

# Embarking on Adventurous Journeys Through Some Subjects in Pure Mathematics

## Part II First Edition

### Extension of Fundamental Trigonometric Functions and Hyperbolic Functions

$$\sin(x, y) = \frac{1}{2i} (y^{ix} - y^{-ix})$$

$$\cos(x, y) = \frac{1}{2} (y^{ix} + y^{-ix})$$

$$\tan(x, y) = \frac{1}{i} \cdot \frac{y^{ix} - y^{-ix}}{y^{ix} + y^{-ix}}$$

$$\cot(x, y) = i \cdot \frac{y^{ix} + y^{-ix}}{y^{ix} - y^{-ix}}$$

$$\sinh(x, y) = \frac{1}{2} (y^x - y^{-x})$$

$$\cosh(x, y) = \frac{1}{2} (y^x + y^{-x})$$

$$\tanh(x, y) = \frac{y^x - y^{-x}}{y^x + y^{-x}}$$

$$\coth(x, y) = \frac{y^x + y^{-x}}{y^x - y^{-x}}$$

### Unraveling the Mystery of the Riemann Hypothesis: Toward a Complete Proof

All possible nontrivial zeros to  $\zeta(s) = \zeta(\sigma+i\tau) = 0$  for  $0 < \sigma < 1$  and  $\sigma \neq \frac{1}{2}$  lie within the narrow bands  $6.28318534... < \tau \leq 6.28983598...$  or  $-6.28983598... \leq \tau < -6.28318534....$

Tue Vu & Trista Vu



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## Preface

In Part I, we explored two subjects: ***Nested Radicals and Algebraic Identities*** in Chapter 1 and ***Innovative Approaches to Explore Polynomial Equation Solving: Trigonometric and Hyperbolic Insights*** in Chapters 2. As we enter Part II, we will delve into two new subjects: ***Extension of Fundamental Trigonometric and Hyperbolic Functions***, and ***Unraveling the Mystery of the Riemann Hypothesis: Toward a Complete Proof***, starting with Chapter 3 and Chapter 4.

In this Part II, we present our work as a narrative of self-discovery, letting ideas unfold naturally with the journey of insight. The book is intended for readers who love mathematics and are comfortable with fundamental concepts. It is available free of charge on **Series Math Study** ([www.seriesmathstudy.com](http://www.seriesmathstudy.com)).

In modern mathematics, elementary functions of a single variable form the foundation of our analytical tools. When graphed on a two dimensional coordinate plane, each function produces a characteristic curve. Extending these functions into three dimensions does not fundamentally change their nature; the resulting surfaces are simply extrusions of the original curves. Although combinations of one variable elementary functions can theoretically represent any shape in the universe<sup>[1]</sup>, expressing complex structures in this way is extremely difficult and often impractical.

In the natural world, many complex curved surfaces are not to be easily or adequately described using traditional three dimensional coordinates. This limitation motivates the extension of elementary mathematics into three dimensions, where basic functions are defined with two variables. A set of new elementary two variable functions in a certain aspect can greatly simplify the representation of complex geometries, provided these functions are systematically studied and understood.

Trigonometric and hyperbolic functions illustrate both the strengths and the limitations of our current framework. Their familiar forms, namely sine, cosine, tangent, and their hyperbolic counterparts, are deeply embedded in mathematics,

physics, and engineering. These functions have become standardized, each serving a distinct and essential role. Collectively, they form a mature and saturated framework: for most purposes, the existing elementary functions are sufficient, and there is extremely difficult with little motivation to introduce entirely new ones.

A more feasible approach is to extend familiar one variable functions into two variable forms, creating a new layer built upon existing elementary functions. In this approach, we easily maintain conceptual continuity and avoid unnecessary complexity, all while opening the door to fresh mathematical possibilities.

This book introduces such extensions in Chapter 3, generalizing classical trigonometric and hyperbolic functions and elevating them to a broader conceptual level. These new forms allow dynamic behavior that was previously inaccessible due to computational complexity, opening the door to reshaping classical functions and exploring entirely new families of elementary functions.

**How do we develop these extended functions?** Trigonometric functions are traditionally defined using the unit circle and right-angled triangles, while hyperbolic functions arise from the geometry of the hyperbola. Both families can also be expressed in terms of the exponential constant  $e$  through Euler's formula, which provides a unified algebraic framework. By replacing the exponential constant  $e$  with a variable  $y$ , we obtain functions capable of representing not only traditional curves in the plane but also genuinely two variable objects in space.

This extension creates a richer mathematical framework for studying natural curved surfaces. Classical trigonometric functions such as  $\sin(x)$  and  $\cos(x)$  produce identical cross sections when plotted in three dimensions, yielding surfaces that replicate the same curve for every value of  $y$ . In contrast, the new two variable trigonometric functions generate curves that vary with  $y$ , producing surfaces that differ across the domain and giving rise to distinct geometric objects rather than simple extrusions.

Through this approach, we establish a broader collection of elementary functions capable of describing three dimensional structures in a unified and dynamic way. This expanded viewpoint deepens our understanding of trigonometry and geometry and lays the groundwork for future developments in pure mathematics. By embracing these extended functions, we gain access to a wider spectrum of natural shapes and mathematical behaviors, opening promising new directions for exploration and innovation.

In Chapter 4, we present the proof of the **Riemann Hypothesis**, one of the most intriguing and significant problems in mathematics. We explore known findings and

properties of the Riemann zeta function, “rewrite” the zeta functional equation, and present a proof of the Riemann Hypothesis using an elementary algebraic approach. **In Section 4-4, we demonstrate that all possible nontrivial zeros lie in the narrow bands  $6.28318534... < \text{Im}(s) \leq 6.28983598... \text{ or } -6.28983598... \leq \text{Im}(s) < -6.28318534...$  for  $0 < \text{Re}(s) < 1$  and  $\text{Re}(s) \neq 1/2$ . However, none of them serve as solutions to  $\zeta(s) = 0$  because their imaginary parts are below the first nontrivial zero.** This approach aims to clarify the assertion that all nontrivial zeros of the zeta function lie on the vertical line where the real part is  $1/2$ .

Though we offer this book at a free of charge, unless you wish to acquire a physical copy, in which case you will need to cover the printing and paper costs through Amazon. We assume no liability in connection with its contents.

*A few words from the authors: We express our gratitude for the support received from our parents. A special thanks to Tue Vu's three daughters, Thao Vu, Thanh-Tu Vu, and Mai-Tran Vu, who dedicated their spare time to reviewing the book.*

While this book may contain variations in writing style influenced by cultural factors, as well as possible errors such as incorrect results, typos, or other imperfections, we sincerely welcome constructive feedback and contributions from readers to improve the overall quality. Your valuable input is greatly appreciated. Our contact emails are [sms@seriesmathstudy.com](mailto:sms@seriesmathstudy.com) and [tuevu2003@gmail.com](mailto:tuevu2003@gmail.com). If you wish to support our books and efforts, we gratefully accept donations without refunds in any form through the provided PayPal link,

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## Chapter 3

### ***Extension of Fundamental Trigonometric Functions and Hyperbolic Functions***

#### **Section 3-1. Background and Motivation for Two Variable Elementary Functions**

We have known since Leonhard Euler introduced the formula

$$e^{ix} = \cos(x) + i\sin(x) \tag{3.1}$$

in his 1748 publication, *De Formulis Exponentialibus ac Logarithmis*<sup>[2]</sup>, that it brought about a profound revolution not only in mathematics but also in related fields such as physics, chemistry, and engineering. Formula (3.1) later became known as Euler's identity, which has a close connection with the trigonometric functions sine and cosine, as well as the exponential function  $e$  over the complex number  $i$ . It not only extended the definition of trigonometric functions beyond the measurement of triangle angles to numerical analysis but also enabled the representation of angles using complex numbers, laying the foundation for modern analysis and complex algebra in pure mathematics.

On a deeper level, the complex number  $i$  forms a crucial bridge between trigonometric and hyperbolic functions, unifying them within a single analytic framework. Hyperbolic functions themselves trace back to the independent work of Vincenzo Riccati and Johann Heinrich Lambert in the 1760s. Traditionally, both trigonometric and hyperbolic functions have been treated as elementary single variable functions and visualized in two and three dimensional space. Yet even in 3D, these functions remain fundamentally one dimensional, unable to express the richer curvilinear structures that appear throughout nature.

Much of pure mathematics has already uncovered its most immediate and “obvious” laws. Without expanding our foundational assumptions, the discovery of new elementary functions risks stagnation. For centuries, we have relied on single variable frameworks; however, functions such as  $\cos(x)$  merely generate a one dimensional curve that, when extended into 3D, becomes a simple extruded surface rather than a naturally evolving spatial form. To move beyond these limitations, we propose extending elementary functions to two variables,  $x$  and  $y$ , creating new “atomic” functions that inherently curve and evolve within three dimensional space. These functions are not mere extensions of existing forms; they introduce fundamentally new geometric behaviors that cannot be captured by traditional single-variable systems.

Gödel’s Incompleteness Theorem reminds us that any formal system contains truths that cannot be proven within its own boundaries. This insight compels us to seek new conceptual structures rather than remain confined to existing ones. Just as a child transitions from crawling on a flat surface to standing and perceiving the full volume of a room, mathematics must transition from one variable “crawling” to two variable “standing” in order to access a broader landscape of possibilities.

Modern computational tools now make it feasible to construct and explore such two variable elementary functions while preserving their distinct identities. Building on Euler’s identity as a foundational guide, we aim to generalize classical functions from a single variable  $x$  to the pair  $(x,y)$ . Although this endeavor may challenge established conventions, any new mathematical branch must remain coherent with existing knowledge. Our immediate goal is therefore the rigorous development of fundamental two variable elementary functions that can serve as the basis for this expanded framework.

In the following sections, we review both the trigonometric and hyperbolic functions. We introduce notation for several extended functions and then examine their properties in detail, accompanied by graphs that illustrate their behavior.

## Section 3-2. Review Trigonometric Functions<sup>[6]</sup>

In this section, we briefly review the fundamental trigonometric functions by showing how their familiar formulas can be derived using Euler’s identity.

We start by replacing  $x$  with  $-x$  in (3.1), and obtain

$$e^{-ix} = \cos(x) - i\sin(x), \tag{3.2}$$

Next, we show that the trigonometric functions, and their corresponding domains and ranges, can be derived by adding and subtracting equations (3.1) and (3.2).

### a. Cosine Function

Adding (3.1) and (3.2) gives

$$e^{ix} + e^{-ix} = 2 \cos(x).$$

Solving for  $\cos(x)$  from the expression yields

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}). \quad (3.3)$$

The domain and range of the cosine function are:

- Domain:  $x \in (-\infty, \infty)$  or  $x \in \mathbb{R}$
- Range:  $-1 \leq \cos(x) \leq 1$  or  $\cos(x) \in [-1, 1]$ .

### b. Sine Function

Similarly, by subtracting (3.2) from (3.1), we obtain

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \quad (3.4)$$

The domain and range are:

- Domain:  $x \in (-\infty, \infty)$  or  $x \in \mathbb{R}$
- Range:  $-1 \leq \sin(x) \leq 1$  or  $\sin(x) \in [-1, 1]$ .

### c. Tangent Function

The tangent function is defined as the ratio of  $\sin(x)$  to  $\cos(x)$ :

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{1}{2i}(e^{ix} - e^{-ix})}{\frac{1}{2}(e^{ix} + e^{-ix})}$$

$$= \frac{1}{i} \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}. \quad (3.5)$$

The domain and range are:

- Domain:  $x \in \mathbb{R}, x \neq \frac{\pi}{2} + \pi k$  and  $k \in \mathbb{Z}$
- Range:  $-\infty < \tan(x) < \infty$ .

