work. We presented the behavior of a photon sent to the front of the bubble in the original and symmetric Natario warp drive vectors with the lapse function in these new parallel $3+1$ ADM-MTW formalisms using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provided here the step by step mathematical calculations in order to outline the final results found in our work which are the following ones: For the case of the lapse function or the $3+1$ spacetime in these parallel ADM-MTW formalisms the Horizon do not exists at all.Due to the extra terms in the lapse function or the presence of the $3D$ dimensions that affects the whole spacetime geometry these solutions allows to circumvent the problem of the Horizon.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates in [2]).(see Appendix A for the complete mathematical demonstration of the Natario calculations for the Hodge Star in [2]).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$ or better:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{1}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{2}$$

However Polar Coordinates are not real $3D$ coordinates since it uses only the two dimensional Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

We adopted the second expression above taken from Natario (pg 5 in [1]) to define an alternative warp drive vector that do not uses the Hodge Star but retains all the Natario requirements as will be demonstrated in this work.The final form of the alternative warp drive vector nWD is given by:

$$nWD = 2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{3}$$

Note that this alternative warp drive vector nWD is symmetrical when compared to the second original Natario warp drive vector in the shift vector and Canonical Basis $X^\theta e_\theta$.In the Natario case $X^\theta e_\theta$ is negative $[-v_s(2f + rf')\sin\theta \mathbf{e}_\theta]$ while in the new case $X^\theta e_\theta$ is positive $[+v_s(2f + rf')\sin\theta \mathbf{e}_\theta]$.The symmetry in this case lies over the shift vector X^θ where in the Natario case is $X^\theta = [-v_s(2f + rf')\sin\theta]$ and in our case is $X^\theta = [+v_s(2f + rf')\sin\theta]$

Note also that the symmetrical warp drive vector nWD above uses a constant speed because it was derived from the original Natario warp drive vector nX also with a constant speed.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another alternative warp drive vector must be calculated.Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about $3D$ Spherical Coordinates in [2]).

In this work we present the alternative warp drive vector nWD in $3D$ Spherical Coordinates for variable speeds.Remember that a real spaceship must accelerate or de-accelerate.

The warp drive work that predates Natario by 7 years was written by Alcubierre in 1994.(see [14])

Alcubierre([16]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([15]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pg 67 in [16],see also eq 1 pg 3 in [14]) and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.In this work concerning the ADM formalism we adopt the Alcubierre methodology.

The ADM equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq (21.40) pg 507 in [15]).

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{4}$$

In the equation above α is the so-called lapse function,γ_{ij} is the $3D$ diagonalized induced metric and β^i and β^j are the so-called shift vectors.

In this work we present two new $3+1$ ADM-MTW formalisms:One is the parallel contravariant in which all the shift vector components in its mathematical structure are completely contravariant and the other one is the parallel covariant in which all the shift vector components in its mathematical structure are completely covariant.

The first proposed equation in the $3+1$ parallel contravariant ADM formalism with signature $(-,+,+,+)$ is given by:(see Appendices A and B)

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{5}$$

Expanding all the terms of this equation we will get only contravariant components.

The second proposed equation is the $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ is given by:(see Appendices C and D)

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{6}$$

Expanding all the terms of this equation we will get only covariant components.

We apply these parallel $3+1$ ADM formalisms to the original and symmetrical Natario warp drive vectors in $3D$ spherical coordinates with variable speeds because a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ and remember that a real spaceship must accelerate or de-accelerate.

In a dimensional reduction from $3+1$ to a $1+1$ spacetime these parallel ADM formalisms becomes equal to the original ADM formalism.

In this work we also discuss the Horizon problem for both the original and symmetric Natario warp drive vectors in the parallel $3 + 1$ ADM formalisms contravariant and covariant in Spherical coordinates with the lapse function and variable velocities and we arrive at the conclusion that in the $3 + 1$ spacetime these new 3+1 parallel ADM formalisms dont suffers from the problem of the Horizon.(see Appendices F and H)

Horizons were deeply covered in the warp drive literature but always for constant velocities and without lapse functions in the $1 + 1$ spacetime.(see pg 6 in [1],pg 34 in [23],pgs 268 in [24]).The behavior of a photon sent to the front of the warp bubble in the case of the original and symmetric warp drives with variable velocity and a lapse function) is one of the main purposes of this work.We present the behavior of a photon sent to the front of the bubble in the original and symmetric Natario warp drives in the parallel $3 + 1$ ADM formalisms in Spherical coordinates with the lapse function at variable velocities using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline(or underline or reinforce) the final results found in our work which are the following ones:

In the parallel $3 + 1$ ADM formalisms in Spherical Coordinates the Horizon do not exists at all!!.

In the solutions with $3 + 1$ dimensions with the lapse function the whole spacetime geometries are affected by presence of the $3 + 1$ dimensions and have different results when compared to the solutions with only $1 + 1$ dimensions.These $3 + 1$ dimensions affects the behavior of the Horizon.For results in the $1 + 1$ dimensions see sections 4 and 5 in [13].The behavior of a photon moving in a $3 + 1$ dimensions is completely different than the behavior the same photon moving in a $1+1$ dimensions affecting the Horizon.

Remember that we are presenting our results using step by step mathematics in order to better illustrate our point of view. For the solutions of the quadratic forms in parallel $3+1$ ADM formalisms see the Appendices F and H.These solutions are different than the ones obtained only in $1 + 1$ spacetimes.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes.

This presentation about Horizons is a complement to our works in [13],[34],[35] and [36].The symmmetric Natario warp drive have geometrical advantages in terms of Horizons when compared with the original Natario warp drive.

In order to fully understand the main idea presented in this work(a symmetric warp drive vector nWD in $3D$ Spherical Coordinates obtained independently from the Natario Hodge Star but retaining all the Natario physical features and properties) in the new parallel $3+1$ ADM formalisms acquaintance or familiarity with the Natario original warp drive paper in [1] or familiarity with our works in [2],[13],[34],[35] and [36] are a previous reading requirement.We provide useful mathematical demonstrations QED(Quod Erad Demonstratum) of the process Natario used to obtain the original warp drives using the Hodge Star in the Appendices of these works.

Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates so a photon sent to the front of the bubble fundamentally moves in a $3D$ spacetime.

The Arnowitt-Dresner-Misner(ADM) formalism using the approaches of Misner-Thorne-Wheeler(MTW)([15]) and Alcubierre([16] are a fundamental requirement and we provide in this work all the mathematical demonstrations QED(Quod Erad Demonstratum) in the Appendices from A to D.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical references and Appendices required for the background needed to understand the process used to generate these symmetric warp drive vectors independently from the Natario Hodge Star but retaining all the Natario physical features and properties and also the behavior of a photon sent to the front of the bubble in the original and symmetric Natario warp drive vectors in the new parallel $3+1$ ADM formalisms with the lapse function at variable velocities.

We hope our paper is suitable to fill this proposed task.

This work was designed as a companion work to our works in [2],[12],[13],[34],[35] and [36].

2 The equation of the original Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{7}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result all in [35])

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{8}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{9}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{10}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{11}$$

Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * dvs(t)$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1]).

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x - axis$(like Natario did) or the equatorial plane $x - y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Also the equatorial plane $x - y$ makes an angle of 90 degrees with the $z - axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components reduces to:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{12}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{13}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = 0 \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 0 \tag{14}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = 0 \rightarrow \cos\phi = 0 \tag{15}$$

The remaining contravariant components are:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{16}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{17}$$

In a $1+1$ spacetime the equatorial plane we get¿:

$$nX = X^t e_t + X^r e_r \tag{18}$$

$$X^t = 2rf(r)a \tag{19}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at \tag{20}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(r)at \tag{21}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $f(r)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $f'(r)$ of the shape function $f(r)$ is zero and the shift vector $X^{rs} = 2[2f(r)^2]at$ with $X^r = 0$ inside the bubble and $X^{rs} = 2[2f(r)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

Note that in the dimensional reduction from $3+1$ to a $1+1$ spacetime both spherical coordinates $3D$ and polar coordinates vectors produces the same result.

3 The equation of the symmetrical Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a

The equation of the alternative Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nWD is given by:

$$nWD = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{22}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result all in [36])

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{23}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{24}$$

$$X^\theta = +(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{25}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{26}$$

Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nWD generates a warp drive spacetime if $nWD = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nWD = vs(t) * dx + x * dvs(t)$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1]).

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$(like Natario did) or the equatorial plane $x-y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Also the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components reduces to:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{27}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{28}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = 0 \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 0 \tag{29}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = 0 \rightarrow \cos\phi = 0 \tag{30}$$

The remaining contravariant components are:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{31}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{32}$$

In a $1+1$ spacetime the equatorial plane we get¿:

$$nWD = X^t e_t + X^r e_r \tag{33}$$

$$X^t = 2rf(r)a \tag{34}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at \tag{35}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(r)at \tag{36}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $f(r)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $f'(r)$ of the shape function $f(r)$ is zero and the shift vector $X^{rs} = 2[2f(r)^2]at$ with $X^r = 0$ inside the bubble and $X^{rs} = 2[2f(r)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

Note that in the dimensional reduction from $3+1$ to a $1+1$ spacetime both spherical coordinates $3D$ and polar coordinates vectors produces the same result.

4 Original Natario Vectors and original Natario vectors

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{37}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result all in [35])

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{38}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{39}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{40}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{41}$$

The equation of the Natario warp drive vector nX in polar coordinates with a variable speed vs due to a constant acceleration a is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{42}$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector are defined by(see Appendices A and B for pedagogical purposes and C for the final result all in [35]):

$$X^t = 2f(r)rcos\theta a \tag{43}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{44}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{45}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates.

