THE NONTRIVIAL ZEROES OF THE RIEMANN ZETA FUNCTION ARE TRIVIALLY EXPRESSED BY THE EULER PRODUCT

RYAN HACKBARTH
E-MAIL: RHACKBAR@PURDUE.EDU

Abstract. Leonard Euler considered the Zeta function for real values input of s, and in doing so he famously derived the Euler Product formulation of the Zeta function, which linked the function to prime numbers. The connection between the prime numbers and the Zeta function was further pursued by Bernhard Riemann, who constructed an analytic continuation of the function which remained valid for the entire complex plane. In doing so, he was able to link the zeroes of the Zeta function to the distribution of prime numbers. In this paper, I show that the zeroes of the Zeta Function can be identified directly from the Euler Product, and in doing so, suggest a trivial link between the primes and the zeroes of the Zeta function while bypassing the more complex machinery of analytic continuation.

1. Introduction

Let the Zeta Function be defined as :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \tag{1}$$

Let the Dirichlet Eta Function be defined as the alternating version of (1):

$$\eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} \tag{2}$$

Equation (1) can be shown to be split into a sum of odd and even integers through factoring akin to Euler's method for deriving the Euler Product (Meyer).

$$\zeta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) + \frac{1}{2^s}\zeta(s) \tag{3}$$

A simple sign change produces the Dirichlet Eta function:

$$\eta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{1}{2^s}\zeta(s) \tag{4}$$

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2. Power Series Representation

When $s = \frac{1}{2} + bi$, this difference can be expressed as a complex conjugate division:

$$\eta(0.5+bi) = -\left(\frac{1-\frac{1}{2^{0.5+bi}}}{\frac{1}{2^{0.5+bi}}}\right)^{*} \zeta(0.5+bi) \tag{5}$$

Which can be rearranged into a more appealing form:

$$\eta(0.5+bi) = \left(\frac{\frac{1-\frac{1}{2^{0.5+bi}}}{1-2^{0.5-bi}}}{1-\frac{1-\frac{1}{2^{0.5+bi}}}{1-2^{0.5-bi}}}\right)^{*} \zeta(0.5+bi) \tag{6}$$

Whose appeal is made obvious by a simple variable substitution, where the magnitude of x is always less than 1.

$$\eta(0.5+bi) = \left(\frac{x}{1-x}\right)^{*} \zeta(0.5+bi) \tag{7}$$

Which can be expressed as a Power Series:

$$\eta(0.5+bi) = \sum_{n=1}^{\infty} a\,(x^{*})^{n} \tag{8}$$

Where the coefficient a, is the Riemann Zeta Function, and more specifically, the Euler Product representation of the Riemann Zeta function (Riemann).

$$\prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{(p_n)^{0.5+bi}}}$$

And

$$x = \frac{1-\frac{1}{2^{0.5+bi}}}{1-2^{0.5-bi}}$$

When the power series (8) is fully evaluated, we recover the well-known relation between the Zeta and the Eta Function.

$$\left(1-2^{1-s}\right)\zeta(s) = \eta(s)$$

The benefit of this representation is it allows us to identify the zeroes of the Zeta Function as the values of b whose first term most closely approximates the total sum of the power series:

$$\frac{\left(\frac{1-\frac{1}{2^{0.5+bi}}}{1-2^{0.5-bi}}\right)^{*}}{1-2^{0.5-bi}} \prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{(p_n)^{0.5+bi}}} \tag{9}$$

Which simplifies nicely to:

$$\frac{1}{2-2^{0.5-bi}} \prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{(p_n)^{0.5+bi}}} \tag{10}$$

Thus, the zeroes of the Zeta Function are trivially identified by evaluating the real and imaginary results of (10), displayed in Figure 1.

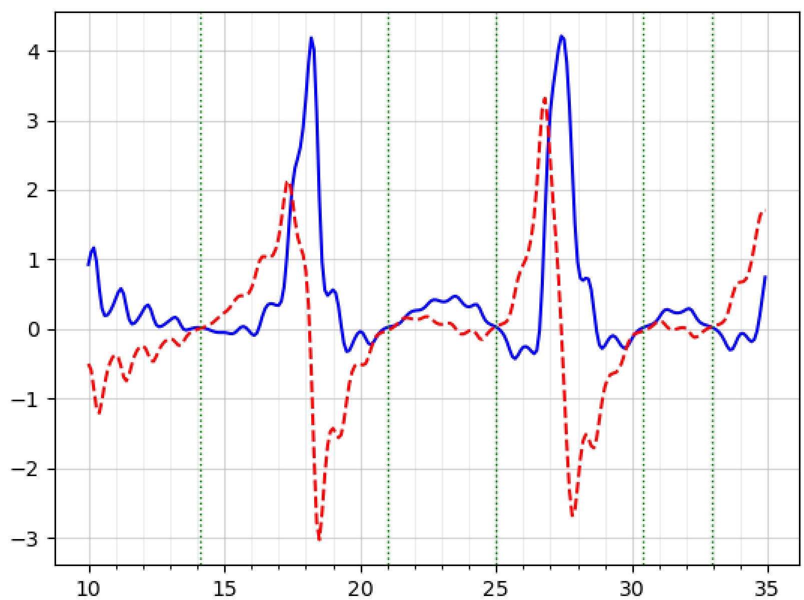

FIGURE 1. Real (blue) and Imaginary (Red) output of equation (10), considering all primes up to 10,000 in the Euler Product for computation. First 5 zeroes readily identified and marked as dotted green line.

REFERENCES

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