#### d. Cotangent Function

The cotangent function is defined as

$$\begin{aligned} \cot(x) &= \frac{\cos(x)}{\sin(x)} = \frac{\frac{1}{2}(e^{ix} + e^{-ix})}{\frac{1}{2i}(e^{ix} - e^{-ix})} \\ &= i \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}. \end{aligned} \quad (3.6)$$

The domain and range are

- Domain:  $x \in \mathbb{R}, x \neq \pi k$  and  $k \in \mathbb{Z}$
- Range:  $-\infty < \cot(x) < \infty$ .

#### e. Secant Function

The secant function is defined as

$$\sec(x) = \frac{1}{\cos(x)} = \frac{2}{e^{ix} + e^{-ix}} \quad (3.3a)$$

- Domain:  $x \in \mathbb{R}, x \neq \pi/2 + \pi k$  and  $k \in \mathbb{Z}$
- Range:  $|\sec(x)| \geq 1$ .

#### e. Cosecant Function

The cosecant function is defined as

$$\csc(x) = \frac{1}{\sin(x)} = \frac{2i}{e^{ix} - e^{-ix}} \quad (3.4 a)$$

- Domain:  $x \in \mathbb{R}, x \neq \pi k$  and  $k \in \mathbb{Z}$
- Range:  $|\csc(x)| \geq 1$ .

### Section 3-3. Review Hyperbolic Functions

We follow an approach analogous to that of Section 3-2 to derive the hyperbolic functions in this section.

Substituting  $x$  with  $ix$  in (3.1) and using the identities  $\cos(ix) = \cosh(x)$  and  $\sin(ix) = i\sinh(x)$ , we obtain

$$e^{-x} = \cosh(x) - \sinh(x). \quad (3.7)$$

Replacing  $x$  with  $-x$  in (3.7) gives

$$e^x = \cosh(x) + \sinh(x). \quad (3.8)$$

#### a. Hyperbolic Cosine Function

Adding (3.7) and (3.8) gives

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}). \quad (3.9)$$

The domain and range of the hyperbolic cosine are:

- Domain:  $x \in \mathbb{R}$
- Range:  $1 \leq \cosh(x) < \infty$ .

#### b. Hyperbolic Sine Function

Similarly, by subtracting (3.8) from (3.7), we obtain the hyperbolic sine:

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}). \quad (3.10)$$

- Domain:  $x \in \mathbb{R}$
- Range:  $-\infty < \sinh(x) < \infty$ .

### c. Hyperbolic Tangent Function

The hyperbolic tangent is defined as

$$\begin{aligned}\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}}.\end{aligned}\tag{3.11}$$

- Domain:  $x \in \mathbb{R}$
- Range:  $-1 < \tanh(x) < 1$ .

### d. Hyperbolic Cotangent Function

The hyperbolic cotangent is defined as the ratio of  $\cosh(x)$  to  $\sinh(x)$ :

$$\begin{aligned}\coth(x) &= \frac{\cosh(x)}{\sinh(x)} = \frac{\frac{1}{2}(e^x + e^{-x})}{\frac{1}{2}(e^x - e^{-x})} \\ &= \frac{e^x + e^{-x}}{e^x - e^{-x}}.\end{aligned}\tag{3.12}$$

The domain and range of the hyperbolic cotangent are:

- Domain:  $x \in \mathbb{R}, x \neq 0$
- Range:  $\coth(x) < -1$  or  $\coth(x) > 1$ .

### e. Hyperbolic Secant Function

The hyperbolic secant is defined as

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}.\tag{3.9a}$$

The domain and range of the hyperbolic cosecant are:

- Domain:  $x \in \mathbb{R}$
- Range:  $0 < \operatorname{sech}(x) \leq 1$ .

### f. Hyperbolic Cosecant Function

The hyperbolic cosecant is defined as

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}. \quad (3.10 a)$$

The domain and range of the hyperbolic cosecant are:

- Domain:  $x \in \mathbb{R}, x \neq 0$
- Range:  $|\operatorname{csch}(x)| > 0$ .

## Section 3-4. Notations of Extended Trigonometric Functions

We introduce new bivariable trigonometric and hyperbolic functions as extensions of traditional trigonometric and hyperbolic geometry. We propose a framework for analyzing enhanced trigonometric and hyperbolic functions. Our exploration encompasses their domains, properties, identities, and inverses within the contexts of trigonometry and geometry. By integrating these extensions with modern computational advancements, we aim to provide a more comprehensive understanding of trigonometry and hyperbolic geometry, evolving from a curve in 2D to a surface in 3D.

The logarithmic constant  $e$  plays a fundamental role in all trigonometric and hyperbolic functions. For instance, the definitions of trigonometric functions like  $\cos(x)$  and hyperbolic functions like  $\cosh(x)$  inherently rely on  $e$ , as demonstrated in the formulas (3.3) and (3.9). We observe that the functions  $e^x$  and  $e^{-x}$  are present in all trigonometric and hyperbolic functions where the base  $e$  remains constant and unchanging. However, what if we generalize the exponential function, allowing the base to vary? When we desire the base  $e$  to be variable, we represent it as a variable  $y$ . Throughout this chapter, the notations and definitions for the extended forms of trigonometric and hyperbolic functions are presented as follows:

Table 1 – Notations for Extended Trigonometric &amp; Hyperbolic Functions

Extended Trigonometric Functions	Extended Hyperbolic Functions
1. $\cos(x, y) \equiv \frac{1}{2}(y^{ix} + y^{-ix})$	7. $\cosh(x, y) \equiv \frac{1}{2}(y^x + y^{-x})$
2. $\sin(x, y) \equiv \frac{1}{2i}(y^{ix} - y^{-ix})$	8. $\sinh(x, y) \equiv \frac{1}{2}(y^x - y^{-x})$
3. $\tan(x, y) \equiv \frac{1}{i} \frac{y^{ix} - y^{-ix}}{y^{ix} + y^{-ix}}$	9. $\tanh(x, y) \equiv \frac{y^x - y^{-x}}{y^x + y^{-x}}$
4. $\cot(x, y) \equiv i \cdot \frac{y^{ix} + y^{-ix}}{y^{ix} - y^{-ix}}$	10. $\coth(x, y) \equiv \frac{y^x + y^{-x}}{y^x - y^{-x}}$
5. $\sec(x, y) \equiv \frac{2}{y^{ix} + y^{-ix}}$	11. $\operatorname{sech}(x, y) \equiv \frac{2}{y^x + y^{-x}}$
6. $\operatorname{csc}(x, y) \equiv \frac{2i}{y^{ix} - y^{-ix}}$	12. $\operatorname{csch}(x, y) \equiv \frac{2}{y^x - y^{-x}}$

By introducing these new bivariable notations, we define  $\sin(x,y)$ ,  $\cos(x,y)$ ,  $\tan(x,y)$ ,  $\cot(x,y)$ ,  $\sec(x,y)$  and  $\operatorname{csc}(x,y)$  as extended trigonometric functions or trigonometric-like functions, while  $\sinh(x,y)$ ,  $\cosh(x,y)$ ,  $\tanh(x,y)$ ,  $\coth(x,y)$ ,  $\operatorname{sech}(x,y)$  and  $\operatorname{csch}(x,y)$  are termed extended hyperbolic functions or hyperbolic-like functions. The terms 'extended trigonometric functions' and 'trigonometric-like functions' will be used interchangeably in this book.

*It is natural to question whether the newly introduced extended functions from (1) to (11) behave analogously to that of the trigonometric functions and hyperbolic functions.*

We will demonstrate in the next sections that each of these bivariable functions maintains a connection to similar conventional notation of the trigonometric or hyperbolic function. We also exhibit unique characteristics, including distinct domains, ranges, inverse, periodicity and other properties.

### Remark 3.4.1.

- **Extend notations:** We greatly respect the foundational work of earlier mathematicians in establishing standard notations. In this book, we reuse the symbols of the trigonometric and hyperbolic functions. This adaptation serves only to provide a natural and intuitive extension within the context of our

work.

- **Real values:** Although these notations (definitions) involve the imaginary unit  $i$ , the extended trigonometric functions such as  $\sin(x,y)$ ,  $\cos(x,y)$ ,  $\tan(x,y)$ ,  $\cot(x,y)$ ,  $\sec(x,y)$  and  $\csc(x,y)$  are real values for all real  $x$  and  $y$ .
- **Principal branch:** All the extended expressions in Table 1 assume the use of the principal branch of the complex exponential function. This ensures the proper behavior of the function and that the values of the extended trigonometric functions remain real for real  $x$  and  $y$ .
- **Complex exponentials:** Note that  $y^{ix}$  and  $y^{-ix}$ , where  $y$  is real base, are complex exponentials defined using the principal branch of the complex exponential function.
- **Transition between trigonometric and hyperbolic functions:** We use the imaginary unit  $i$  to transition between trigonometric and hyperbolic functions. While the imaginary unit facilitates the transformation, the values of both trigonometric and hyperbolic functions remain real when the argument  $x$  is real.
- **Convention:** Throughout this book, whenever a solution  $x$  of a trigonometric equation depends on an integer  $k \in \mathbb{Z}$ , we denote it by  $x_k$ . That is, any solution of the form, for instance

$$x = \frac{(1+2k)\pi}{2|\ln y|}, k \in \mathbb{Z}$$

will be understood and written as  $x_k(y)$  or

$$x_k = \frac{(1+2k)\pi}{2|\ln y|}, k \in \mathbb{Z}.$$

We will use the notation  $x_k$  to refer to these solutions in later sections.

- The symbols "=" and "≡" may appear interchangeably, although they traditionally carry different meanings. The sign "=" denotes ordinary equality, while "≡" is often used for identities or statements true for all values of the variables. Since many texts do not strictly enforce this distinction, we follow the common convention of using both symbols in contexts where the meaning is clear from the expression itself.

### Section 3-5. Foundations of the Extended Cosine Function $\cos(x,y)$

We will follow the same procedure as shown Section 3-2 to derive the extended cosine function  $\cos(x,y)$ . Let  $x$  and  $y > 0$  be real numbers. By replacing  $x$  with  $x \ln(y)$  in (3.1), we have:

$$e^{ix \ln y} = \cos(x \ln y) + i \sin(x \ln y) \quad (3.13)$$

Using  $e^{ix \ln y} = e^{\ln y^{ix}} = y^{ix}$ , therefore, we obtain the extended Euler formula with the base  $y$  in  $y^{ix}$  as

$$y^{ix} = \cos(x \ln y) + i \sin(x \ln y) \quad (3.14)$$

Replacing  $x$  with  $-x$  into (3.14) gives a general form of Euler's function in terms of  $x$  and  $y$ :

$$y^{-ix} = \cos(x \ln y) - i \sin(x \ln y) \quad (3.15)$$

#### 3-5-1. Definition and Theorem

**Definition.** The extended cosine function, denoted as  $\cos(x,y)$ , is defined by

$$\cos(x, y) \equiv \frac{1}{2} (y^{ix} + y^{-ix}) \text{ for } x, y \in \mathbb{R} \text{ with } y > 0. \quad (3.16)$$

**Theorem 1.** The extended cosine function satisfies the relation

$$\cos(x, y) \equiv \frac{1}{2} (y^{ix} + y^{-ix}) = \cos(x \ln y), \text{ for real } x, y \text{ and } y > 0. \quad (3.17)$$

#### Proof

By adding (3.14) and (3.15), we have:

$$\begin{aligned} y^{ix} + y^{-ix} &= 2 \cos(x \ln y) \\ \Rightarrow \cos(x \ln y) &= \frac{1}{2} (y^{ix} + y^{-ix}). \end{aligned}$$

Combining the above outcome with definition (3.16) gives

$$\cos(x, y) \equiv \frac{1}{2}(y^{ix} + y^{-ix}) = \cos(x \ln y).$$

### 3-5-2. Basic Properties of $\cos(x, y)$

This section uses knowledge of the elementary trigonometric functions to determine the maxima, minima, and zeros and other properties of  $\cos(x, y)$  without using calculus tools.