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a constant speed vs nX is given by::

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{46}$$

With the contravariant shift vector components X^r, X^θ and X^ϕ given by:
(see Appendices F and G for details all in [35])

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{47}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{48}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{49}$$

The equation of the Natario warp drive vector nX in polar coordinates with a constant speed is given by(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{50}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [35])

$$X^r = 2v_s f(r)\cos\theta \tag{51}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{52}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates

In the dimensional reduction from $3+1$ to a $1+1$ spacetime the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates and both spherical coordinates $3D$ and polar coordinates vectors produces the same and identical result.This is due to the fact that Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$ only (like Natario did) or the equatorial plane $x-y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1])

The remaining Natario warp drive vector in polar coordinates $1+1$ spacetime with variable velocities is:

$$nX = X^t e_t + X^r e_r \tag{53}$$

$$X^t = 2rf(r)a \tag{54}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at \tag{55}$$

The remaining Natario warp drive vector nX in polar coordinates $1+1$ spacetime with a constant speed is:

$$nX = X^r e_r \tag{56}$$

$$X^r = 2v_s f(r) \tag{57}$$

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).

The Hodge Star actually must be taken over the product (xvs) giving the expression $nX = *(xvs) = vs * (dx) + x * (dvs)$ but due to a constant speed vs the term $x * d(vs) = 0$.In this work we examine what happens with the Natario vector when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

In this work we already presented new warp drive vectors for variable speeds $nX = vs * (dx) + x * (dvs)$.

The warp drive vector in polar coordinates with constant speed was presented in the Appendices A and B all in [35] and the warp drive vector in $3D$ spherical coordinates with constant speed was presented in the Appendices F and G all in [35]. The warp drive vector in polar coordinates with variable speeds speed was presented in the Appendix C in [35] and the warp drive vector in $3D$ spherical coordinates with variable speeds was presented in the Appendix H in [35].

When in the warp drive vector whether in polar or spherical coordinates the velocity becomes constant the term $x * d(vs)$ disappears and the remaining term is $vs * (dx)$.Note that this term $vs * (dx)$ exists in the constant speed and in the variable speeds warp drive vectors.

In this section we demonstrated the possibility of a dimensional reduction from $3D$ spherical coordinates to polar coordinates in the geometry of warp drive vectors.

We also pointed out that a variable warp drive vector $vs * (dx) + x * (dvs)$ can be reduced to a constant speed warp drive vector $vs * (dx)$ because for constant velocities the term $x * d(vs)$ disappears.

The Appendices M,N,O,P and Q all in [35] outlines the problem of the negative energy density distribution for the Natario warp drive in polar coordinates with constant speeds.

This negative energy is in front of the ship able to deflect incoming hazardous objects from the interstellar space avoiding dangerous collisions between the ship and the Interstellar Medium IM.Doppler blueshifted photons or space dust or debris would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [17])

We dont have the negative energy density distribution for the $3D$ spherical or accelerated warp drive vectors but since the Natario warp drive in polar coordinates with constant speeds is a particular case of these new warp drive vectors we hope that in these warp drive spacetimes the negative energy density also remains in front of the ship.

Otherwise we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors if these calculations are made "by the hand"..

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pg 342 in [15], pg 276 in [19],pgs $454, 457, 560$ in [20] pg 98 in [21],pg 178 in [22]).

Appendix C pgs $551 - 555$ in [20] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3 + 1$ spacetime metric using $Mathematica$.[1]

[1]Unfortunately we dont have access to anyone of these programs so we have our hands "tied up"

5 Symmetrical Natario Vectors and symmetrical Natario vectors

The equation of the alternative Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nWD is given by:

$$nWD = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{58}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result all in [36])

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{59}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{60}$$

$$X^\theta = +(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{61}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{62}$$

The equation of the alternative Natario warp drive vector nWD in polar coordinates with a variable speed vs due to a constant acceleration a is given by:

$$nWD = X^t e_t + X^r e_r + X^\theta e_\theta \tag{63}$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector are defined by(see Appendices A and B for pedagogical purposes and C for the final result all in [36]):

$$X^t = 2f(r)rcos\theta a \tag{64}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{65}$$

$$X^\theta = +2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{66}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates.

The equation of the alternative Natario warp drive vector in $3D$ spherical coordinates with a constant speed vs nWD is given by::

$$nWD = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{67}$$

With the contravariant shift vector components X^r, X^θ and X^ϕ given by:
(see Appendices F and G for details all in [36])

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{68}$$

$$X^\theta = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{69}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{70}$$

The equation of the alternative Natario warp drive vector nWD in polar coordinates with a constant speed is given by:

$$nWD = X^r e_r + X^\theta e_\theta \tag{71}$$

With the contravariant shift vector components X^r and X^θ given by:(see Appendices A and B for details all in [36])

$$X^r = 2v_s f(r)\cos\theta \tag{72}$$

$$X^\theta + -v_s(2f(r) + (r)f'(r))\sin\theta \tag{73}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates

In the dimensional reduction from $3+1$ to a $1+1$ spacetime the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates and both spherical coordinates $3D$ and polar coordinates vectors produces the same and identical result.This is due to the fact that Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$ only (like Natario did) or the equatorial plane $x-y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1])

The remaining alternative Natario warp drive vector in polar coordinates $1+1$ spacetime with variable velocities is:

$$nWD = X^t e_t + X^r e_r \tag{74}$$

$$X^t = 2rf(r)a \tag{75}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at \tag{76}$$

The remaining alternative Natario warp drive vector nWD in polar coordinates $1+1$ spacetime with a constant speed is:

$$nWD = X^r e_r \tag{77}$$

$$X^r = 2v_s f(r) \tag{78}$$

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).

The Hodge Star actually must be taken over the product (xvs) giving the expression $nX = *(xvs) = vs * (dx) + x * (dvs)$ but due to a constant speed vs the term $x * d(vs) = 0$.In this work we examine what happens with the Natario vector when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

In this work we already presented alternative warp drive vectors for variable speeds $nX = vs*(dx)+x*(dvs)$.

The alternative warp drive vector in polar coordinates with constant speed was presented in the Appendices A and B all in [36] and the alternative warp drive vector in $3D$ spherical coordinates with constant speed was presented in the Appendices F and G all in [36].The warp drive vector in polar coordinates with variable speeds speed was presented in the Appendix C in [36] and the warp drive vector in $3D$ spherical coordinates with variable speeds was presented in the Appendix H in [36].

When in the warp drive vector whether in polar or spherical coordinates the velocity becomes constant the term $x * d(vs)$ disappears and the remaining term is $vs * (dx)$.Note that this term $vs * (dx)$ exists in the constant speed and in the variable speeds warp drive vectors.

In this section we demonstrated the possibility of a dimensional reduction from $3D$ spherical coordinates to polar coordinates in the geometry of warp drive vectors.The alternative Natario vector in the $1+1$ spacetime is equal to the original Natario vector also in the $1+1$ spacetime.

We also pointed out that a variable alternative warp drive vector $vs*(dx)+x*(dvs)$ can be reduced to a constant speed alternative warp drive vector $vs*(dx)$ because for constant velocities the term $x*d(vs)$ disappears.

The Appendices M,N,O,P and Q in [36] outlines the problem of the negative energy density distribution for the original Natario warp drive in polar coordinates with constant speeds.

This negative energy is in front of the ship able to deflect incoming hazardous objects from the interstellar space avoiding dangerous collisions between the ship and the Interstellar Medium IM.Doppler blueshifted photons or space dust or debris would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [17])

We dont have the negative energy density distribution for the $3D$ spherical or accelerated alternative warp drive vectors but since the Natario warp drive in polar coordinates with constant speeds is a particular case of these new warp drive vectors when dimensionally reduced from $3+1$ to $1+1$ spacetimes we hope that in these alternative warp drive spacetimes the negative energy density also remains in front of the ship.

Otherwise we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors if these calculations are made "by the hand"..

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pg 342 in [15], pg 276 in [19],pgs $454, 457, 560$ in [20] pg 98 in [21],pg 178 in [22]).

Appendix C pgs $551-555$ in [20] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3+1$ spacetime metric using $Mathematica$.[2]

Consider motion in the $x-axis$ only (like Natario did) or the equatorial plane $x-y$ in r where $\theta=0$ $\sin(\theta)=0$ and $\cos(\theta)=1$.(see pgs 4,5 and 6 in [1]) and grouping together both the original and alternative Natario warp drive vectors:

$$nX = 2v_s f\cos\theta\mathbf{e}_r - v_s(2f+rf')\sin\theta\mathbf{e}_\theta \tag{79}$$

$$nWD = 2v_s f\cos\theta\mathbf{e}_r + v_s(2f+rf')\sin\theta\mathbf{e}_\theta \tag{80}$$

When $\sin(\theta)=0$ it is easy to see why the alternative Natario vector in the $1+1$ spacetime is equal to the original Natario vector also in the $1+1$ spacetime.

[2]Unfortunately we dont have access to anyone of these programs so we have our hands "tied up"

6 Horizons in the parallel contravariant $3+1$ ADM-MTW-Alcubierre formalism with the lapse function α in Spherical Coordinates

In the Appendix F we solved the null-like geodesics $ds^2 = 0$ for the Horizons in the case of the parallel contravariant $3+1$ ADM equations with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i + \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X^1 + X^2 + X^3 + \alpha) \quad (81)$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i - \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X^1 + X^2 + X^3 - \alpha) \quad (82)$$

The original Natario warp drive vector contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ (see Appendices F and G for pedagogical purposes and H for the final result all in [35]) in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX are given by:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \quad (83)$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = X^1 \quad (84)$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = X^2 \quad (85)$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = X^3 \quad (86)$$

The symmetrical Natario warp drive vector contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ (see Appendices F and G for pedagogical purposes and H for the final result all in [36]) in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nWD are given by:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \quad (87)$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = X^1 \quad (88)$$

$$X^\theta = +(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = X^2 \quad (89)$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = X^3 \quad (90)$$

Remember also from Appendices A and B that the induced metric $dl^2 = \gamma_{ii}dx^idx^i = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$ with $\gamma_{11} = \gamma_{rr} = 1$, $\gamma_{22} = \gamma_{\theta\theta} = r^2$ and $\gamma_{33} = \gamma_{\phi\phi} = r^2\sin^2\theta$.

