

From Rényi Entropy to General Relativity

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Abstract

In this sequel to [4-6], we argue that the equations of General Relativity (GR) follow from constrained maximization of Rényi Entropy. This finding is entirely consistent with our earlier derivation of Feynman's Path Integrals from constrained maximization of Rényi Entropy. An unforeseen conclusion is that three fundamental constants of Nature, (Planck's, Newton's and cosmological constants) emerge as *Lagrange multipliers* of the maximization procedure.

Key words: Rényi Entropy, General Relativity, multifractal geometry, constrained optimization, Lagrange multipliers.

[1. Introduction

1.1 Motivation

Despite overwhelming experimental support, General Relativity (GR) remains conceptually isolated from Statistical Physics, Information Theory, and Quantum Field Theory. Attempts to bridge this gap often invoke entropy or information as auxiliary constructs rather than as primary dynamic principles. In this work, we advance a stronger claim: *GR and classical spacetime geometry are emergent manifestations of the Rényi Entropy flow in the infrared limit of long distances.*

1.2 Distinction from Approaches Based on Entropic Gravity

The entropic gravity program [9-10] interprets gravity as an entropic outcome arising from coarse-grained microscopic degrees of freedom. Such approaches:

1. Start from the baseline geometry of classical spacetime,

2. Treat entropy gradients as secondary, emergent effects,
3. Do not derive Einstein's equations as extrema of the entropy flow.

By contrast, the framework introduced herein:

- Does not assume a preexisting spacetime geometry,
- Derives the metric, curvature, and Einstein equations from an entropy extremization principle,
- Identifies gravity as an IR fixed point of entropy flow, not as a derived dynamical property.

1.3 Distinction from Information-Based Approaches to GR

Information-based reconstructions of GR typically rely on Shannon entropy, relative entropy, or mutual information defined on pre-existing manifolds.

These approaches implicitly assume:

- Euclidean or pseudo-Riemannian backgrounds,
- Additivity and extensivity of entropy.

By contrast, our framework differs fundamentally in the following sense,

- We begin with a *multifractal configuration space*, suitable for the description of physics in the ultraviolet (UV) sector,
- Probabilities are defined via resolution-dependent measures, not metric determinants,
- Rényi Entropy replaces Shannon entropy as primitive object, allowing *dimensional flow* and *non-equilibrium behavior* to naturally enter the picture.

We caution that, in order to make the presentation transparent to a large audience, heavy technical jargon is generally avoided, and – whenever possible - the formal content is kept at the minimum necessary. As such, this introductory study is neither fully rigorous nor complete.

2. Rényi Entropy on a Multifractal Support

We start from the plausible conjecture that, in the early Universe, spacetime dimensions are *random variables* undergoing sustained fluctuations in

amplitude. No appeal to the action principle can be made, as this principle *is likely to break down* in far-from-equilibrium settings [7].

2.1 Definition

Let Ω stand for a generic set of microstates endowed with a multifractal structure, otherwise called a *multifractal configuration space*. Let $\{p_i(\ell)\}$ denote probabilities associated with a resolution scale having linear size ℓ . The Rényi entropy of continuous order q is [1]

$$S_q(\ell) = \frac{1}{1-q} \ln \sum_i p_i(\ell)^q. \quad (2.1)$$

2.2 Generalized Dimensions

The generalized Rényi dimension D_q is defined by [1, 4]

$$D_q = \lim_{\ell \rightarrow 0} \frac{1}{q-1} \frac{\ln \sum_i p_i(\ell)^q}{\ln \ell}. \quad (2.2)$$

Note that this formulation **does not assume** a preexisting geometric manifold; generalized dimensions (2.2) are emergent and scale-dependent,

as both linear resolution and index q flow with the observation scale. Stated differently, (2.1) and (2.2) imply no metric, no curvature and no gravity in the GR sense. Rényi Entropy is defined *purely by probability measures*, not by conventional geometry.