**1. Domain:**  $x \in (-\infty, \infty)$ ,  $y > 0$ .

**2. Range:**  $-1 \leq \cos(x, y) \equiv \cos(x \ln y) \leq 1$ .

**3. Function type:** Symmetry in  $x$ :  $\cos(x, y)$  is even for all  $y > 0$ . Symmetry in  $y$ : The function is neither even nor odd in  $y$ .

#### Proof

3. We consider two cases:

a. Symmetry in  $x$ : We have  $\cos(-x, y) = \cos[(-x)(\ln(y))] = \cos(x \ln(y)) = \cos(x, y)$ . Therefore, the function is even in  $x$  for all  $y > 0$ .

b. Symmetry in  $y$ : Since  $\ln(y)$  is defined only for  $y > 0$ ,  $\cos(x, -y)$  is undefined because  $\ln(-y)$  is not real. The function is neither even nor odd in  $y$ .

#### Note:

$$\begin{cases} \ln(y) \geq 0, & \text{for } y \geq 1 \\ \ln(y) < 0, & \text{for } 0 < y < 1 \end{cases}$$

### 4. Maxima, Minima and Zeros of $\cos(x, y)$ with Respect to $x$

**Theorem 2.** The maxima, minima and zeros of  $\cos(x, y)$  with respect to  $x$  are:

#### a. Maxima

$$x = \frac{2\pi k}{|\ln y|}, k \in \mathbb{Z}. \quad (3.18)$$

**b. Minima**

$$x = \frac{\pi(1+2k)}{|\ln y|}, k \in \mathbb{Z}. \quad (3.19)$$

**c. Zeros**• **Form 1:**

$$x = \frac{(\pm 1 + 4k)\pi}{2|\ln y|}, k \in \mathbb{Z}. \quad (3.20)$$

• **Form 2:**

$$x = \frac{(1+2k)\pi}{2|\ln y|}, k \in \mathbb{Z}. \quad (3.21)$$

**Proof****a. Maxima in x**

The maxima of the function  $\cos(x, y) \equiv \cos(x \ln y)$  with respect to  $x$  occur when  $x \ln y$  equals  $\pi k$  and  $k \in \mathbb{Z}$ , since  $\cos(\theta)$  achieves its maximum value 1 at those points. Solving  $x \ln y = 2\pi k$  for  $x$  in terms of  $y$  and  $k$  gives the general solution

$$x = \frac{2\pi k}{|\ln y|}, k \in \mathbb{Z}.$$

**b. Minima in x**

The function  $\cos(x \ln y)$  reaches its minimum when  $x \ln y = \pi + 2\pi k$  (because  $\cos(x \ln y) = -1$ ). Solving for  $x$  in terms of  $y$  and  $k$  yields

$$x = \frac{\pi(1+2k)}{|\ln y|}, k \in \mathbb{Z}.$$

**c. Zeros of  $\cos(x, y)$  on x-axis**

The zeros of  $\cos(x, y)$  on x-axis occur when  $\cos(x, y) \equiv \cos(x \ln y) = 0$ . There are two equivalent ways to express the solutions:

• **Form 1:**

Start with the identity

$$\cos(x \ln y) = \cos\left(\pm \frac{\pi}{2} + 2\pi k\right), k \in \mathbb{Z}.$$

Solving for  $x$  gives the general solution

$$x = \frac{(\pm 1 + 4k)\pi}{2|\ln y|}, k \in \mathbb{Z}.$$

• **Form 2:**

Alternatively, use the fact that the zeros of cosine occur every  $\pi$  units starting at  $\pi/2$ :

$$\cos(x \ln y) = \cos\left(\frac{\pi}{2} + \pi k\right), k \in \mathbb{Z}.$$

Thus, solving for  $x$  gives

$$x = \frac{(1 + 2k)\pi}{2|\ln y|}, k \in \mathbb{Z}.$$

**Remark 3.4.5.**

- *If we remove the absolute value, then we can adjust the sign of  $k$  to ensure the compensation of the sign of  $x \ln y$ .*
- *As we introduce the convention early, the solution of form (3.21) can be written as*

$$x_k = \frac{(1 + 2k)\pi}{2|\ln y|}, k \in \mathbb{Z}$$

**5. Periodicity<sup>[4]</sup> and Wavelength<sup>[5]</sup> of  $\cos(x, y)$  with Respect to  $x$**

**5A. Periodicity in  $x$**

**Theorem 3.** For each fixed  $y > 0$ , the period of  $\cos(x, y)$  with respect to  $x$  is given by

$$T_x(y) = \frac{2\pi}{|\ln(y)|}. \tag{3.22}$$

**Proof**

Assume  $T_x(y) \equiv T_x$  is the period of  $\cos(x,y)$  with respect to  $x$ . Then  $T_x$  must satisfy the equation

$$\begin{aligned}\cos(x+T_x, y) &= \cos(x, y) \\ \Rightarrow \cos((x+T_x) \ln(y)) &= \cos(x \ln y + 2\pi k), \quad k \in \mathbb{Z} \\ \Rightarrow (x+T_x) \ln(y) &= x \ln y + 2\pi k \\ \Rightarrow T_x \ln(y) &= 2\pi k\end{aligned}$$

Solving for  $T_x$  in terms of  $y$  and  $k$  gives

$$T_x(y) = \frac{2\pi k}{\ln(y)}, \quad k \in \mathbb{Z}.$$

Recall that the period of  $\cos(ax)$  is  $2\pi/|a|$ , for any  $a$ . Hence, the period of the function  $\cos(x,y) \equiv \cos(\ln(y) x)$  is

$$T_x(y) = \frac{2\pi k}{|\ln(y)|}, \quad k \in \mathbb{Z}.$$

We complete the proof.

**5B. Wavelength of  $\cos(x,y)$  in  $x$  (for fixed  $y > 0$ )**

**Theorem 4 (Wavelength in  $x$ ).** For  $\cos(x,y) = \cos(x \ln y)$ , the wavelength  $D_x(y)$  between two consecutive extrema or between two consecutive zeros in  $x$  is

**a) Maxima:**

$$D_x(y) = \frac{2\pi}{|\ln y|} \tag{3.23}$$

**b) Minima:**

$$D_x(y) = \frac{2\pi}{|\ln y|} \tag{3.24}$$

**c) Zeros:**

$$D_x(y) = \frac{2\pi}{|\ln y|} \quad (3.25)$$

**Proof**

a) Wavelength between two consecutive maxima:

From formula (3.18), the critical points for maxima are

$$x_k = \frac{2\pi k}{|\ln y|}, k \in \mathbb{Z}.$$

Therefore, the wavelength between two consecutive maxima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{2\pi(k+1)}{|\ln y|} - \frac{2\pi k}{|\ln y|} \\ &= \frac{2\pi}{|\ln y|}. \end{aligned}$$

b) Wavelength between two consecutive minima:

From formula (3.19), the critical points for minima are

$$x_k = \frac{\pi(1+2k)}{|\ln y|}, k \in \mathbb{Z}.$$

Hence, the wavelength between two consecutive minima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{\pi(1+2(k+1))}{|\ln y|} - \frac{\pi(1+2k)}{|\ln y|} \\ &= \frac{2\pi}{|\ln y|}. \end{aligned}$$

c) Wavelength between two consecutive zeros of  $\cos(x,y)$  within one full cycle:

From formula (3.20), the critical points are given by

$$x_k = \frac{(1+4k)\pi}{2|\ln y|}, k \in \mathbb{Z}.$$

Then the wavelength between two consecutive zeros of  $\cos(x,y)$  is determined by

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{\pi(1+4(k+1))}{2|\ln y|} - \frac{\pi(1+4k)}{2|\ln y|} \\ &= \frac{2\pi}{|\ln y|}. \end{aligned}$$

Thus, the wavelength between two consecutive maxima, minima, or zeros is identical, each equals to  $D_x = 2\pi/|\ln y|$ .

**Remark 3.4.6.**

- Observe that for each specific value of  $y$  (a 'slice' of  $y$ ), the period of the function  $\cos(x,y)$  in  $x$  remains constant. However, when the value of  $y$  varies, the period of  $\cos(x,y)$  also varies; it remains periodic with a repeating cycle because its natural characteristic still belongs to the cosine function, resulting in a non-constant overall period for  $\cos(x,y)$ .
- Recall that if  $f(x+T_x) = f(x)$ , then  $T_x$  is a period of  $f$ , and  $T_x$  does not require  $|T_x|$ . Indeed, we can define if  $f(x-T_x) = f(x)$ , then  $T_x$  is a period of  $f$ . It follows that if  $T_x$  is a period, then  $-T_x$  is also a period. Formula (3.22) uses  $|\ln y|$  to ensure the period is positive. That's why we use the absolute sign in formula (3.22).
- Note that period tells us how long one cycle takes (repetition in time); wavelength tells us how far the wave travels during one cycle (repetition in space).

**Example**

Given the function  $\cos(x,y)$ , determine the following for slices at  $y = \frac{1}{2}$  and  $y = 3$  with respect to  $x$ :

- The wavelength between two consecutive maxima.
- The wavelength between two consecutive minima.
- The wavelength between two consecutive zeros within one full cycle.

### Solution

#### 1. Slice at $y = 1/2$

- From part(a) of Theorem 4, we have the critical points for maxima:

$$x_k = \frac{2\pi k}{|\ln(y)|} = \frac{2\pi k}{|\ln(1/2)|} = \frac{2\pi k}{\ln(2)}, \quad k \in \mathbb{Z}.$$

Therefore, the wavelength between two consecutive maxima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ D_x(y) &= \frac{2\pi}{|\ln y|} \\ &= \frac{2\pi}{\ln(2)}. \end{aligned}$$

- Similarly, from part (b) of Theorem 4, the critical points for minima in x-axis are

$$x_k = \frac{\pi(1+2k)}{|\ln y|} = \frac{\pi(1+2k)}{\ln 2}, \quad k \in \mathbb{Z}.$$

And the wavelength between two consecutive minima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ D_x(y) &= \frac{2\pi}{|\ln y|} \\ &= \frac{2\pi}{\ln(2)}. \end{aligned}$$

- From part (c) of Theorem 4, the critical points are

$$x_k = \frac{\pi(1+2k)}{|\ln y|} = \frac{\pi(1+2k)}{\ln 2}, k \in \mathbb{Z}.$$

And the wavelength between two consecutive zeros of  $\cos(x, y)$  is

$$\begin{aligned} D_x(y) &= \frac{2\pi}{|\ln y|} \\ &= \frac{2\pi}{\ln(2)}. \end{aligned}$$

## 2. Slice at $y = 3$

a. From part (a) of Theorem 4, the critical points for maxima are

$$x_k = \frac{2\pi k}{|\ln(y)|} = \frac{2\pi k}{|\ln(3)|} = \frac{2\pi k}{\ln(3)}, k \in \mathbb{Z}.$$

The wavelength between two consecutive maxima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{2\pi(k+1)}{\ln(3)} - \frac{2\pi k}{\ln(3)} \\ &= \frac{2\pi}{\ln(3)}. \end{aligned}$$

b. The critical points for minima in  $x$  are

$$x_k = \frac{\pi(1+2k)}{|\ln y|} = \frac{\pi(1+2k)}{\ln 3}, k \in \mathbb{Z}.$$

The wavelength between two consecutive minima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{\pi(1+2(k+1))}{\ln 3} - \frac{\pi(1+2k)}{\ln 3} \\ &= \frac{2\pi}{\ln(3)}. \end{aligned}$$

c. The critical points for zeros are

$$x_k = \frac{\pi(1+2k)}{|\ln y|} = \frac{\pi(1+2k)}{\ln 3}, \quad k \in \mathbb{Z}.$$