Still from Appendix B we know that the lapse function α can be given by:

$$\alpha = (1 - X^t) \quad (91)$$

We now have all the key ingredients needed to compute the Horizons in a tridimensional spacetime:

$$\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}=\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X^1+X^2+X^3+\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi+\alpha) \tag{92}$$

$$\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}=\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X^1+X^2+X^3-\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi-\alpha) \tag{93}$$

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi+\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi+(1-X^t)) \tag{94}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi-\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi-(1-X^t)) \tag{95}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi+(1-X^t))=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi+1-X^t) \tag{96}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi-(1-X^t))=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X^r+X^\theta+X^\phi-1+X^t)) \tag{97}$$

The situation in which a photon is sent to the front of the bubble is illustrated by the terms containing $-\alpha$ or $-(1-X^t)$.In the parallel contravariant $3+1$ ADM-MTW-Alcubierre formalism with the lapse function α in Spherical Coordinates the Horizon never occurs because the term $\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}$ is never zero hence a photon sent to the front of the bubble never stops and the Horizon never appears.The term $\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}$ cannot be zero otherwise an Horizon would appear.

The symmetrical Natario warp drive behaves better than its original counterpart because the term X^θ is always positive.

The Horizon occurs in the parallel contravariant $1+1$ ADM-MTW-Alcubierre formalism with constant speeds as depicted in the end of the Appendix E because in this case $\frac{dr}{dt}=0$ in some circumstances and the photon stops.

The $3+1$ dimensions and the lapse function affects the whole spacetime geometry avoiding the appearance of the Horizons.

Of course this point of view about the Horizons reflects only the geometrical point of view of the original and symmetrical Natario warp drive equations for variable speed vs in a $3+1$ spacetime.But we expect in these cases that the negative energy density may covers the entire bubble.Since the negative energy density have repulsive gravitational behavior(see pg 116 in [17]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.The original and symmetrical Natario warp drive equations in the $1+1$ spacetime are equivalent possessing negative energy in front of the ship.

The solution that allows contact with the bubble walls was presented in pg 83 in [18].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [18] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [23],pg 268 in [24] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

7 Horizons in the parallel covariant $3+1$ ADM-MTW-Alcubierre formalism with the lapse function α in Spherical Coordinates

In the Appendix H we solved the null-like geodesics $ds^2 = 0$ in the case of the parallel covariant $3+1$ ADM equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i+\alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1+X_2+X_3+\alpha) \tag{98}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i-\alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1+X_2+X_3-\alpha) \tag{99}$$

The original Natario warp drive vector covariant shift vector components X_t,X_{rs}, X_θ and X_ϕ (Obtained from the contravariant components of the previous section multiplied by the induced metric coefficients $X_i = \gamma_{ii}X^i$) in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX are given by:

$$X_t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{100}$$

$$X_r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = X_1 \tag{101}$$

$$X_\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta)(r^2) = X_2 \tag{102}$$

$$X_\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta)(r^2\sin^2\theta) = X_3 \tag{103}$$

The symmetrical Natario warp drive vector covariant shift vector components X_t,X_{rs}, X_θ and X_ϕ (Obtained from the contravariant components of the previous section multiplied by the induced metric coefficients $X_i = \gamma_{ii}X^i$) in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nWD are given by:

$$X_t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{104}$$

$$X_r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = X_1 \tag{105}$$

$$X_\theta = +(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta)(r^2) = X_2 \tag{106}$$

$$X_\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta)(r^2\sin^2\theta) = X_3 \tag{107}$$

Remember also from Appendices A and B that the induced metric $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ with $\gamma_{11} = \gamma_{rr} = 1$, $\gamma_{22} = \gamma_{\theta\theta} = r^2$ and $\gamma_{33} = \gamma_{\phi\phi} = r^2\sin^2\theta$.

Still from Appendix D we know that the lapse function α can be given by:

$$\alpha = (1 - X_t) \tag{108}$$

We now have all the key ingredients needed to compute the Horizons in a tridimensional spacetime:

$$\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}=\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1+X_2+X_3+\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi+\alpha) \quad (109)$$

$$\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}=\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1+X_2+X_3-\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi-\alpha) \quad (110)$$

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi+\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi+(1-X_t)) \quad (111)$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi-\alpha)=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi-(1-X_t)) \quad (112)$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi+(1-X_t))=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi+1-X_t) \quad (113)$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi-(1-X_t))=\frac{\sqrt{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}}{\gamma_{rr}+\gamma_{\theta\theta}+\gamma_{\phi\phi}}(X_r+X_\theta+X_\phi-1+X_t)) \quad (114)$$

The situation in which a photon is sent to the front of the bubble is illustrated by the terms containing $-\alpha$ or $-(1-X_t)$.In the parallel covariant $3+1$ ADM-MTW-Alcubierre formalism with the lapse function α in Spherical Coordinates the Horizon never occurs because the term $\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}$ is never zero hence a photon sent to the front of the bubble never stops and the Horizon never appears.The term $\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}$ cannot be zero otherwise an Horizon would appear.

The symmetrical Natario warp drive behaves better than its original counterpart because the term X_θ is always positive.

The Horizon occurs in the parallel covariant $1+1$ ADM-MTW-Alcubierre formalism with constant speeds as depicted in the end of the Appendix G because in this case $\frac{dr}{dt}=0$ in some circumstances and the photon stops.

The $3+1$ dimensions and the lapse function affects the whole spacetime geometry avoiding the appearance of the Horizons.

Of course this point of view about the Horizons reflects only the geometrical point of view of the original and symmetrical Natario warp drive equations for variable speed vs in a $3+1$ spacetime.But we expect in these cases that the negative energy density may covers the entire bubble.Since the negative energy density have repulsive gravitational behavior(see pg 116 in [17]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.The original and symmetrical Natario warp drive equations in the $1+1$ spacetime are equivalent possessing negative energy in front of the ship.

The solution that allows contact with the bubble walls was presented in pg 83 in [18].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [18] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [23],pg 268 in [24] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

8 Conclusion

In 1994 Alcubierre developed the first warp drive theory using the original $3+1$ ADM-MTW formalism.The original ADM-MTW formalism(Arnowitt-Dresner-Misner)(Misner-Thorne-Wheeler) uses both contravariant and covariant shift vector components in its mathematical structure.It possesses mixed shift vector components.

Seven years later in 2001 the same original $3+1$ ADM-MTW formalism appeared in the first part of the second warp drive theory developed by Natario.(The second part of Natario theory uses the Hodge Star).

In this work we present two new $3+1$ ADM-MTW formalisms:One is the parallel contravariant in which all the shift vector components in its mathematical structure are completely contravariant and the other one is the parallel covariant in which all the shift vector components in its mathematical structure are completely covariant.

We describe both the original and symmetric Natario warp drive vectors using the mathematical techniques of these parallel formalisms.We focused ourselves in the $3D$ spherical coordinates for variable speeds.

Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.Also a real warp drive must accelerate and de-accelerate.

One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble. The behavior of a photon sent to the front of the warp bubble in the case of the original and symmetric Natario warp drive vectors with variable speeds and a lapse function was also one of the main purposes of this work.

We presented the behavior of a photon sent to the front of the bubble in the original and symmetric Natario warp drive vectors with the lapse function in these new parallel $3+1$ ADM-MTW formalisms using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provided here the step by step mathematical calculations in order to outline the final results found in our work which are the following ones:

For the case of the lapse function or the $3+1$ spacetime in these parallel ADM-MTW formalisms the Horizon do not exists at all.Due to the extra terms in the lapse function or the presence of the $3D$ dimensions that affects the whole spacetime geometry these solutions allows to circumvent the problem of the Horizon.

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and the Natario spacetime offers an excellent protection to the crew members as depicted in the works [7],[8],[9] and [10].However since these works were based in the original Natario 2001 paper this line of reason must be extended to encompass the alternative $3D$ spherical coordinates symmetrical warp drive vector

In the original Natario warp drive in polar coordinates the negative energy density with repulsive gravitational behavior lies in front of the ship protecting the ship from hazardous interstellar collisions even in a $1+1$ spacetime.(see Appendices M,N,O,P and Q both in [35] and [36]).

Our proposed alternative Natario warp drive vectors in $3D$ spherical coordinates when reduced to a $1+1$ spacetime gives the same results of the original Natario warp drive in polar coordinates in a $1+1$ spacetime so it is reasonable to suppose that the negative energy density with repulsive gravitational behavior lies in front of the ship also in these $3+1$ spacetimes.

In a dimensional reduction from the $3+1$ to a $1+1$ spacetimes the parallel ADM-MTW formalisms are equivalent to the original ADM-MTW formalism.

We dont know the negative energy density distribution of both the original and symmetric warp drive vectors in $3D$ spherical coordinates using these parallel ADM-MTW formalisms because we dont have the $Maple$ or $Mathematica$ programs needed to handle all the tensor algebra

For the $Maple$ or $Mathematica$ see pg 342 in [15], pg 276 in [19],pgs $454, 457, 560$ in [20] pg 98 in [21],pg 178 in [22],see also Appendix C pgs $551-555$ in [20].

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pg 98 in [25] and pg 377 in [26]).

For the experimental verification of the acceleration of the Universe see for example the bottom of pg 355 and top of pg 356 eq 8.155 in [28].