3. Entropy Saturation and the IR Fixed Point

By analogy with conventional Statistical Physics, the partition function of multifractal geometry is defined as [4],

$$\chi = \sum_i p_i^q(l) \propto l^{-\tau(q)}$$

where,

$$\tau(q) = (1-q)D_q$$

As shown in [4, 11], the generalized dimension of ordinary spacetime is recovered in the infrared (IR) limit $q=1/2$ as $D_{1/2}=4$. In this case, the partition function becomes

$$\sum_i \sqrt{p_i(\ell)} \sim \ell^{-D_{1/2}/2}. \quad (3.1)$$

Entropy saturation requires this sum (3.1) to remain finite and non-vanishing in the IR regime. In this regime, entropy turns into an extensive and stable quantity, which means that the four-dimensional spacetime represents an *IR fixed point* of the entropy flow.

4. Identification of the Metric from Probability Measures

4.1 Coarse-graining the Probability Measure

We first define a *coarse-grained measure*

$$\mu(\ell) = \sum_i p_i(\ell) \delta V_i(\ell), \quad (4.1)$$

where $\delta V_i(l) \propto l^{D_{1/2}}$. The definition implies that, in $D_{1/2}=4$ spacetime dimensions, the sum of probabilities scales as density of the coarse-grained measure taken over the four-volume element l^4 .

4.2 Emergence of the Metric from the IR Entropy

The goal of this section is to suggest that, in the IR limit,

$$\mu(\ell) \rightarrow \int d^4x \sqrt{-g}. \quad (4.2)$$

where g is the determinant of the GR metric. This identification means that the GR metric can be interpreted as emerging from a continuum representation of the entropy-based, coarse-grained measure (4.1).

To develop this point and arrive at (4.2), consider a generic space with $n \neq 4$ Euclidean dimensions and note that, when the number of microstates becomes large, one can replace

$$\sum_i \rightarrow \int d^n x \mu(x)$$

Probabilities must be invariant under relabeling of microstates, which means that

$$\mu(x) d^n x = \mu(x') d^n x'.$$

This implies the existence of a *Jacobian factor* $J(x)$ assuming the role of a scalar density and defined via

$$d\mu \Rightarrow J(x) d^n x.$$

Any positive scalar density can be written as the square root of a determinant. Thus,

$$J(x) \equiv \sqrt{\det h_{ab}(x)}.$$

Note that, when $n \neq 4$, h_{ab} is strictly an *entropy-based metric*, devoid of any geometric content. The underlying hypothesis is, as the next section indicates, it is only when Rényi Entropy is maximized that the Euclidean dimension flows into its IR fixed point $n = D_{1/2} = 4$, and $h_{ab}(x)$ emerge as components of the GR metric, that is, $h_{ab}(x) \rightarrow g_{ab}(x)$.

5. Constrained Maximization of Rényi Entropy

The fundamental object in this framework is Rényi Entropy, not an action.

At the IR fixed point, the entropy functional (2.2) is

$$S_{1/2} \equiv 2 \ln \left(\sum_i \sqrt{p_i} \right). \quad (5.1)$$

In line with the Maximum Entropy Principle (MEP) [2], we next proceed to the constrained maximization of (5.1).

5.1 Why are constraints required?

Entropy maximization without constraints is trivial and physically meaningless. One must impose constraints that encode the properties of all macroscopic, coarse-grained observables. Recall that the underlying assumption is that, at large distances matching the approach to the IR fixed point, entropy saturates and admits a continuum, geometry-based description. This leads to two *unavoidable* constraints:

A. Normalization of the Overall Measure

According to (4.2), at the IR fixed point $D_{1/2} = 4$, the coarse-grained entropy measure becomes

$$\mu(\ell) \rightarrow \int d^4x \sqrt{-g}.$$

Fixing the total measure requires,

$$\int d^4x \sqrt{-g} = \text{const.} \quad (5.2)$$

This constraint is enforced below by a Lagrange multiplier λ_1 .

B. Local Distribution of Probability

Entropy saturation alone does not fix how probability is distributed locally.