The wavelength between two zeros of  $\cos(x, y)$  in  $x$  is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{\pi(1+2(k+1))}{\ln 3} - \frac{\pi(1+2k)}{\ln 3} \\ &= \frac{2\pi}{\ln(3)}. \end{aligned}$$

## 6. Maxima, Minima and Zeros of $\cos(x, y)$ with Respect to $y$

### 6A. Maxima in $y$

**Theorem 5.** The maxima of  $\cos(x, y)$  in  $y$  are given by

$$y = e^{2\pi k/|x|}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.26)$$

#### Proof

The maxima of  $\cos(x, y)$  in  $y$  occur when  $\cos(x, y) \equiv \cos(x \ln y) = 1$ . Because  $\cos(x \ln y + 2\pi k) = \cos(x \ln y)$  and  $k \in \mathbb{Z}$ , it follows that

$$\begin{aligned} x \ln y_k &= 2\pi k, \quad k \in \mathbb{Z}. \\ \Rightarrow \ln y_k &= \frac{2\pi k}{|x|}, \quad x \neq 0, k \in \mathbb{Z}. \end{aligned}$$

Solving for  $y_k$  gives

$$y_k = e^{\frac{2\pi k}{|x|}}, \quad x \neq 0, k \in \mathbb{Z}.$$

### 6B. Minima in $y$

**Theorem 6.** The minima of  $\cos(x, y)$  in  $y$  are

$$y_k = e^{\pi(1+2k)/|x|}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.27)$$

**Proof**

The minima of  $\cos(x,y)$  in  $y$  occur when  $\cos(x,y) \equiv \cos(x \ln y) = -1$ . Because  $\cos(x \ln y + 2\pi k) = \cos(x \ln y)$ , it follows that

$$x \ln y_k = \pi + 2\pi k, \quad k \in \mathbb{Z}.$$

$$\Rightarrow \ln y_k = \frac{\pi(1+2k)}{|x|}, \quad x \neq 0, k \in \mathbb{Z}.$$

Applying the exponential function to both sides yields

$$y_k = e^{\pi(1+2k)/|x|}, \quad x \neq 0, k \in \mathbb{Z}.$$

**6C. Zeros of  $\cos(x,y)$  in  $y$ , for fixed  $x$** 

**Theorem 7.** For fixed  $x$ , the zeros of  $\cos(x,y)$  respect to  $y$  are given by in two equivalent forms:

- **Form 1:**

$$y = e^{\pi(\pm 1 + 4k)/2|x|}, \quad x \neq 0, k \in \mathbb{Z} \quad (3.28)$$

- **Form 2:**

$$y = e^{\pi(1+2k)/2|x|}, \quad x \neq 0, k \in \mathbb{Z} \quad (3.29)$$

**Proof**

For fixed  $x$ , the zeros of  $\cos(x,y)$  occur when  $\cos(x,y) \equiv \cos(x \ln y) = 0$ . There are two equivalent forms to express the solutions:

- **Form 1:**

$$x \ln y_k = \pm \frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}.$$

Solving for  $y_k$  in terms of  $x$  and  $k$  gives

$$y_k = e^{\pi(\pm 1 + 4k)/2|x|}, \quad x \neq 0, k \in \mathbb{Z}.$$

- **Form 2:**

$$x \ln y_k = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$

Solving for  $y_k$  gives

$$y_k = e^{\pi(1+2k)/2|x|}, \quad x \neq 0, k \in \mathbb{Z}.$$

## 7. Periodicity and Wavelength (Distance) with Respect to y

### 7A. Periodicity in y

**Theorem 8.** For fixed  $x$ , the period of  $\cos(x,y)$  with respect to  $y$  is given by

$$T_y(x, y) = \left( e^{\frac{2\pi k}{|x|}} - 1 \right) y, \quad x \neq 0, k \in \mathbb{Z}, \quad (3.30)$$

#### Proof

Assuming  $T_y(x, y) \equiv T_y$  is the period of  $\cos(x, y)$  in  $y$ . Then  $T_y$  must satisfies the equation

$$\begin{aligned} \cos(x, y+T_y) &= \cos(x, y) \\ \Rightarrow \cos(x \ln(y+T_y)) &= \cos(x \ln y) && \text{(Use } \cos(x, y) = \cos(x \ln y)) \\ \Rightarrow x \ln(y+T_y) &= x \ln y + 2\pi k, \quad k \in \mathbb{Z} && \text{(Use } \cos(x \ln y) = \cos(x \ln y + 2\pi k)) \\ \Rightarrow x \ln\left(\frac{y+T_y}{y}\right) &= 2\pi k && \text{(Move } y \text{ terms to the left side of } =) \\ \Rightarrow \ln\left(\frac{y+T_y}{y}\right) &= \frac{2\pi k}{|x|} \\ \Rightarrow \frac{y+T_y}{y} &= e^{2\pi k/|x|} \end{aligned}$$

Solving for  $T_y$  in terms of  $|x|$ ,  $y$  and  $k$  gives

$$T_y(x, y) = y(e^{2\pi k/|x|} - 1), \quad k \in \mathbb{Z}.$$

Since  $T_y(x, y)$  depends on  $x$ ,  $y$  and  $k$  in  $y$ , this means that  $T(x, y)$  is not a constant

period for each slice  $x$  but rather a function of repeating itself for values of  $k$  after a different shift for different values of  $y$ . We conclude that ***the function  $\cos(x,y)$  is still repetitive in terms of  $x$  and  $y$ , but the repetition is not uniform across the entire domain  $y$ .***

### 7B. Wavelength between Two Consecutive Maxima with Respect to $y$

**Theorem 8.** The wavelength (distance) between two consecutive maxima in  $y$ , denoted  $D_y(x)$ , of  $\cos(x,y)$  is

$$D_y(x) = e^{2\pi k/|x|} (e^{2\pi/|x|} - 1), x \neq 0, k \in \mathbb{Z}. \quad (3.31)$$

#### Proof

From (3.26), we obtain the maxima in terms of  $k$ :

$$y_k = e^{2\pi k/|x|}, x \neq 0, k \in \mathbb{Z}$$

The wavelength between two consecutive maxima in  $y$  is given by

$$\begin{aligned} D_y(x) &= y_{k+1} - y_k \\ &= e^{2\pi(k+1)/|x|} - e^{2\pi k/|x|} \\ &= e^{2\pi k/|x|} (e^{2\pi/|x|} - 1). \end{aligned}$$

### 7C. Wavelength between Two Consecutive Minima in $y$ (for fixed $x$ )

**Theorem 9.** For fixed  $x$ , the wavelength between two consecutive minima of  $\cos(x,y)$  in  $y$  is

$$D_y(x) = e^{\pi(1+2k)/|x|} (e^{2\pi/|x|} - 1), x \neq 0, k \in \mathbb{Z}. \quad (3.32)$$

#### Proof

Use (3.27) to obtain the minima in terms of  $k$ :

$$y_k = e^{\pi(1+2k)/|x|}, x \neq 0, k \in \mathbb{Z}$$

The wavelength of  $\cos(x,y)$  in  $y$  between two consecutive minima is

$$D_y(x) = y_{k+1} - y_k.$$

Substituting  $y_k$  and  $y_{k+1}$  into the expression  $D_y(x)$  gives

$$\begin{aligned} D_y(x) &= e^{\pi(1+2(k+1))/|x|} - e^{\pi(1+2k)/|x|} \\ &= e^{\pi(1+2k)/|x|} (e^{2\pi/|x|} - 1). \end{aligned}$$

### 7D. Wavelength between Two Consecutive Zeros of $\cos(x,y)$ in $y$

**Theorem 10.** For fixed  $x$ , the wavelength between two consecutive zeros of  $\cos(x,y)$  in  $y$  within one full cycle is

#### Form 1:

$$D_y(x) = e^{\pi(\pm 1+4k)/2|x|} (e^{2\pi/|x|} - 1) \quad (3.33)$$

#### Form 2:

$$D_y(x) = e^{\pi(1+2k)/2|x|} (e^{\pi/|x|} - 1), \quad x \neq 0, k \in \mathbb{Z} \quad (3.34)$$

### Proof

#### Form 1:

For fixed  $x \neq 0$  and by applying (3.28), the zeros of  $\cos(x,y)$  are

$$y_k = e^{\pi(\pm 1+4k)/2|x|}, \quad x \neq 0, k \in \mathbb{Z}$$

The wavelength between two consecutive zeros of  $\cos(x,y)$  in  $y$  is given by

$$D_y(x) = y_{k+1} - y_k.$$

Substituting  $y_k$  and  $y_{k+1}$  into the expression  $D_y(x)$  gives

$$\begin{aligned} D_y(x) &= e^{\pi(\pm 1+4(k+1))/2|x|} - e^{\pi(\pm 1+4k)/2|x|} \\ &= e^{\pi(\pm 1+4k)/2|x|} (e^{2\pi/|x|} - 1). \end{aligned}$$

#### Form 2:

Using (3.29) gives

$$y_k = e^{\pi(1+2k)/2|x|}, \quad x \neq 0, k \in \mathbb{Z}.$$

The wavelength between two consecutive zeros of  $\cos(x,y)$  in  $y$  is

$$D_y(x) = y_{k+1} - y_k$$

Substituting  $y_k$  and  $y_{k+1}$  in the expression  $D_y$  gives

$$\begin{aligned} D_y(x) &= e^{\pi(1+2(k+1))/2|x|} - e^{\pi(1+2k)/2|x|} \\ &= e^{\pi(1+2k)/2|x|} (e^{\pi/|x|} - 1). \end{aligned}$$

### Example

Given the function  $\cos(x,y)$ , determine the following for slices at  $x = 1/2$  with respect to  $y$ :

- The distance (wavelength) between two consecutive maxima.
- The distance between two consecutive minima.
- The distance between two consecutive zeros of  $\cos(x,y)$  within one full cycle.

Additionally, verify whether the distances between two consecutive maxima, between two consecutive minima, and between two consecutive zeros within one full cycle are the same for each of slice.

### Solution

#### 1. Slice at $x = 1/2$

- According to formula (3.31), the wavelength between two consecutive maxima in  $y$  is

$$D_y(x) = e^{2\pi k/|x|} (e^{2\pi/|x|} - 1).$$

Substituting  $x = 1/2$  in the expression  $D_y(x)$  gives

$$= e^{4\pi k} (e^{4\pi} - 1), \quad k \in \mathbb{Z}.$$

- Similarly, based on formula (3.32), the distance between two consecutive minima is

$$D_y(x) = e^{\pi(1+2k)/|x|} (e^{2\pi/|x|} - 1).$$

Substituting  $x = 1/2$  gives

$$= e^{2\pi(1+2k)} (e^{4\pi} - 1), \quad k \in \mathbb{Z}.$$

c. Based on Form 1 of Theorem 10, the distance between two consecutive zeros within one full cycle is

$$\begin{aligned} D_y(x) &= e^{\pi(\pm 1+4k)/2|x|} (e^{2\pi/|x|} - 1) \\ &= e^{\pi(\pm 1+4k)} (e^{4\pi} - 1), \quad k \in \mathbb{Z}. \end{aligned}$$

The results from parts (a), (b), and (c) show that the distances between two consecutive maxima, between two consecutive minima, and between two consecutive zeros within one full cycle at  $x = 1/2$  are not identical.