9 Appendix A:The parallel contravariant $3+1$ ADM formalism according to MTW and Alcubierre for a constant speed vs

A $3+1$ ADM contravariant formalism parallel to the original $3+1$ ADM formalism according with the equation (21.40) pg [507] in [15][3]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{115}$$

using the signature $(-,+,+,+)$ can be given by:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{116}$$

Note that in the equation above all the essential 3 elements of the original $3+1$ ADM formalism are also present.These elements are:

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)$.$\beta^i$ is known as the contravariant shift vector.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{117}$$

$$(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + (\beta^i dt)^2 \tag{118}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + (\beta^i dt)^2 \tag{119}$$

$$ds^2 = -\alpha^2 dt^2 + (\beta^i dt)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}(dx^i)^2 \tag{120}$$

$$ds^2 = (-\alpha^2 + [\beta^i]^2)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{121}$$

$$ds^2 = (-\alpha^2 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{122}$$

[3]we adopt the Alcubierre notation here

Then the equations of the parallel contravariant $3+1$ ADM formalism are given by:

$$ds^2 = (-\alpha^2 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{123}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{124}$$

The components of the inverse metric are given by the matrix inverse :[4]

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{125}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-\alpha^2 + \beta^i\beta^i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\alpha^2 + \beta^i\beta^i \end{pmatrix} \tag{126}$$

Suppressing the lapse function $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{127}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{128}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{129}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-1 + \beta^i\beta^i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -1 + \beta^i\beta^i \end{pmatrix} \tag{130}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ we should expect for:

$$ds^2 = (1 - \beta^i\beta^i)dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{131}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{132}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{133}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i\beta^i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta^i \times -\sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i\beta^i \end{pmatrix} \tag{134}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i\beta^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i\beta^i \end{pmatrix} \tag{135}$$

[4]see Wikipedia:the free Encyclopedia on inverse or invertible matrices

The equations of the parallel contravariant $3+1$ ADM formalism are given by:

$$ds^2 = (1-\beta^i\beta^i)dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{136}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1-\beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{137}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00}\times g_{ii}) - (g_{i0}\times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{138}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1-\beta^i\beta^i]\times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta^i \times -\sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1-\beta^i\beta^i \end{pmatrix} \tag{139}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1-\beta^i\beta^i]\times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1-\beta^i\beta^i \end{pmatrix} \tag{140}$$

obeys the generic equation of a warp drive in the parallel contravariant $3+1$ ADM formalism:

$$ds^2 = dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{141}$$

For a generic coordinates system in contravariant form we must employ the equation given by the parallel contravariant $3+1$ ADM formalism as being:

$$ds^2 = dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X^i dt)^2 \tag{142}$$

Note that $\beta^i = -X^i$ and $\beta^i\beta^i = X^iX^i$ with X^i being the generic original or alternative Natario contravariant shift vectors components for constant speeds obtained from sections 4 and 5.Hence we have:

$$ds^2 = (1-X^iX^i)dt^2 + 2\sqrt{\gamma_{ii}}X^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{143}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1-X^iX^i & \sqrt{\gamma_{ii}}X^i \\ \sqrt{\gamma_{ii}}X^i & -\gamma_{ii} \end{pmatrix} \tag{144}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00}\times g_{ii}) - (g_{i0}\times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{145}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1-X^iX^i]\times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}X^i \times \sqrt{\gamma_{ii}}X^i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}}X^i \\ -\sqrt{\gamma_{ii}}X^i & 1-X^iX^i \end{pmatrix} \tag{146}$$

Note that all the shift vectors in the equations above are the original or alternative Natario contravariant shift vector components.This is the reason why we call this formalism as the parallel contravariant $3+1$ ADM.

10 Appendix B:The parallel contravariant $3+1$ ADM formalism according to MTW and Alcubierre for a variable speed vs

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65] in [16].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65] in [16])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66] in [16]) illustrated by the equation :[5]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{147}$$

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)$.$\beta^i$ is known as the contravariant shift vector.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3 + 1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{148}$$

$$ds^2 = (-\alpha^2 + \beta^i \beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{149}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i \beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{150}$$

[5] we adopt also the Alcubierre notation here

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = (\sqrt{\gamma_{tt}} + \beta^t)^2 = (\gamma_{tt} + 2\sqrt{\gamma_{tt}}\beta^t + \beta^t\beta^t) \tag{151}$$

Remember that here we are working with geometrized units in which $c = 1$ so $\gamma_{tt} = 1$

$$\alpha^2 = (1 + 2\beta^t + \beta^t\beta^t) \tag{152}$$

$$\alpha^2 = (1 + \beta^t)^2 \tag{153}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)^2 \tag{154}$$

$$ds^2 = (-\alpha^2 + \beta^i\beta^i) dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii} dx^i dx^i \tag{155}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{156}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -(1 + 2\beta^t + \beta^t\beta^t) dt^2 + (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)^2 \tag{157}$$

$$ds^2 = (-(1 + 2\beta^t + \beta^t\beta^t) + \beta^i\beta^i) dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii} dx^i dx^i \tag{158}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -(1 + 2\beta^t + \beta^t\beta^t) + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{159}$$

Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \alpha^2 dt^2 - (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)^2 \tag{160}$$

$$ds^2 = (\alpha^2 - \beta^i\beta^i) dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{161}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{162}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (1 + 2\beta^t + \beta^t\beta^t) dt^2 - (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)^2 \tag{163}$$

$$ds^2 = ((1 + 2\beta^t + \beta^t\beta^t) - \beta^i\beta^i) dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{164}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} (1 + 2\beta^t + \beta^t\beta^t) - \beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{165}$$

For a generic coordinates system we must employ the equation that obeys the parallel contravariant $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} (\sqrt{\gamma_{ii}} dx^i - X^i dt)^2 \tag{166}$$

Note that $\beta^t = -X^t$,$\beta^i = -X^i$ and $\beta^i \beta^i = X^i X^i$ with X^i being the generic original or alternative Natario contravariant shift vectors components for variable speeds obtained from sections 4 and 5.

The term α^2 is now defined by the following expression:

$$\alpha^2 = (\sqrt{\gamma_{tt}} - X^t)^2 = (\gamma_{tt} - 2\sqrt{\gamma_{tt}} X^t + X^t X^t) \tag{167}$$

Suppressing the summing convention parallel contravariant $3+1$ ADM formalism becomes:

$$ds^2 = (\sqrt{\gamma_{tt}} - X^t)^2 dt^2 - (\sqrt{\gamma_{ii}} dx^i - X^i dt)^2 \tag{168}$$

Note that in the equation given above a remarkable mathematical structure appears:the term $\sqrt{\gamma_{tt}}$ is associated with the time coordinate while the term $\sqrt{\gamma_{ii}}$ is associated with the remaining spatial coordinates.

Remember that here we are working with geometrized units in which $c = 1$ so $\gamma_{tt} = 1$ hence the term α^2 can be given by:

$$\alpha^2 = (1 - X^t)^2 = (1 - 2X^t + X^t X^t) \tag{169}$$

The parallel contravariant $3+1$ ADM formalism now becomes:

$$ds^2 = (1 - X^t)^2 dt^2 - (\sqrt{\gamma_{ii}} dx^i - X^i dt)^2 \tag{170}$$

$$ds^2 = (1 - 2X^t + X^t X^t - X^i X^i) dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{171}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X^t + X^t X^t - X^i X^i & \sqrt{\gamma_{ii}} X^i \\ \sqrt{\gamma_{ii}} X^i & -\gamma_{ii} \end{pmatrix} \tag{172}$$

$$ds^2 = (\alpha^2 - X^i X^i) dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{173}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - X^i X^i & \sqrt{\gamma_{ii}} X^i \\ \sqrt{\gamma_{ii}} X^i & -\gamma_{ii} \end{pmatrix} \tag{174}$$

Note that all the shift vectors in the equations above are the original or alternative Natario contravariant shift vector components.This is the reason why we call this formalism as the parallel contravariant $3+1$ ADM.

11 Appendix C:The parallel covariant $3+1$ ADM formalism according to MTW and Alcubierre for a constant speed vs

A $3+1$ ADM covariant formalism parallel to the original $3+1$ ADM formalism according with the equation (21.40) pg [507] in [15][6]

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{175}$$

using the signature $(-,+,+,+)$ can be given by:

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{176}$$

Note that in the equation above all the essential 3 elements of the original $3+1$ ADM formalism are also presen.These elements are:

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β_i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)$.$\beta_i$ is known as the covariant shift vector defined as : $\beta_i = \gamma_{ij}\beta^j$.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{177}$$

$$(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + (\beta_i dt)^2 \tag{178}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + (\beta_i dt)^2 \tag{179}$$

$$ds^2 = -\alpha^2 dt^2 + (\beta_i dt)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}(dx^i)^2 \tag{180}$$

$$ds^2 = (-\alpha^2 + [\beta_i]^2)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{181}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{182}$$

[6] we adopt also the Alcubierre notation here

Then the equations of the parallel covariant $3+1$ ADM formalism are given by:

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{183}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{184}$$

The components of the inverse metric are given by the matrix inverse :[7]

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{185}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-\alpha^2 + \beta_i\beta_i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\alpha^2 + \beta_i\beta_i \end{pmatrix} \tag{186}$$

Suppressing the lapse function $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{187}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{188}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{189}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-1 + \beta_i\beta_i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -1 + \beta_i\beta_i \end{pmatrix} \tag{190}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ we should expect for:

$$ds^2 = (1 - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{191}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{192}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{193}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta_i \times -\sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{194}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{195}$$

[7] see Wikipedia:the free Encyclopedia on inverse or invertible matrices

The equations of the parallel covariant $3+1$ ADM formalism given by:

$$ds^2 = (1 - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{196}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{197}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{198}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta_i \times -\sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{199}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{200}$$

obeys the generic equation of the parallel covariant $3+1$ ADM formalism:

$$ds^2 = dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{201}$$

For a generic coordinates system in covariant form we must employ the equation given by the parallel covariant $3+1$ ADM formalism as being:

$$ds^2 = dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{202}$$

with $X_i = \gamma_{ii}X^i$

Note that $\beta_i = -X_i$ and $\beta_i\beta_i = X_iX_i$ with X_i being the generic original or alternative Natario covariant shift vectors components for constant speeds obtained from sections 4 and 5.Hence we have:

$$ds^2 = (1 - X_iX_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{203}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_iX_i & \sqrt{\gamma_{ii}}X_i \\ \sqrt{\gamma_{ii}}X_i & -\gamma_{ii} \end{pmatrix} \tag{204}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{205}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X_iX_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}X_i \times \sqrt{\gamma_{ii}}X_i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}}X_i \\ -\sqrt{\gamma_{ii}}X_i & 1 - X_iX_i \end{pmatrix} \tag{206}$$

Note that all the shift vectors in the equations above are the original or alternative Natario covariant shift vector components.This is the reason why we call this formalism as the parallel covariant $3+1$ ADM.