The leading scalar that measures *inhomogeneity of coarse-grained probability flow* is the analog of the Ricci scalar R . This is because,

- It is the *unique scalar* built from second derivatives of the emergent metric,
- It is the *lowest-order invariant* compatible with diffeomorphism invariance. Recall that diffeomorphism invariance demands that the mathematical form of physical laws and predictions stay unchanged under any smooth, invertible coordinate transformation.

In light of these considerations, we constrain the integrated “curvature”:

$$\int d^4x \sqrt{-g} R = \text{const.} \quad (5.3)$$

This is enforced below by a second Lagrange multiplier λ_2 .

Putting everything together, the object to be extremized is *not an action*, but the constrained entropy functional:

$$\boxed{\mathcal{F} = S_{1/2} - \lambda_1 \int d^4x \sqrt{-g} - \lambda_2 \int d^4x \sqrt{-g} R} \quad (5.4)$$

Few important conceptual points regarding (5.4):

1. $S_{1/2}$ is dimensionless,
2. λ_1 and λ_2 are parameters carrying dimensions,
3. No principle of stationary action has been invoked,
4. Geometry is emergent at the IR fixed point as $h_{ab}(x) \rightarrow g_{ab}(x)$.

Since microscopic probabilities p_i are already saturated at the IR fixed point, the only remaining degrees of freedom are the variables defining the

emergent metric. The functional variation is thus taken with respect to the first order variations in the emergent metric:

$$\delta\mathcal{F} = 0 \quad \text{under} \quad \delta g_{\mu\nu}. \quad (5.5)$$

Since entropy is already maximized at the IR fixed point $D_{1/2} = 4$, we have

$$\delta S_{1/2} = 0 \quad (5.6)$$

6. Explicit Variation of the Constraints

The standard identity gives

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \quad (6.1a)$$

Thus,

$$\delta(d^4x\sqrt{-g}) = -\frac{1}{2}\int d^4x\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \quad (6.1b)$$

Using standard differential geometry along with the Lovelock theorem [8], we obtain,

$$\delta(\sqrt{-g}R) = \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} + \text{boundary terms.} \quad (6.1c)$$

A convenient setting is to cancel boundary terms under fixed-boundary conditions. Collecting all terms leads to

$$\delta\mathcal{F} = \int d^4x \sqrt{-g} \left[\lambda_2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} \lambda_1 g_{\mu\nu} \right] \delta g^{\mu\nu}. \quad (6.1d)$$

Since $\delta g^{\mu\nu}$ are arbitrary, the bracket must vanish, that is,

$$\lambda_2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} \lambda_1 g_{\mu\nu} = 0. \quad (6.2)$$

7. Identification with Einstein's Equations

Next, divide (6.2) by λ_2 and define

$$\Lambda \equiv \frac{\lambda_1}{2\lambda_2}. \quad (7.1)$$

Hence,

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0} \quad (7.2)$$

which is the *vacuum Einstein equation*. To include deviations of the entropy flow from the IR fixed point, we allow

$$\delta S_{1/2} \neq 0, \quad (7.3)$$

which generates an *effective stress-energy tensor*:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}^{(\text{eff})}} \quad (7.4)$$

8. Interpretation of Constants

Identifying

$$\boxed{\lambda_2 = \frac{1}{16\pi G}, \quad \lambda_1 = \frac{\Lambda}{8\pi G}} \quad (8.1)$$

Shows that G and Λ lose their status as fundamental constants and simply become Lagrange multipliers of (5.4).

9. Key Takeaways

We conclude by listing the main findings of our work,

1. GR and classical spacetime geometry are emergent manifestations of the Rényi Entropy flow near the IR fixed point,
2. One can reasonably infer that Dark Matter and Dark Energy correspond to deviations from $\delta S_{1/2} = 0$, in line with our conjecture that the Dark Sector emerges from dimensional deviations from four spacetime-dimensions, that is, $\delta S_{1/2} = \delta D_{1/2} = 0 \Rightarrow \varepsilon = 4 - D_{1/2} = 0$ [12].
3. Along with Planck's constant emerging from the constrained maximization of Rényi Entropy in field theory [3, 5], Newton's and cosmological constants *arise as Lagrange multipliers* of the entropy maximization procedure.

References

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