### 3-5-3. Graph of $\cos(x,y)$

The 3D plot of  $\cos(x,y)$  is shown in Figures 3.1a and 3.1b. Since  $\cos(x,y) = \cos(x \ln y)$ , its behavior follows that of the cosine function. The graph appears to have local maxima and minima but no saddle points. In addition, the range of  $\cos(x,y)$  lies in  $[-1, 1]$  for all  $x \in \mathbb{R}$ .

**3D Surface of  $z = \cos(x,y)$**

Graph:  $\cos(x,y)$  for  $x$  in  $(-4\pi, 4\pi)$

Graph:  $\cos(x,y)$  for  $x$  in  $(-\pi, \pi)$

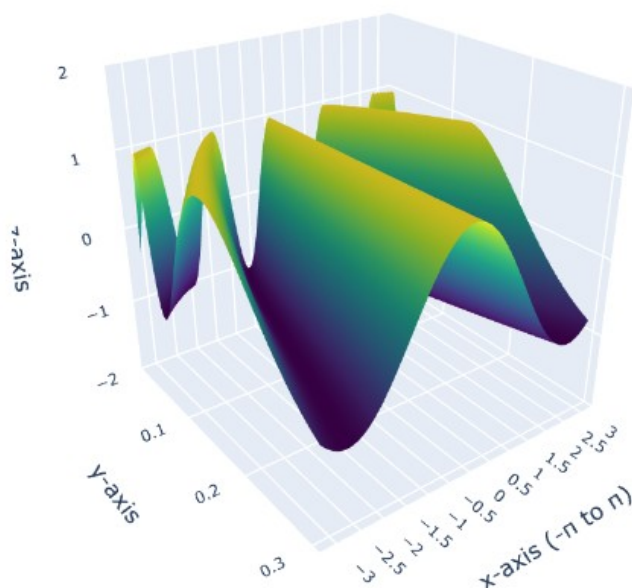
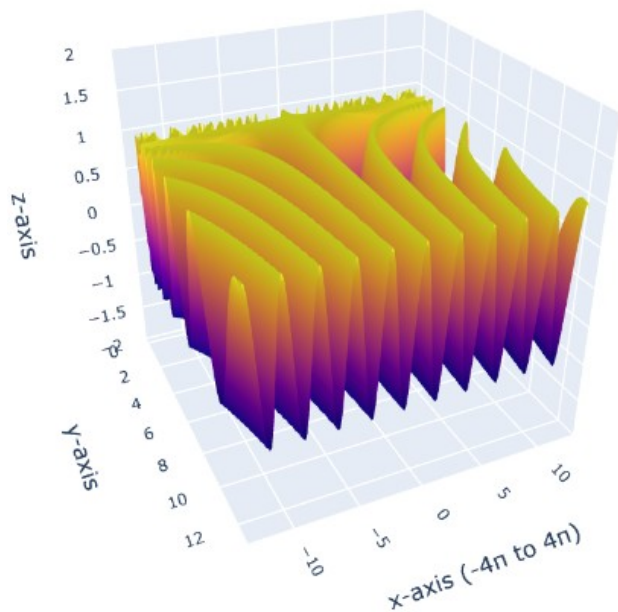


Figure 3.1a shows graphs of  $\cos(x,y) \equiv \frac{1}{2}(y^{ix} + y^{-ix})$  on different domains.

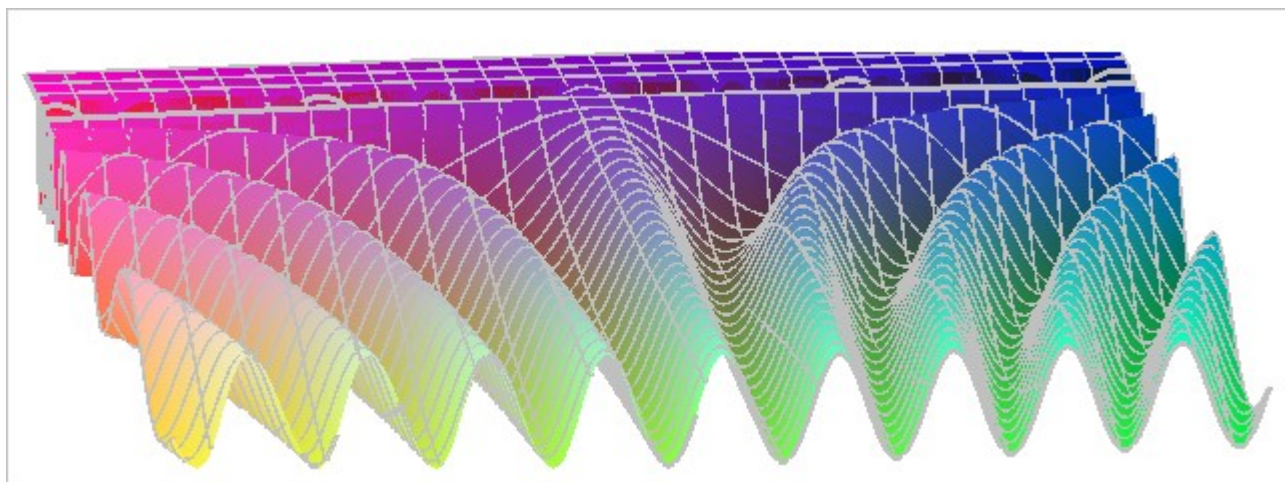


Figure 3.1b shows overall behavior of  $\cos(x,y) \equiv \frac{1}{2}(y^{ix} + y^{-ix})$ .

### 3-5-4. Local Behavior of $\cos(x,y)$ via Calculus

#### A. Critical Points: Local Maxima, Minima and Saddle Points (Review)

We use the Hessian discriminant<sup>[3]</sup> to determine the critical points of the function  $f(x,y)$  as follows:

*i. First Derivative Test (finding critical points):* Calculate the first partial derivatives,  $f_x$  and  $f_y$ . Then set both derivatives equal to zero ( $f_x = 0$  and  $f_y = 0$ ), and solve the resulting system to obtain all ordered pairs  $(x,y)$  where both derivatives vanish. Also, identify any points where  $f_x$  and  $f_y$  are undefined.

*ii. Second Derivative Test (classify the critical point  $(x_0, y_0)$ ):*

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

\*> If  $D > 0$  and  $f_{xx}(x_0, y_0) < 0$ , then  $(x_0, y_0)$  is a local maximum of  $f$ .

\*> If  $D > 0$  and  $f_{xx}(x_0, y_0) > 0$ , then  $(x_0, y_0)$  is a local minimum of  $f$ .

\*> If  $D < 0$ , then  $(x_0, y_0)$  is a saddle point of  $f$ .

\*> If  $D = 0$ , the test is inconclusive (no conclusion).

#### B. Find Local Maxima, Local Minima and Saddle Points of $\cos(x,y)$

We will find the critical points of  $f(x, y) = \cos(x, y)$  on the domain  $y > 0$ , then classify them as maxima, minima, or saddle points.

##### (i). First partial derivatives

We have

$$f_x = \frac{d}{dx} \cos(x \ln y) = -\sin(x \ln y) \cdot \ln y$$

$$f_y = \frac{d}{dy} \cos(x \ln y) = -\sin(x \ln y) \cdot x \cdot \frac{1}{y} = -\frac{x}{y} \sin(x \ln y)$$

##### (ii). Critical points

The critical points occur when both partial derivatives are zero:

$$f_x = 0 \Rightarrow \sin(x \ln y) \cdot \ln y = 0 \Rightarrow \begin{cases} \sin(x \ln y) = 0 & \Rightarrow (a) \quad x \ln y = \pi k, \text{ for } k \in \mathbb{Z} \\ \ln y = 0 & \Rightarrow (b) \quad y = 1 \end{cases}$$

$$f_y = 0 \Rightarrow \frac{x}{y} \sin(x \ln y) = 0 \Rightarrow x \sin(x \ln y) = 0 \Rightarrow \begin{cases} (c) \quad x = 0 \\ \sin(x \ln y) = 0 \Rightarrow (d) \quad x \ln y = \pi k, \text{ for } k \in \mathbb{Z} \end{cases}$$

The results can be summarized in three cases:

- Case 1: At  $y = 1$ , it is true for all real  $x$ . Therefore, it provides a set of critical points along the line  $y_0 = 1$ , meaning that every point of the form  $(x, 1)$  is considered a critical point.
- Case 2: At  $x = 0$ , it is true for all real  $y > 0$ . Then every point of the form  $(0, y)$  is a critical point.
- Case 3: At  $x \ln y = k\pi$ , for  $y > 0$  and  $k \in \mathbb{Z}$ , it provides a full set of the critical points along  $x \ln y = k\pi$ . Notice that the result indicates the set also covers in Case 1 and Case 2 (when  $k = 0$ ).

Thus, the full set of critical points is given by the family of the curves

$$x \ln y = k\pi, \quad y > 0, \quad k \in \mathbb{Z}.$$

### (iii). Second order partial derivatives

$$f_{xx} = -\cos(x \ln y) \cdot \ln^2 y$$

$$f_{yy} = -\frac{x \cos(x \ln y) \frac{x}{y} \cdot y - x \sin(x \ln y)}{y^2} = \frac{x \sin(x \ln y) - x^2 \cos(x \ln y)}{y^2}$$

$$f_{xy} = -\frac{x}{y} \cdot \ln y \cos(x \ln y) - \frac{1}{y} \sin(x \ln y)$$

### (iv). Calculate discriminant D

At critical curve defined by  $x_0 \ln y_0 = \pi k$  for  $y_0 > 0$ , then  $\sin(\pi k) = 0$ , and we have:

$$\begin{aligned} D(x_0, y_0) &= (f_{xx})(f_{yy}) - (f_{xy})^2 \\ &= (-\cos(\pi k) \ln^2 y_0) \left( \frac{x_0 \sin(\pi k) - x_0^2 \cos(\pi k)}{y_0^2} \right) - \left( -\frac{x_0}{y_0} \ln y_0 \cos(\pi k) - \frac{1}{y_0} \sin(\pi k) \right)^2 \\ &\hspace{15em} \text{(using the substitution } \sin(\pi k) = 0) \\ &= (-\cos(\pi k) \ln^2 y_0) \left( \frac{-x_0^2 \cos(\pi k)}{y_0^2} \right) - \left( -\frac{x_0}{y_0} \ln y_0 \cos(\pi k) \right)^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{x_0^2}{y_0^2} \ln^2 y_0 \cos^2(\pi k) - \frac{x_0^2}{y_0^2} \ln^2 y_0 \cos^2(\pi k) \\
&= 0.
\end{aligned}$$

Since  $D(x_0, y_0) = 0$ , the Second Derivative Test is inconclusive, and it does not classify the local extrema of  $f(x, y) = \cos(x \ln y)$ . Along the critical curves  $x_0 \ln y_0 = \pi k$ , we have

$$f(x_0, y_0) = \cos(\pi k) = (-1)^k.$$

Thus,

- if  $k$  is even ( $k=2k'$ ),  $f(x_0, y_0) = +1$ , so  $f$  attains a local maximum along  $x_0 \ln y_0 = 2\pi k'$ ,  $k' \in \mathbb{Z}$ .
- if  $k$  is odd ( $k=2k'+1$ ),  $f(x_0, y_0) = -1$ , so  $f$  attains a local minimum along  $x_0 \ln y_0 = \pi(2k'+1)$ ,  $k' \in \mathbb{Z}$ .

No saddle points exist, as around each critical point,  $f(x_0, y_0)$  does not demonstrate opposing directional behavior. Rather, perturbations of  $x_0 \ln y_0$  cause  $f(x_0, y_0)$  to shift uniformly away from  $\pm 1$ .