12 Appendix D:The parallel covariant $3+1$ ADM formalism according to MTW and Alcubierre for a variable speed vs

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65] in [16].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65] in [16])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66] in [16]) illustrated by the equation :[8]

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{207}$$

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.
- 3)-the relative velocity β_i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)$.$\beta_i$ is known as the covariant shift vector defined as : $\beta_i = \gamma_{ij}\beta^j$.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{208}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{209}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{210}$$

[8]we adopt also the Alcubierre notation here

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = (\sqrt{\gamma_{tt}} + \beta_t)^2 = (\gamma_{tt} + 2\sqrt{\gamma_{tt}}\beta_t + \beta_t\beta_t) \tag{211}$$

Remember that here we are working with geometrized units in which $c = 1$ so $\gamma_{tt} = 1$

$$\alpha^2 = (1 + 2\beta_t + \beta_t\beta_t) \tag{212}$$

$$\alpha^2 = (1 + \beta_t)^2 \tag{213}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{214}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{215}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{216}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -(1 + 2\beta_t + \beta_t\beta_t)dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{217}$$

$$ds^2 = (-(1 + 2\beta_t + \beta_t\beta_t) + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{218}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -(1 + 2\beta_t + \beta_t\beta_t) + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{219}$$

Changing the signature from $(-, +, +, +)$ to signature $(+, -, -, -)$ we have:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \alpha^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{220}$$

$$ds^2 = (\alpha^2 - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{221}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{222}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (1 + 2\beta_t + \beta_t\beta_t)dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{223}$$

$$ds^2 = ((1 + 2\beta_t + \beta_t\beta_t) - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{224}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} (1 + 2\beta_t + \beta_t\beta_t) - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{225}$$

For a generic coordinates system we must employ the equation that obeys the parallel covariant $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{226}$$

The term α^2 is now defined by the following expression:

$$\alpha^2 = (\sqrt{\gamma_{tt}} - X_t)^2 = (\gamma_{tt} - 2\sqrt{\gamma_{tt}}X_t + X_t X_t) \tag{227}$$

with $X_i = \gamma_{ii}X^i$

Note that $\beta_i = -X_i$ and $\beta_i\beta_i = X_i X_i$ with X_i being the generic original or alternative Natario covariant shift vectors components for variable speeds obtained from sections 4 and 5.Hence we have:

Suppressing the summing convention then the equation in the parallel covariant $3+1$ ADM formalism now becomes:

$$ds^2 = (\sqrt{\gamma_{tt}} - X_t)^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{228}$$

Note that in the equation given above a remarkable mathematical structure appears:the term $\sqrt{\gamma_{tt}}$ is associated with the time coordinate while the term $\sqrt{\gamma_{ii}}$ is associated with the remaining spatial coordinates.

Remember that here we are working with geometrized units in which $c=1$ so $\gamma_{tt}=1$ hence the term α^2 can be given by:

$$\alpha^2 = (1 - X_t)^2 = (1 - 2X_t + X_t X_t) \tag{229}$$

The equation in the parallel covariant $3+1$ ADM formalism now becomes:

$$ds^2 = (1 - X_t)^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{230}$$

$$ds^2 = (1 - 2X_t + X_t X_t - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{231}$$

$$ds^2 = (\alpha^2 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{232}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2Xt + X_t X_t - X_i X_i & \sqrt{\gamma_{ii}}X_i \\ \sqrt{\gamma_{ii}}X_i & -\gamma_{ii} \end{pmatrix} \tag{233}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - X_i X_i & \sqrt{\gamma_{ii}}X_i \\ \sqrt{\gamma_{ii}}X_i & -\gamma_{ii} \end{pmatrix} \tag{234}$$

Note that all the shift vectors in the equations above are the original or alternative Natario covariant shift vector components.This is the reason why we call this formalism as the parallel covariant $3+1$ ADM.

13 Appendix E:Generic quadratic forms in the parallel contravariant $3+1$ ADM formalism without the lapse function.

Using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity,Horizons can be easily computed for the dimensionally reduced $1+1$ spacetime versions of the equations because only the quadratic form dr^2 exists but in the $3+1$ spacetime we have the presence of 3 quadratic forms respectively dr^2,$r^2 d\theta^2$ and $r^2 \sin^2\theta d\phi^2$.Algebraic solutions for the null-like geodesics $ds^2 = 0$ of General Relativity of the $3+1$ equations above are extremely difficult due to the presence of these 3 quadratic forms considering solutions for each quadratic form dr^2 or $r^2 d\theta^2$ or $r^2 \sin^2\theta d\phi^2$ isolated.

The best effort to solve the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above is to find out a solution that encompasses all the 3 quadratic forms dr^2 and $r^2 d\theta^2$ and $r^2 \sin^2\theta d\phi^2$ grouped together.

We will demonstrate all the required mathematics step by step.

Back to the parallel contravariant $3+1$ ADM formalism compact generic equation given below:(see Appendix A)

$$ds^2 = dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X^i dt)^2 \tag{235}$$

Expanding the equation above we have:

$$ds^2 = (1 - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}}X^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{236}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (1 - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}}X^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{237}$$

Dividing by dt^2 we have:

$$0 = (1 - X^i X^i) + 2\sqrt{\gamma_{ii}}X^i \frac{dx^i dt}{dt^2} - \gamma_{ii}\frac{(dx^i)^2}{dt^2} \tag{238}$$

$$0 = (1 - X^i X^i) + 2\sqrt{\gamma_{ii}}X^i \frac{dx^i}{dt} - \gamma_{ii}(\frac{dx^i}{dt})^2 \tag{239}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{240}$$

We have now a generic quadratic form in the term U^i:

$$0 = (1 - X^i X^i) + 2\sqrt{\gamma_{ii}}X^i U^i - \gamma_{ii}(U^i)^2 \tag{241}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2\sqrt{\gamma_{ii}}X^i U^i - (1 - X^i X^i) = 0 \tag{242}$$

$$\gamma_{ii}(U^i)^2 - 2\sqrt{\gamma_{ii}}X^iU^i + (X^iX^i - 1) = 0 \tag{243}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2\sqrt{\gamma_{ii}}X^i \pm \sqrt{[-2\sqrt{\gamma_{ii}}X^i]^2 - 4[\gamma_{ii}(X^iX^i - 1)]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X^i \pm \sqrt{4\gamma_{ii}[X^i]^2 - 4[\gamma_{ii}(X^iX^i] + 4[\gamma_{ii}]}}{2\gamma_{ii}} \tag{244}$$

$$U^i = \frac{2\sqrt{\gamma_{ii}}X^i \pm \sqrt{4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X^i \pm 2\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{\sqrt{\gamma_{ii}}X^i \pm \sqrt{[\gamma_{ii}]}}{\gamma_{ii}} = \frac{\sqrt{[\gamma_{ii}]}}{\gamma_{ii}}(X^i \pm 1) \tag{245}$$

At last we have the final solution of this generic quadratic form in the term U^i given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X^i \pm 1) \tag{246}$$

But this expression actually means:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i \pm 1)] \tag{247}$$

The subscript γ_{ii} as being a sum $\sum_{i=1}^{3}\gamma_{ii}$ is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [27]).Then $\sum_{i=1}^{3}\sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3}\gamma_{ii}}$.The same is valid for X^i also as a sum $\sum_{i=1}^{3}X^i$ implying also in $\sum_{i=1}^{3}\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{\sqrt{\sum_{i=1}^{3}\gamma_{ii}}}{\sum_{i=1}^{3}\gamma_{ii}}$. Remember the Einstein summing convention in which $X^ie_i = \sum_{i=1}^{3}X^ie_i = X^1e_1 + X^2e_2 + X^3e_3$ and $\sqrt{X^ie_i} = \sqrt{X^1e_1 + X^2e_2 + X^3e_3}$.

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i \pm 1)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 \pm 1) \tag{248}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X^i + 1) \tag{249}$$

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X^i - 1) \tag{250}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i + 1)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 + 1) \tag{251}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i - 1)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 - 1) \tag{252}$$

$$\sum_{i=1}^{3}[U^i]=\sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i+1)]=\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X^1+X^2+X^3+1) \tag{253}$$

$$\sum_{i=1}^{3}[U^i]=\sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i-1)]=\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X^1+X^2+X^3-1) \tag{254}$$

We solved the null-like geodesics $ds^2=0$ in the case of the parallel contravariant $3+1$ ADM equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}$.

For the parallel contravariant $2+1$ ADM equations we have:

$$\sum_{i=1}^{2}[U^i]=\sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X^i+1)]=\frac{dx^1}{dt}+\frac{dx^2}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}}}{\gamma_{11}+\gamma_{22}}(X^1+X^2+1) \tag{255}$$

$$\sum_{i=1}^{2}[U^i]=\sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X^i-1)]=\frac{dx^1}{dt}+\frac{dx^2}{dt}=\frac{\sqrt{\gamma_{11}+\gamma_{22}}}{\gamma_{11}+\gamma_{22}}(X^1+X^2-1) \tag{256}$$

We solved the null-like geodesics $ds^2=0$ in the case of the parallel contravariant $2+1$ ADM equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt}+\frac{dx^2}{dt}$.

For the parallel contravariant $1+1$ ADM equations we have:

$$\sum_{i=1}^{1}[U^i]=\sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X^i+1)]=\frac{dx^1}{dt}=\frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X^1+1) \tag{257}$$

$$\sum_{i=1}^{1}[U^i]=\sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X^i-1)]=\frac{dx^1}{dt}=\frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X^1-1) \tag{258}$$

We solved the null-like geodesics $ds^2=0$ in the case of the parallel contravariant $1+1$ ADM equations given above with the solution that encompasses the quadratic form $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

Take the original or alternative Natario vectors which in $1+1$ spacetime are identical and use the polar coordinates constant velocity vectors given in sections 4 and 5.It is easy to see that in the $1+1$ spacetime we would recover the problem of the Horizon.Compare with the final result of section 4 in [13].

In the $1+1$ spacetime without the lapse function the parallel contravariant $1+1$ ADM formalism suffers also the problem of the Horizon as in the $1+1$ original ADM formalism but in the $2+1$ and $3+1$ spacetimes the Horizon do not occurs.

14 Appendix F:Generic quadratic forms in the parallel contravariant $3+1$ ADM formalism with the lapse function.

This Appendix is a continuation of the Appendix E but this time we consider the lapse function.We provide all the step by step mathematical calculations.

Back to the $3+1$ ADM formalism compact generic equation with the lapse function given below:(see Appendix B)

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} (\sqrt{\gamma_{ii}} dx^i - X^i dt)^2 \tag{259}$$

Expanding the equation above we have:

$$ds^2 = (\alpha^2 - X^i X^i) dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{260}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (\alpha^2 - X^i X^i) dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{261}$$

Dividing by dt^2 we have:

$$0 = (\alpha^2 - X^i X^i) + 2\sqrt{\gamma_{ii}} X^i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{262}$$

$$0 = (\alpha^2 - X^i X^i) + 2\sqrt{\gamma_{ii}} X^i \frac{dx^i}{dt} - \gamma_{ii} (\frac{dx^i}{dt})^2 \tag{263}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{264}$$

We have now a generic quadratic form in the term U^i:

$$0 = (\alpha^2 - X^i X^i) + 2\sqrt{\gamma_{ii}} X^i U^i - \gamma_{ii} (U^i)^2 \tag{265}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii} (U^i)^2 - 2\sqrt{\gamma_{ii}} X^i U^i - (\alpha^2 - X^i X^i) = 0 \tag{266}$$

$$\gamma_{ii}(U^i)^2 - 2\sqrt{\gamma_{ii}}X^iU^i + (X^iX^i - \alpha^2) = 0 \tag{267}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2\sqrt{\gamma_{ii}}X^i \pm \sqrt{[-2\sqrt{\gamma_{ii}}X^i]^2 - 4[\gamma_{ii}(X^iX^i - \alpha^2)]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X^i \pm \sqrt{4\gamma_{ii}[X^i]^2 - 4[\gamma_{ii}(X^iX^i] + 4[\alpha^2\gamma_{ii}]}}{2\gamma_{ii}} \tag{268}$$

$$U^i = \frac{2\sqrt{\gamma_{ii}}X^i \pm \sqrt{4[\alpha^2\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X^i \pm 2\sqrt{[\alpha^2\gamma_{ii}]}}{2\gamma_{ii}} = \frac{\sqrt{\gamma_{ii}}X^i \pm \alpha\sqrt{[\gamma_{ii}]}}{\gamma_{ii}} = \frac{\sqrt{[\gamma_{ii}]}}{\gamma_{ii}}(X^i \pm \alpha) \tag{269}$$

At last we have the final solution of this generic quadratic form in the term U^i given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X^i \pm \alpha) \tag{270}$$

But this expression actually means:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i \pm \alpha)] \tag{271}$$

The subscript γ_{ii} as being a sum $\sum_{i=1}^{3}\gamma_{ii}$ is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [27]).Then $\sum_{i=1}^{3}\sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3}\gamma_{ii}}$.The same is valid for X^i also as a sum $\sum_{i=1}^{3}X^i$ implying also in $\sum_{i=1}^{3}\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{\sqrt{\sum_{i=1}^{3}\gamma_{ii}}}{\sum_{i=1}^{3}\gamma_{ii}}$. Remember the Einstein summing convention in which $X^ie_i = \sum_{i=1}^{3}X^ie_i = X^1e_1 + X^2e_2 + X^3e_3$ and $\sqrt{X^ie_i} = \sqrt{X^1e_1 + X^2e_2 + X^3e_3}$.

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i \pm \alpha)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 \pm \alpha) \tag{272}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X^i + \alpha) \tag{273}$$

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X^i - \alpha) \tag{274}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i + \alpha)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 + \alpha) \tag{275}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i - \alpha)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 - \alpha) \tag{276}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i + \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 + \alpha) \quad (277)$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X^i - \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X^1 + X^2 + X^3 - \alpha) \quad (278)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel contravariant $3+1$ ADM equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

For the parallel contravariant $2+1$ ADM equations we have:

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X^i + \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}}(X^1 + X^2 + \alpha) \quad (279)$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X^i - \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}}(X^1 + X^2 - \alpha) \quad (280)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel contravariant $2+1$ ADM equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

For the parallel contravariant $1+1$ ADM equations we have:

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X^i + \alpha)] = \frac{dx^1}{dt} = \frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X^1 + \alpha) \quad (281)$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X^i - \alpha)] = \frac{dx^1}{dt} = \frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X^1 - \alpha) \quad (282)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel contravariant $1+1$ ADM equations given above with the solution that encompasses the quadratic form $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

Take the original or alternative Natario vectors which in $1+1$ spacetime are identical and use the polar coordinates variable velocity vectors given in sections 4 and 5.It is easy to see that in the $1+1$ spacetime the problem of the Horizon with a lapse function do not occurs.Compare with the final result of section 6 in [35].

In the $1+1$ spacetime with the lapse function the parallel contravariant $1+1$ ADM formalism do not suffers the problem of the Horizon.

15 Appendix G:Generic quadratic forms in the parallel covariant $3+1$ ADM formalism without the lapse function.

Using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity,Horizons can be easily computed for the dimensionally reduced $1+1$ spacetime versions of the equations because only the quadratic form dr^2 exists but in the $3+1$ spacetime we have the presence of 3 quadratic forms respectively dr^2,$r^2 d\theta^2$ and $r^2 \sin^2\theta d\phi^2$.Algebraic solutions for the null-like geodesics $ds^2 = 0$ of General Relativity of the $3+1$ equations above are extremely difficult due to the presence of these 3 quadratic forms considering solutions for each quadratic form dr^2 or $r^2 d\theta^2$ or $r^2 \sin^2\theta d\phi^2$ isolated.

The best effort to solve the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above is to find out a solution that encompasses all the 3 quadratic forms dr^2 and $r^2 d\theta^2$ and $r^2 \sin^2\theta d\phi^2$ grouped together.

We will demonstrate all the required mathematics step by step.

Back to the parallel covariant $3+1$ ADM formalism compact generic equation given below:(see Appendix C)

$$ds^2 = dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{283}$$

Expanding the equation above we have:

$$ds^2 = (1 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{284}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (1 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{285}$$

Dividing by dt^2 we have:

$$0 = (1 - X_i X_i) + 2\sqrt{\gamma_{ii}}X_i \frac{dx^i dt}{dt^2} - \gamma_{ii}\frac{(dx^i)^2}{dt^2} \tag{286}$$

$$0 = (1 - X_i X_i) + 2\sqrt{\gamma_{ii}}X_i \frac{dx^i}{dt} - \gamma_{ii}(\frac{dx^i}{dt})^2 \tag{287}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{288}$$

We have now a generic quadratic form in the term U^i:

$$0 = (1 - X_i X_i) + 2\sqrt{\gamma_{ii}}X_i U^i - \gamma_{ii}(U^i)^2 \tag{289}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2\sqrt{\gamma_{ii}}X_i U^i - (1 - X_i X_i) = 0 \tag{290}$$

$$\gamma_{ii}(U^i)^2 - 2\sqrt{\gamma_{ii}}X_iU^i + (X_iX_i - 1) = 0 \tag{291}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2\sqrt{\gamma_{ii}}X_i \pm \sqrt{[-2\sqrt{\gamma_{ii}}X_i]^2 - 4[\gamma_{ii}(X_iX_i - 1)]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X^i \pm \sqrt{4\gamma_{ii}[X_i]^2 - 4[\gamma_{ii}(X_iX_i] + 4[\gamma_{ii}]}}{2\gamma_{ii}} \tag{292}$$

$$U^i = \frac{2\sqrt{\gamma_{ii}}X_i \pm \sqrt{4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X_i \pm 2\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{\sqrt{\gamma_{ii}}X_i \pm \sqrt{[\gamma_{ii}]}}{\gamma_{ii}} = \frac{\sqrt{[\gamma_{ii}]}}{\gamma_{ii}}(X_i \pm 1) \tag{293}$$

At last we have the final solution of this generic quadratic form in the term U^i given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X_i \pm 1) \tag{294}$$

But this expression actually means:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i \pm 1)] \tag{295}$$

The subscript γ_{ii} as being a sum $\sum_{i=1}^{3}\gamma_{ii}$ is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [27]).Then $\sum_{i=1}^{3}\sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3}\gamma_{ii}}$.The same is valid for X_i also as a sum $\sum_{i=1}^{3}X_i$ implying also in $\sum_{i=1}^{3}\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{\sqrt{\sum_{i=1}^{3}\gamma_{ii}}}{\sum_{i=1}^{3}\gamma_{ii}}$. Remember the Einstein summing convention in which $X^ie_i = \sum_{i=1}^{3}X^ie_i = X^1e_1 + X^2e_2 + X^3e_3$ and $\sqrt{X^ie_i} = \sqrt{X^1e_1 + X^2e_2 + X^3e_3}$.