**Remark 3.4.2.** Furthermore, if we regard saddle points as analogous to inflection points, then the inflection points of the function  $\cos(x)$ , for instance, are situated at  $x = \pi/2 + \pi k$ , where  $k \in \mathbb{Z}$ . Consequently, there exist points that are similar to saddle points where  $\cos(x \ln(y)) = 0$ , particularly at  $x \ln(y) = \pi/2 + \pi k$ ,  $k \in \mathbb{Z}$ . This outcome indicates the existence of geometric lines of inflections to generalize the equation  $f(x_0, y_0) = \cos(x_0 \ln(y_0)) = 0$ .

#### (v). Summary of all types of critical points of $\cos(x, y)$

- Critical points: All  $(x_0, y_0)$  with  $x_0 \ln y_0 = \pi k$ ,  $y > 0$ ,  $k \in \mathbb{Z}$ .
- Local maxima:  $x_0 \ln y_0 = 2\pi k$ ,  $k \in \mathbb{Z}$ .
- Local minima:  $x_0 \ln y_0 = \pi(2k+1)$ ,  $k \in \mathbb{Z}$ .
- Saddle points: None (which is based on the saddle definition. However, there exist lines of inflections which are the multiple analogues of the zero-crossings.)

**Remark 3.4.3.** For  $f(x,y) = \cos(x \ln y)$ , local extrema coincide with global extrema, and they are simply 1 and -1, occurring infinitely often.

**Remark 3.4.4.** Although the Second Derivative Test cannot determine the local extrema of the function  $f(x,y)$ , we can still identify them based on the critical points and the nature properties of the cosine function. We will further examine these points and the characteristics of the function  $f(x,y)$  in the coming sections.

### 3-5-5. Brief Overview Example from Calculus

In calculus, when we differentiate or integrate  $\sin(xy)$  with respect to  $x$  or  $y$ , the procedure is the same as for the single-variable sine function, except that we treat the other variable as a constant. For example:

$$1. \quad \frac{d}{dx} \cos(x, y) = \frac{d}{dx} \cos(x \ln y) = -\ln(y) \sin(x \ln y)$$

$$2. \quad \frac{d}{dy} \cos(x, y) = \frac{d}{dy} \cos(x \ln y) = -\frac{x}{y} \sin(x \ln y)$$

$$3. \quad \int \cos(x, y) dx = \int \cos(x \ln y) dx = \frac{1}{\ln(y)} \sin(x \ln y) + C$$

$$4. \quad \int \cos(x, y) dy = \int \cos(x \ln y) dy = \frac{y}{x^2+1} (\cos(x \ln y) + x \sin(x \ln y)) + C$$

We offer only a brief overview example from calculus, and we do not discuss the multivariable calculus in detail here, as that is a separate topic.

If the reader is interested in how we obtain the result in #4, the derivation is given below.

Let  $u = \ln(y)$ ,  $y = e^u$ ,  $dy = e^u du$ . By substitution, the given integral becomes

$$\int \cos(x, y) dy = \int \cos(x \ln y) dy = \int \cos(xu) e^u du$$

By applying the formula<sup>[7]</sup>  $\int \cos(a \cdot u) e^u du = \frac{e^u}{a^2+1} (\cos(au) + a \sin(au))$  and substituting back  $u = \ln(y)$  and  $e^u = y$ , we obtain

$$\int \cos(x \ln y) dy = \frac{y}{x^2+1} (\cos(x \ln y) + x \sin(x \ln y)) + C$$

### 3-5-6. Summary of $\cos(x,y)$

We have introduced a novel function  $\cos(x,y)$ , which extends the classical cosine function with one variable into a two variable context. Formula (3.17) defines the extended cosine function with two arguments as  $\cos(x,y) \equiv \cos(x \ln y)$ . The function  $\cos(x,y)$  has domain  $x \in (-\infty, \infty)$  with  $y > 0$ , and range  $[-1,1]$ , therefore, its entire graph exists only on one side of the  $x$ -axis. Since  $\cos(0) = 1$ , there is a constant peak at a height of 1 running along this  $x$ -axis where  $y = 1$  or along the  $y$ -axis where  $x = 0$ . It has a constant period with respect to  $x$  for each fixed  $y$  and a non-constant period with respect to  $y$  for each fixed  $x$ . The function represents a cosine-like surface in 3D in which it has local extrema and no saddle points. It is clear that when  $y$  equals  $e$ , the function  $\cos(x, e)$  is equal to  $\cos(x)$ . While retaining similarities to the traditional cosine function, this extended cosine function has well-defined derivatives and integrals, and operates in three dimensional space, resulting in a surface characterized by its unique interplay between the variables  $x$  and  $y$ .

**Note:** From now on, we will use the phrases "new two variable function" and "extended two variable function" interchangeably throughout the text. For example, this means that "new  $\cos(x,y)$ " and "extended  $\cos(x,y)$ " or "new cosine" and "extended cosine" will be used interchangeably.

### Section 3-6. Foundations of the Extended Function $\sin(x, y)$

We will follow the same procedure as shown Section 3-5 to define definition and derive the properties of the extended function  $\sin(x,y)$ .

#### 3-6-1. Definition and Theorem of $\sin(x,y)$

**Definition.** The extended sine function, denoted as  $\sin(x,y)$ , is defined by

$$\sin(x, y) \equiv \frac{1}{2i} (y^{ix} - y^{-ix}) \text{ for } x, y \in \mathbb{R} \text{ and } y > 0. \quad (3.35)$$

**Theorem 11.** The extended sine function  $\sin(x,y)$  satisfies the relation

$$\sin(x, y) \equiv \frac{1}{2i} (y^{ix} - y^{-ix}) = \sin(x \ln y) \text{ for real } x, y \text{ and } y > 0. \quad (3.36)$$

**Proof**

By subtracting (3.14) from (3.13), we have

$$y^{ix} - y^{-ix} = 2i \sin(x \ln y)$$

$$\Rightarrow \sin(x \ln y) = \frac{1}{2i} (y^{ix} - y^{-ix}).$$

Combining this result with definition (3.35) yields

$$\sin(x, y) \equiv \frac{1}{2i} (y^{ix} - y^{-ix}) = \sin(x \ln y).$$

**3-6-2. Basic Properties of  $\sin(x, y)$** 

**1. Domain:**  $x \in (-\infty, \infty)$ ,  $y > 0$ .

**2. Range:**  $-1 \leq \sin(x, y) \equiv \sin(x \ln y) \leq 1$ .

**3. Function type:** Symmetry in  $x$ :  $\sin(x, y)$  is even for all  $y > 0$ . Symmetry in  $y$ : The function is neither even nor odd in  $y$ .

**Proof**

3. We consider two cases:

a. Symmetry in  $x$ : We have  $\sin(-x, y) = \sin[(-x)(\ln(y))] = -\sin(x \ln(y)) = -\sin(x, y)$ . Therefore, the function is odd in  $x$  for all  $y > 0$ .

b. Symmetry in  $y$ : Since  $\ln(y)$  is defined only for  $y > 0$ , the function  $\sin(x, -y)$  is undefined because  $\ln(-y)$  is not real. The function is neither even nor odd in  $y$ .

**4. Maxima, Minima and Zeros of  $\sin(x, y)$  with Respect to  $x$** **4A. Maxima of  $\sin(x, y)$  in  $x$** 

**Theorem 12.** For fixed  $y$ , the maxima of  $\sin(x, y)$  with respect to  $x$  are given by

$$x = \frac{\pi(1+4k)}{2|\ln y|}, k \in \mathbb{Z}. \quad (3.37)$$

**Proof**

For fixed  $y$ , the maxima of  $\sin(x,y)$  in  $x$  occur when  $\sin(x,y) \equiv \sin(x \ln y) = 1$ . It follows that

$$\begin{aligned}\sin(x \ln y) &= \sin\left(\frac{\pi}{2} + 2\pi k\right), k \in \mathbb{Z} \\ \Rightarrow x \ln y &= \frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}\end{aligned}$$

Solving for  $x$  in terms of  $y$  and  $k$  gives

$$x = \frac{\pi(1+4k)}{2|\ln y|}, \quad k \in \mathbb{Z}.$$

**4B. Minima of  $\sin(x,y)$  in  $x$** 

**Theorem 13.** For fixed  $y$ , the minima of  $\sin(x,y)$  with respect to  $x$  are given by

$$x = \frac{\pi(-1+4k)}{2|\ln y|}, \quad k \in \mathbb{Z}. \quad (3.38)$$

**Proof**

For fixed  $y$ , the minima of  $\sin(x,y)$  in  $x$  occur when  $\sin(x,y) \equiv \sin(x \ln y) = -1$ . It follows that

$$x \ln y = -\frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}$$

Solving for  $x$  in terms of  $y$  yields

$$x = \frac{\pi(-1+4k)}{2|\ln y|}, \quad k \in \mathbb{Z}.$$

**4C. Zeros of  $\sin(x,y)$  in  $x$** 

**Theorem 14.** For fixed  $y$ , the zeros of  $\sin(x,y)$  are given by

$$x = \frac{\pi k}{|\ln y|}, \quad k \in \mathbb{Z}. \quad (3.39)$$

**Proof**

The zeros of  $\sin(x, y)$  in  $x$  occur when  $\sin(x, y) \equiv \sin(x \ln y) = 0$ . Because  $\sin(x \ln y + 2\pi k) = \sin(x \ln y)$  and  $k \in \mathbb{Z}$ , it follows that

- **Form 1:**

$$x \ln y = \begin{cases} 2\pi k \\ \pi + 2\pi k \end{cases}, \quad k \in \mathbb{Z}.$$

Solving for  $x$  gives

$$x = \begin{cases} \frac{2\pi k}{|\ln y|} \\ \frac{\pi(2k+1)}{|\ln y|} \end{cases}, \quad k \in \mathbb{Z}.$$

- **Form 2:**

The solutions expressed in Form 1 for integer  $k$  can be combined into a simplified form so that that it represents the zeros at every integer multiple of  $\pi$ :

$$x = \frac{\pi k}{|\ln y|}, \quad k \in \mathbb{Z}.$$

**5. Periodicity in  $x$** 

**Theorem 15.** For fixed  $y > 0$ , the period of  $\sin(x, y)$  with respect to  $x$  is given by

$$T_x(y) = \frac{2\pi}{|\ln(y)|}, \quad y > 0, y \neq 1, k \in \mathbb{Z}. \quad (3.42)$$

**Proof**

Assume  $T_x(y)$  is the period of  $\sin(x, y)$  in  $x$ . Then  $T_x(y)$  must satisfies the equation

$$\begin{aligned} \sin(x + T_x(y), y) &= \sin(x, y) \\ \Rightarrow \sin((x + T_x(y)) \ln(y)) &= \sin(x \ln y) \end{aligned}$$

$$\Rightarrow (x + T_x(y)) \ln(y) = x \ln y + 2\pi k, \quad k \in \mathbb{Z}.$$

Solving for  $T_x(y)$  gives

$$T_x(y) = \frac{2\pi k}{|\ln(y)|}, \quad y > 0, y \neq 1, k \in \mathbb{Z}.$$

Observe that **for each specific value of  $y$  (a 'slice' of  $y$ ), the period  $T_x(y)$  of the function  $\sin(x, y)$  in  $x$  remains constant.** However, when the value of  $y$  varies, the period  $T_x(y)$  of  $\sin(x, y)$  also varies. Despite this variation, the function retains its periodic nature, with a repeating cycle, due to its underlying characteristic as an elementary sine function. As a result, **the overall period of  $\sin(x, y)$  is still repetitive and non-constant. Thus, the  $\sin(x, y)$  is non-constant period in  $x$  and is not uniform across the entire domain  $y$ .**