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i \pm 1)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X_1 + X_2 + X_3 \pm 1) \tag{296}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X_i + 1) \tag{297}$$

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X_i - 1) \tag{298}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i + 1)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X_1 + X_2 + X_3 + 1) \tag{299}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i - 1)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X_1 + X_2 + X_3 - 1) \tag{300}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i+1)] = \frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1+X_2+X_3+1) \quad (301)$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i-1)] = \frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1+X_2+X_3-1) \quad (302)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel covariant $3+1$ ADM equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt}+\frac{dx^2}{dt}+\frac{dx^3}{dt}$.

For the parallel covariant $2+1$ ADM equations we have:

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X_i+1)] = \frac{dx^1}{dt}+\frac{dx^2}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}}}{\gamma_{11}+\gamma_{22}}(X_1+X_2+1) \quad (303)$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X_i-1)] = \frac{dx^1}{dt}+\frac{dx^2}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}}}{\gamma_{11}+\gamma_{22}}(X_1+X_2-1) \quad (304)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel covariant $2+1$ ADM equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt}+\frac{dx^2}{dt}$.

For the parallel covariant $1+1$ ADM equations we have:

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X_i+1)] = \frac{dx^1}{dt} = \frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X_1+1) \quad (305)$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X_i-1)] = \frac{dx^1}{dt} = \frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X_1-1) \quad (306)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel covariant $1+1$ ADM equations given above with the solution that encompasses the quadratic form $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

Take the original or alternative Natario vectors which in $1+1$ spacetime are identical and use the polar coordinates constant velocity vectors given in sections 4 and 5.Obtain the covariant components.It is easy to see that in the $1+1$ spacetime we would recover the problem of the Horizon.Compare with the final result of section 4 in [13].

In the $1+1$ spacetime without the lapse function the parallel covariant $1+1$ ADM formalism suffers also the problem of the Horizon as in the $1+1$ original ADM formalism but in the $2+1$ and $3+1$ spacetimes the Horizon do not occurs.

16 Appendix H:Generic quadratic forms in the parallel covariant $3+1$ ADM formalism with the lapse function.

This Appendix is a continuation of the Appendix G but this time we consider the lapse function.We provide all the step by step mathematical calculations.

Back to the $3+1$ ADM formalism compact generic equation with the lapse function given below:(see Appendix D)

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} (\sqrt{\gamma_{ii}} dx^i - X_i dt)^2 \tag{307}$$

Expanding the equation above we have:

$$ds^2 = (\alpha^2 - X^i X^i) dt^2 + 2\sqrt{\gamma_{ii}} X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{308}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (\alpha^2 - X_i X_i) dt^2 + 2\sqrt{\gamma_{ii}} X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{309}$$

Dividing by dt^2 we have:

$$0 = (\alpha^2 - X_i X_i) + 2\sqrt{\gamma_{ii}} X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{310}$$

$$0 = (\alpha^2 - X_i X_i) + 2\sqrt{\gamma_{ii}} X_i \frac{dx^i}{dt} - \gamma_{ii} (\frac{dx^i}{dt})^2 \tag{311}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{312}$$

We have now a generic quadratic form in the term U^i:

$$0 = (\alpha^2 - X_i X_i) + 2\sqrt{\gamma_{ii}} X_i U^i - \gamma_{ii} (U^i)^2 \tag{313}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii} (U^i)^2 - 2\sqrt{\gamma_{ii}} X_i U^i - (\alpha^2 - X_i X_i) = 0 \tag{314}$$

$$\gamma_{ii}(U^i)^2 - 2\sqrt{\gamma_{ii}}X_iU^i + (X_iX_i - \alpha^2) = 0 \tag{315}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2\sqrt{\gamma_{ii}}X_i \pm \sqrt{[-2\sqrt{\gamma_{ii}}X_i]^2 - 4[\gamma_{ii}(X_iX_i - \alpha^2)]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X_i \pm \sqrt{4\gamma_{ii}[X_i]^2 - 4[\gamma_{ii}(X_iX_i] + 4[\alpha^2\gamma_{ii}]}}{2\gamma_{ii}} \tag{316}$$

$$U^i = \frac{2\sqrt{\gamma_{ii}}X_i \pm \sqrt{4[\alpha^2\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2\sqrt{\gamma_{ii}}X_i \pm 2\sqrt{[\alpha^2\gamma_{ii}]}}{2\gamma_{ii}} = \frac{\sqrt{\gamma_{ii}}X_i \pm \alpha\sqrt{[\gamma_{ii}]}}{\gamma_{ii}} = \frac{\sqrt{[\gamma_{ii}]}}{\gamma_{ii}}(X_i \pm \alpha) \tag{317}$$

At last we have the final solution of this generic quadratic form in the term U^i given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X_i \pm \alpha) \tag{318}$$

But this expression actually means:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i \pm \alpha)] \tag{319}$$

The subscript γ_{ii} as being a sum $\sum_{i=1}^{3}\gamma_{ii}$ is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [27]).Then $\sum_{i=1}^{3}\sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3}\gamma_{ii}}$.The same is valid for X_i also as a sum $\sum_{i=1}^{3}X_i$ implying also in $\sum_{i=1}^{3}\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{\sqrt{\sum_{i=1}^{3}\gamma_{ii}}}{\sum_{i=1}^{3}\gamma_{ii}}$. Remember the Einstein summing convention in which $X^ie_i = \sum_{i=1}^{3}X^ie_i = X^1e_1 + X^2e_2 + X^3e_3$ and $\sqrt{X^ie_i} = \sqrt{X^1e_1 + X^2e_2 + X^3e_3}$.

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i \pm \alpha)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X_1 + X_2 + X_3 \pm \alpha) \tag{320}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X_i + \alpha) \tag{321}$$

$$U^i = \frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}(X_i - \alpha) \tag{322}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i + \alpha)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X_1 + X_2 + X_3 + \alpha) \tag{323}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i - \alpha)] = U^1 + U^2 + U^3 = \frac{\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}}(X_1 + X_2 + X_3 - \alpha) \tag{324}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i + \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1 + X_2 + X_3 + \alpha) \quad (325)$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{3}[(X_i - \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}+\gamma_{33}}}{\gamma_{11}+\gamma_{22}+\gamma_{33}}(X_1 + X_2 + X_3 - \alpha) \quad (326)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel covariant $3+1$ ADM equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

For the parallel covariant $2+1$ ADM equations we have:

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X_i + \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}}}{\gamma_{11}+\gamma_{22}}(X_1 + X_2 + \alpha) \quad (327)$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{2}[(X_i - \alpha)] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{\sqrt{\gamma_{11}+\gamma_{22}}}{\gamma_{11}+\gamma_{22}}(X_1 + X_2 - \alpha) \quad (328)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel covariant $2+1$ ADM equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

For the parallel covariant $1+1$ ADM equations we have:

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X_i + \alpha)] = \frac{dx^1}{dt} = \frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X_1 + \alpha) \quad (329)$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{\sqrt{\gamma_{ii}}}{\gamma_{ii}}]\sum_{i=1}^{1}[(X_i - \alpha)] = \frac{dx^1}{dt} = \frac{\sqrt{\gamma_{11}}}{\gamma_{11}}(X_1 - \alpha) \quad (330)$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the parallel covariant $1+1$ ADM equations given above with the solution that encompasses the quadratic form $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

Take the original or alternative Natario vectors which in $1+1$ spacetime are identical and use the polar coordinates variable velocity vectors given in sections 4 and 5.Obtain the covariant components.It is easy to see that in the $1+1$ spacetime the problem of the Horizon with a lapse function do not occurs.Compare with the final result of section 6 in [35].

In the $1+1$ spacetime with the lapse function the parallel covariant $1+1$ ADM formalism do not suffers the problem of the Horizon.