## 6. Periodicity in $y$

**Theorem 16.** For each fixed  $x$ , the period of  $\sin(x, y)$  with respect to  $y$  is given by

$$T_y(x, y) = y(e^{2\pi k/|x|} - 1), \quad k \in \mathbb{Z}. \quad (3.43)$$

### Proof

Assume  $T_y$  (or  $T_y(x, y)$ ) is the period of  $\sin(x, y)$  in  $y$ . Then  $T_y$  must satisfies the equation

$$\begin{aligned} \sin(x, y + T_y) &= \sin(x, y) \\ \Rightarrow \sin(x \ln(y + T_y)) &= \sin(x \ln y) \\ \Rightarrow x \ln(y + T_y) &= x \ln y + 2\pi k, \quad k \in \mathbb{Z}. \\ \Rightarrow x \ln\left(\frac{y + T_y}{y}\right) &= 2\pi k \\ \Rightarrow \ln\left(\frac{y + T_y}{y}\right) &= \frac{2\pi k}{|x|} \\ \Rightarrow \frac{y + T_y}{y} &= e^{2\pi k/|x|}. \end{aligned}$$

Solving for  $T_y$  in terms of  $|x|$ ,  $y$  and  $k$  gives

$$T_y(x, y) = y(e^{2\pi k/|x|} - 1), \quad x \neq 0, \quad k \in \mathbb{Z}.$$

Since  $T_y(x, y)$  depends on  $x$ ,  $y$  and  $k$  in  $y$ , the period of  $\sin(x, y)$  with respect to  $y$  is not a constant. **The function  $\sin(x, y)$  is still repetitive in terms of  $x$  and  $y$ , but the function is not uniform across the entire domains  $x$  and  $y$ .**

## 7. Wavelength with respect to $x$

**Theorem 17.** For fixed  $y > 0$ , the wavelength  $D_x(y)$  between two consecutive maxima, between two consecutive minima, or between two consecutive zeros of  $\sin(x, y)$  in  $x$  are:

**a) Maxima:**

$$D_x(y) = \frac{2\pi}{|\ln y|} \quad (3.44)$$

**b) Minima:**

$$D_x(y) = \frac{2\pi}{|\ln y|} \quad (3.45)$$

**c) Zeros of  $\sin(x, y)$ :**

$$D_x(y) = \frac{2\pi}{|\ln y|} \quad (3.46)$$

### Proof

a) Wavelength between two consecutive maxima:

From (3.37), the critical points for maxima are

$$x_k = \frac{\pi(1+4k)}{2|\ln y|}, \quad k \in \mathbb{Z}.$$

The wavelength (distance) between two consecutive maxima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{\pi(1+4(k+1))}{2|\ln y|} - \frac{\pi(1+4k)}{2|\ln y|} \end{aligned}$$

$$= \frac{2\pi}{|\ln y|}.$$

b) Distance between two consecutive minima:

From formula (3.38), the critical points for minima are

$$x_k = \frac{\pi(-1+4k)}{2|\ln y|}, \quad k \in \mathbb{Z}.$$

The wavelength between two consecutive minima is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{\pi(-1+4(k+1))}{2|\ln y|} - \frac{\pi(-1+4k)}{2|\ln y|} \\ &= \frac{2\pi}{|\ln y|}. \end{aligned}$$

c) Wavelength between two zeros of  $\sin(x,y)$  within one full cycle:

From formula (3.39), the critical points are

$$x_k = \frac{2\pi k}{|\ln y|}, \quad k \in \mathbb{Z}.$$

The wavelength between two consecutive zeros of  $\sin(x,y)$  in  $x$  is

$$\begin{aligned} D_x(y) &= x_{k+1} - x_k \\ &= \frac{2\pi(k+1)}{|\ln y|} - \frac{2\pi k}{|\ln y|} \\ &= \frac{2\pi}{|\ln y|}. \end{aligned}$$

Thus, the wavelengths between consecutive maxima, between consecutive minima, and between consecutive zeros of  $\sin(x,y)$  are identical, each being  $D_x(y) = 2\pi/|\ln y|$ .

## 8. Maxima, Minima and Zeros of $\sin(x,y)$ with Respect $y$

**Theorem 18.** For fixed  $x$ , the maxima, minima or zeros of  $\sin(x,y)$  with respect to  $y$  are:

### a. Maxima in $y$

$$y = e^{\frac{\pi(1+4k)}{4|x|}}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.47)$$

### b. Minima in $y$

$$y = e^{\frac{\pi(-1+4k)}{4|x|}}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.48)$$

### c. Zeros of $\sin(x,y)$ in $y$

$$y = \begin{cases} e^{\frac{2\pi k}{|x|}} \\ e^{\frac{\pi(1+2k)}{|x|}} \end{cases}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.49)$$

Or we can express it in a simplified form

$$y = e^{\frac{\pi k}{|x|}}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.50)$$

## Proof

### a. Maxima of $\sin(x,y)$ in $y$

The maxima of  $\sin(x,y)$  in  $y$  occur when  $\sin(x,y) \equiv \sin(x \ln y) = 1$ . It follows that

$$\begin{aligned} x \ln y &= \frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z} \\ \Rightarrow \ln y &= \frac{\pi(1+4k)}{4|x|}, \quad x \neq 0, k \in \mathbb{Z}. \end{aligned}$$

Applying the exponential function to both sides yields

$$y = e^{\frac{\pi(1+4k)}{4|x|}}, \quad x \neq 0, k \in \mathbb{Z}.$$

### b. Minima of $\sin(x,y)$ in $y$

The minima of  $\sin(x,y)$  with respect to  $y$  occur when  $\sin(x,y) \equiv \sin(x \ln y) = -1$ . It follows that

$$x \ln y = -\frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$$

$$\Rightarrow \ln y = \frac{\pi(-1+4k)}{4|x|}, x \neq 0, k \in \mathbb{Z}.$$

Applying the exponential function to both sides gives

$$y = e^{\frac{\pi(-1+4k)}{4|x|}}, x \neq 0, k \in \mathbb{Z}.$$

### c. Zeros of $\sin(x, y)$ in $y$

The zeros of  $\sin(x, y)$  in  $y$  occur when  $\sin(x, y) \equiv \sin(x \ln y) = 0$ . It follows that

$$x \ln y = \begin{cases} 2\pi k \\ \pi + 2\pi k. \end{cases}$$

Solving for  $y$  in terms of  $x$  gives

$$y = \begin{cases} e^{\frac{2\pi k}{|x|}} \\ e^{\frac{\pi(1+2k)}{|x|}}, \end{cases} x \neq 0, k \in \mathbb{Z}.$$

The solutions can be combined in a simplified form:

$$y = e^{\frac{\pi k}{|x|}}, x \neq 0, k \in \mathbb{Z}.$$

## 9. Wavelength with Respect to $y$

**Theorem 19.** For fixed  $x$ , the distance or wavelength, denoted  $D_y(x)$ , between two consecutive maxima, minima, or  $y$ -intercepts of  $\sin(x, y)$  is defined as follows:

### a) Wavelength between two consecutive maxima

$$D_y(x) = e^{\frac{\pi(1+4k)}{4|x|}} (e^{\pi/|x|} - 1), x \neq 0, k \in \mathbb{Z}. \quad (3.51)$$

### b) Wavelength between two consecutive minima

$$D_y(x) = e^{\pi(1+2k)/|x|} (e^{2\pi/|x|} - 1), \quad x \neq 0, k \in \mathbb{Z}. \quad (3.52)$$

**c) Wavelength between two consecutive zeros within one full cycle**

$$D_y(x) = \begin{cases} e^{\frac{2\pi k}{|x|}} (e^{2\pi/|x|} - 1) \\ \text{or} \\ e^{\frac{\pi(1+2k)}{|x|}} (e^{2\pi/|x|} - 1) \end{cases}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.53)$$

**Proof**

a. From (3.47), the maxima points in terms of  $k$  are

$$y_k = e^{\frac{\pi(1+4k)}{4|x|}}, \quad x \neq 0, k \in \mathbb{Z}.$$

The distance between two consecutive maxima is determined by

$$\begin{aligned} D_y(x) &= y_{k+1} - y_k \\ &= e^{\frac{\pi(1+4(k+1))}{4|x|}} - e^{\frac{\pi(1+4k)}{4|x|}} \\ &= e^{\frac{\pi(1+4k)}{4|x|}} (e^{\pi/|x|} - 1). \end{aligned}$$

b. From (3.48) the minima points in terms of  $k$  are

$$y_k = e^{\frac{\pi(-1+4k)}{4|x|}}, \quad x \neq 0, k \in \mathbb{Z}.$$

The distance between two consecutive minima is determined by

$$\begin{aligned} D_y(x) &= y_{k+1}(x) - y_k(x) \\ &= e^{\frac{\pi(-1+4(k+1))}{4|x|}} - e^{\frac{\pi(-1+4k)}{4|x|}} \\ &= e^{\frac{\pi(-1+4k)}{4|x|}} (e^{\pi/|x|} - 1) \end{aligned}$$

c. From formula (3.49), the zeros of  $\sin(x,y)$  in terms of  $k$  are

$$y_k = \begin{cases} e^{\frac{2\pi k}{|x|}} \\ e^{\frac{\pi(1+2k)}{|x|}}, \end{cases} \quad x \neq 0, k \in \mathbb{Z}.$$

Therefore, the distance between consecutive zeros of  $\sin(x,y)$  is determined by

$$\begin{aligned} D_y(x) &= y_{k+1}(x) - y_k(x) \\ &= \begin{cases} e^{\frac{2\pi(k+1)}{|x|}} - e^{\frac{2\pi k}{|x|}} \\ e^{\frac{\pi(1+2(k+1))}{|x|}} - e^{\frac{\pi(1+2k)}{|x|}}, \end{cases} \quad k \in \mathbb{Z} \\ &= \begin{cases} e^{\frac{2\pi k}{|x|}} (e^{2\pi/|x|} - 1) \\ e^{\frac{\pi(1+2k)}{|x|}} (e^{2\pi/|x|} - 1) \end{cases}, \quad k \in \mathbb{Z}. \end{aligned}$$

Based on the results (a), (b) and (c), we can observe that the distances  $D_y(x)$  are not identical. It indicates that  $\sin(x,y)$  has a non-constant period in  $y$ .

### Example

Given the function  $\sin(x,y)$ , consider the slice at  $x = 2$  in  $y$ .

- What is the distance between two consecutive maxima?
- What is the spacing between two consecutive minima?
- What is the distance between two consecutive zeros within one full cycle.
- Determine the period of the  $\sin(x,y)$ .