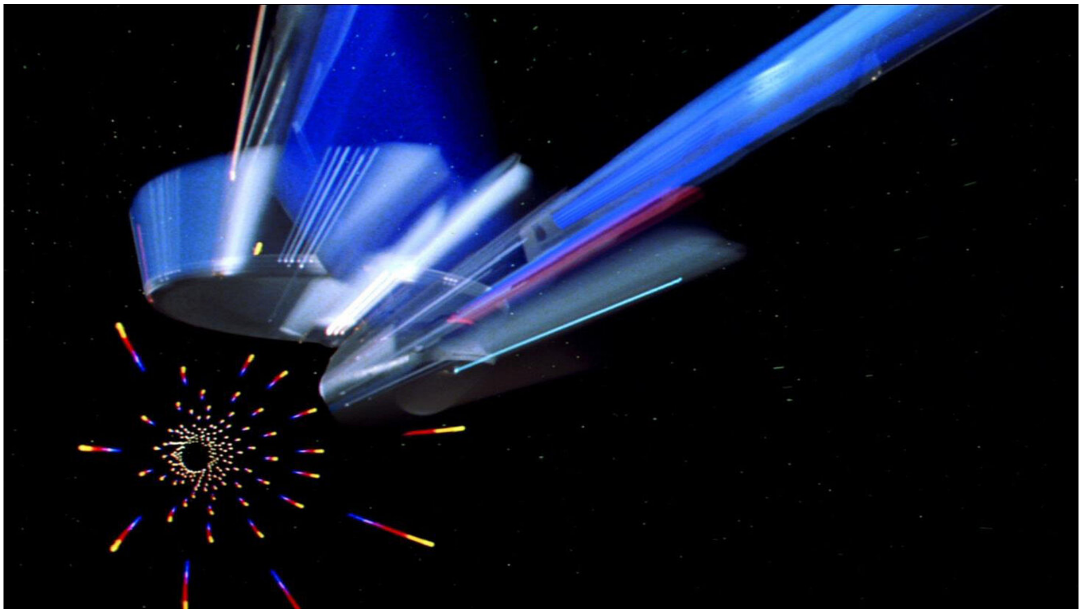

Figure 1: Artistic Presentation of a warp drive spaceship in interstellar space colliding with highly Doppler-Blueshifted photons(Source:Internet)

17 Appendix I:Artistic Presentation of a warp drive spaceship in interstellar space colliding with highly Doppler-Blueshifted Photons

The picture above borrowed from science-fiction depicts one of the most serious(and dangerous) problems a spaceship would confront in a realistic interstellar travel:Collisions with highly Doppler-Blueshifted photons.This problem was first pointed out in 1999 in the work of Chad Clark,Will Hiscock and Shane Larson [32].In 2010 it appeared again in the work of Carlos Barcelo,Stefano Finazzi and Stefano Liberatti [33].In 2012 the same problem of collisions against hazardous IM photons appeared in the work of Brendan McMonigal,Geraint Lewis and Philip O'Byrne [29].

All these works uses the geometry of the original Alcubierre warp drive 1994 paper in [16] and the results outlined in these works are completely correct.

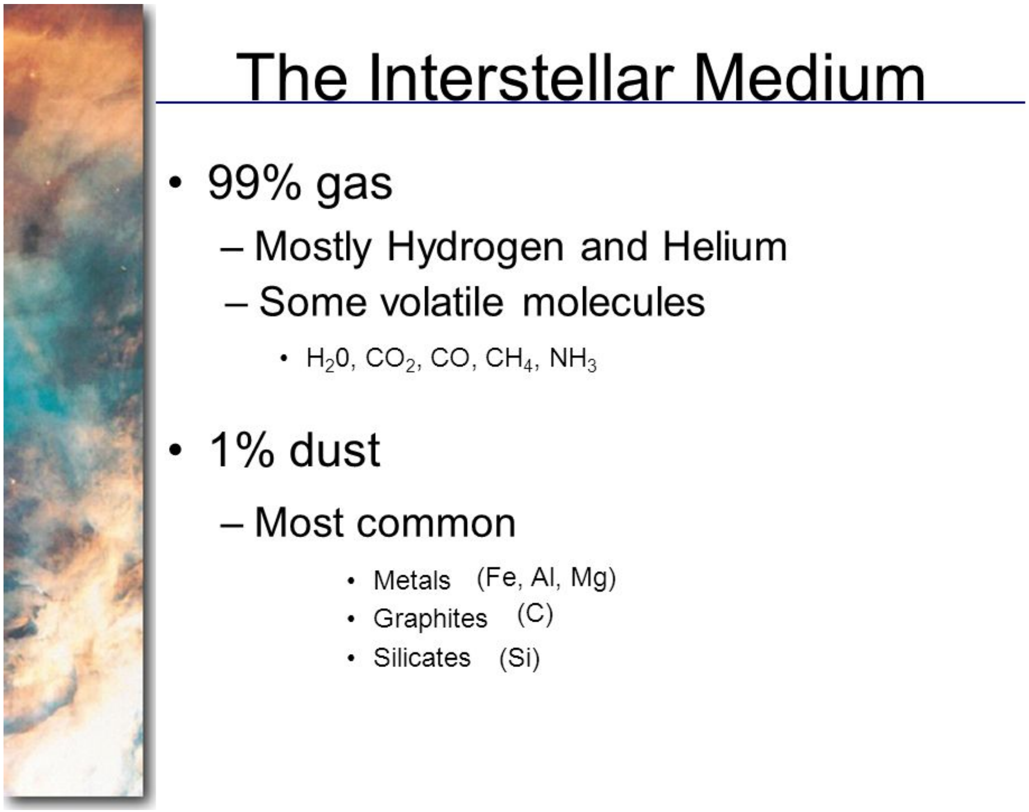

Figure 2: Composition of the Interstellar Medium IM(Source:Internet)

18 Appendix J:Composition of the Interstellar Medium IM

The problem of collisions between a warp drive spaceship moving at superluminal velocity and the potentially dangerous particles from the Interstellar Medium IM is not new.

It was first noticed in 1999 in the work of Chad Clark,Will Hiscock and Shane Larson(see [32]).Later on in 2010 it appeared again in the work of Carlos Barcelo,Stefano Finazzi and Stefano Liberatti(see [33]).In 2012 the same problem of collisions against hazardous IM particles appeared in the work of Brendan McMonigal,Geraint Lewis and Philip O'Byrne(see [29]).

The last work addressing interstellar collisions was the work in ([30]) in 2022.It covers the analysis of Siyu Bian ,Yi Wang,Zun Wang and Mian Zhu.

All these works uses the geometry of the original Alcubierre warp drive 1994 paper in [14] and the results outlined in these works are completely correct.

From the picture above we all can see that the Interstellar Medium IM is not empty and a collision at superluminal speeds is highly dangerous.

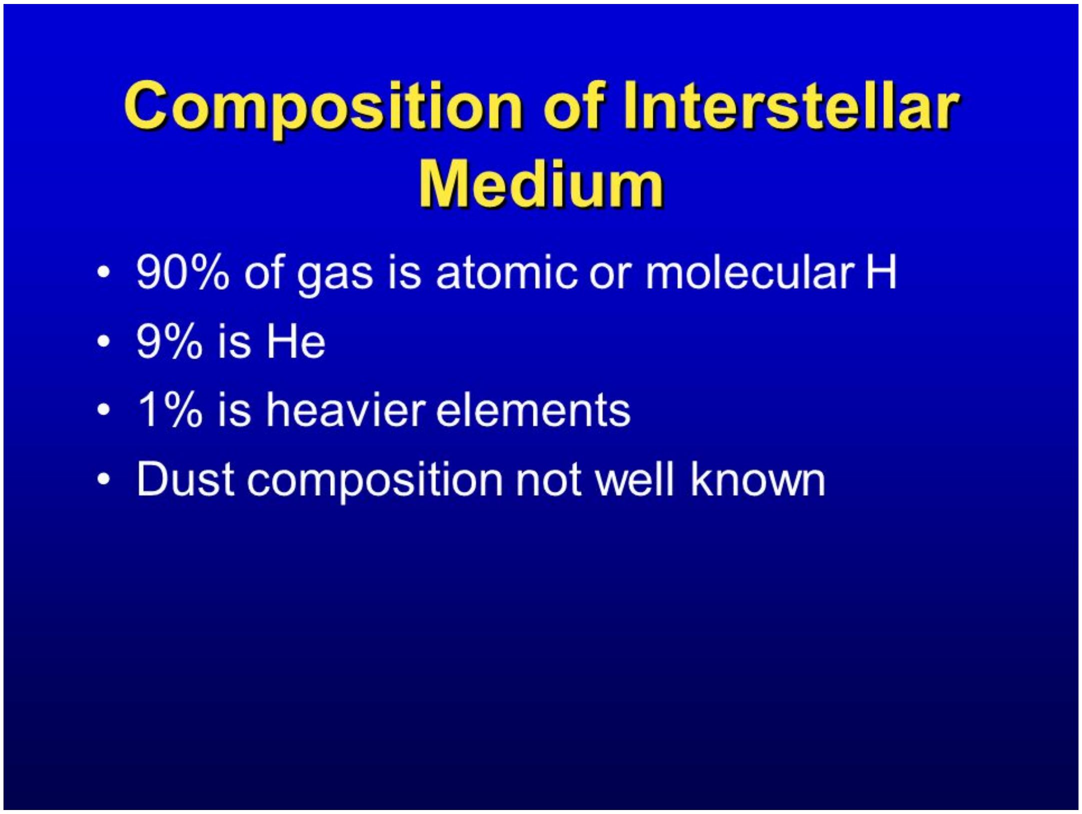

Figure 3: Composition of the Interstellar Medium IM(Source:Internet)

19 Appendix K:Composition of the Interstellar Medium IM

The original Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal interstellar spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and due to a different distribution of the negative energy in front of the ship with repulsive gravitational behavior(see pg 116 in [17]) deflecting all the incoming hazardous particles of the Interstellar Medium.The Natario spacetime offers an excellent protection to the crew members as depicted in the works [7],[8] and specially [9],[10] and [31].

The original and alternative Natario warp drives have the negative energy also in front of the ship even in a $1 + 1$ spacetime protecting the ship and the crew members from collisions against the hazardous components of the Interstellar Medium IM that according with the picture above is not empty.

Figure 4: Artistic Presentation of a Cross-Section C curve(Source:Internet)

20 Appendix L:Another Artistic Presentation of a Natario warp drive in a real faster than light interstellar spaceflight

The image above borrowed from science-fiction depicts a spaceship in an interstellar spaceflight suffering the collision against the IM particles.

Does the C-curve above in front of the spaceship looks familiar?

This figure is exactly similar to one of the figures in [30] and [9] but with a "science fiction look-alike".

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