### Solution

#### Slice at $x = 2$

a. According to the part (a) of Theorem 19, the distance  $T_y(x)$  between two consecutive maxima in  $y$  is

$$\begin{aligned} T_y(x) &= e^{\frac{\pi(1+4k)}{4|x|}} (e^{\pi/|x|} - 1) \\ &= e^{\frac{\pi(1+4k)}{8}} (e^{\pi/2} - 1), \quad k \in \mathbb{Z}. \end{aligned}$$

b. From part (b) of Theorem 19, the distance between the two consecutive minima is

$$\begin{aligned} T_y(x) &= e^{\pi(1+2k)/|x|} (e^{2\pi/|x|} - 1) \\ &= e^{\pi(1+2k)/2} (e^\pi - 1), \quad k \in \mathbb{Z}. \end{aligned}$$

c. From part (c) of Theorem 19, the distance between two zeros of  $\sin(x,y)$  within one full cycle is

$$\begin{aligned} T_y(x) &= \begin{cases} e^{\frac{2\pi k}{|x|}} (e^{2\pi/|x|} - 1) \\ \text{or} \\ e^{\frac{\pi(1+2k)}{|x|}} (e^{2\pi/|x|} - 1) \end{cases} \\ &= \begin{cases} e^{\pi k} (e^\pi - 1) \\ \text{or} \\ e^{\frac{\pi(1+2k)}{2}} (e^\pi - 1) \end{cases}, \quad k \in \mathbb{Z}. \end{aligned}$$

d. The period of  $\sin(x,y)$  in  $y$  is

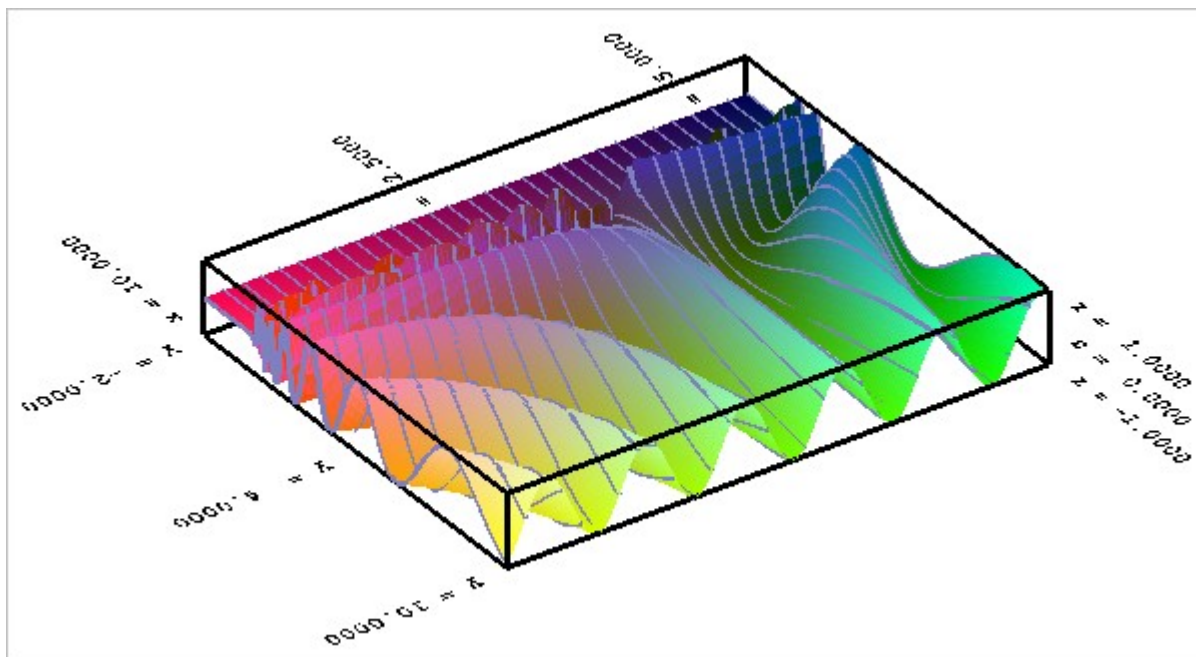
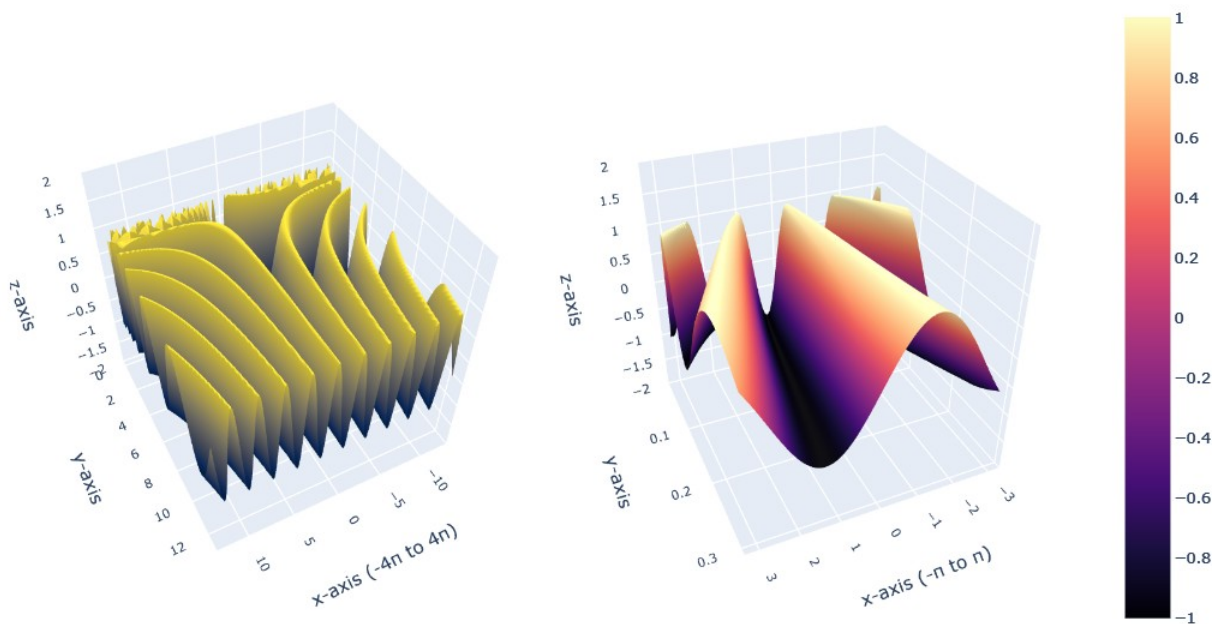
$$T_y(x,y) = y(e^{\pi k} - 1), \quad k \in \mathbb{Z}.$$

The results from parts (a), (b), and (c) show that the distances between two consecutive maxima, between two consecutive minima, and between two consecutive zeros of  $\sin(x,y)$  within one full cycle at  $x = 2$  are not identical.

### 3-6-3. Graph of $\sin(x,y)$

The 3D graph of  $\sin(x,y)$  is shown in Figure 3.2. It behaves like sine function since  $\sin(x,y)$  can be expressed as  $\sin(x \ln y)$ . Its general form is similar to that of  $\cos(x,y)$ , but it exhibits a saddle point at  $(0,1)$ .

3D Surface of  $z = \sin(x,y)$



**Figure 3.2** provides the surface plots of  $\sin(x,y) \equiv \frac{1}{2i}(y^{ix} - y^{-ix})$  on several domains.

**3-6-4. Local Maxima, Minima and Saddle Points of  $\sin(x,y)$  via Calculus**

We will find the extrema and saddle points of  $f(x,y) = \sin(x,y)$ .

### First partial derivatives

$$f_x = \frac{d}{dx} \sin(x \ln y) = \cos(x \ln y) \cdot \ln y$$

$$f_y = \frac{d}{dy} \sin(x \ln y) = \frac{x}{y} \cos(x \ln y)$$

### Critical points

Setting  $f_x = 0$  and  $f_y = 0$ :

$$f_x = 0 \Rightarrow \cos(x \ln y) \cdot \ln y = 0 \Rightarrow \begin{cases} \cos(x \ln y) = 0 & \Rightarrow (a) \quad x \ln y = \pi/2 + \pi k, \text{ for } k \in \mathbb{Z} \\ \ln y = 0 & \Rightarrow (b) \quad y = 1 \end{cases}$$

$$f_y = 0 \Rightarrow \frac{x}{y} \cos(x \ln y) = 0 \Rightarrow x \cos(x \ln y) = 0 \Rightarrow \begin{cases} (c) \quad x = 0 \\ \cos(x \ln y) = 0 \Rightarrow (d) \quad x \ln y = \pi/2 + \pi k, \text{ for } k \in \mathbb{Z} \end{cases}$$

The solutions from (a), (b), (c) and (d) can be summarized in four cases:

- **Case 1:** At  $x = 0$  as resulted from (c), and  $y = 1$ , as obtained from (b), the point  $(x_0, y_0) = (0, 1)$  satisfies both  $f_x = 0$  and  $f_y = 0$ ; therefore, the point  $(0, 1)$  is a critical point.
- **Case 2:** At  $x = 0$  for any  $y > 0$ , it provides a set of the critical points along the line  $x_0 = 0$ .
- **Case 3:** At  $y = 1$  for any  $x$ , it provides a set of the critical points along the line  $y_0 = 1$ .
- **Case 4:** At points  $(x_0, y_0)$  satisfying  $\cos(x \ln y) = 0$ , we derive the set of critical points. This set is given by the family of curves

$$x_0 \ln y_0 = \pi/2 + \pi k, \text{ for } k \in \mathbb{Z}.$$

(Note that this set also covers in Case 2 and Case 3 for when  $k = 0$ .)

### Second order partial derivatives

$$f_{xx} = -\sin(x \ln y) \cdot \ln^2 y$$

$$f_{yy} = -\frac{x \cos(x \ln y) + x^2 \sin(x \ln y)}{y^2}$$

$$f_{xy} = \frac{1}{y} \cos(x \ln y) - \frac{x}{y} \ln y \sin(x \ln y)$$

**Calculate Discriminant D**

- At critical point  $(x_0, y_0) = (0, 1)$ :

$$\begin{aligned}
 D(0,1) &= (f_{xx})(f_{yy}) - (f_{xy})^2 \\
 &= (-\sin(0 \ln 1) \ln^2 1) \frac{(-0 \cos(0 \ln 1) + 0^2 \sin(0 \ln 1))}{1^2} - \left( \frac{1}{1} \cos(0 \ln 1) - \frac{0}{1} \ln 1 \sin(0 \ln 1) \right)^2 \\
 &= (0) \cdot (0) - (1 - 0)^2 = -1
 \end{aligned}$$

Since  $D(0,1) = -1 < 0$ , it is a saddle point.

- At critical curves defined by  $x_0 \ln y_0 = \pi/2 + \pi k$ , for  $k \in \mathbb{Z}$  and  $y_0 > 0$ :

$$\begin{aligned}
 D(x_0, y_0) &= (f_{xx})(f_{yy}) - (f_{xy})^2 \\
 &= (-\sin(x_0 \ln y_0) \cdot \ln^2 y_0) \cdot \left( -\frac{x_0 \cos(x_0 \ln y_0) + x_0^2 \sin(x_0 \ln y_0)}{y_0^2} \right) - \\
 &\quad \left( \frac{1}{y_0} \cos(x_0 \ln y_0) - \frac{x_0}{y_0} \ln y_0 \sin(x_0 \ln y_0) \right)^2 \\
 &= (-\sin(x_0 \ln y_0) \cdot \ln^2 y_0) \cdot \left( -\frac{x_0^2 \sin(x_0 \ln y_0)}{y_0^2} \right) - \left( \frac{x_0^2}{y_0^2} \ln^2 y_0 \sin^2(x_0 \ln y_0) \right) \\
 &= \frac{x_0^2}{y_0^2} \ln^2(y_0) \sin^2(x_0 \ln y_0) - \frac{x_0^2}{y_0^2} \ln^2(y_0) \sin^2(x_0 \ln y_0) \\
 &= 0
 \end{aligned}$$

Since  $D(x_0, y_0) = 0$ , the Second Derivative Test is inconclusive. However, near a critical curve  $x_0 \ln y_0 = \pi/2 + \pi k$ , it indicates that the critical points are local maxima when  $k$  is even, namely

$$x_0 \ln y_0 = \pi/2 + 2\pi k, \text{ for } k \in \mathbb{Z},$$

and the critical points are local minima when  $k$  is odd:

$$x_0 \ln y_0 = \frac{3\pi}{2} + 2\pi k, \text{ for } k \in \mathbb{Z}.$$

### Summary of critical points $\sin(x,y)$

- Local maxima: All  $(x_0, y_0)$  with  $x_0 \ln y_0 = \pi/2 + 2\pi k$ ,  $k \in \mathbb{Z}$ ;  $f(x_0, y_0) = 1$
- Local minima: All  $(x_0, y_0)$  with  $x_0 \ln y_0 = 3\pi/2 + 2\pi k$ ,  $k \in \mathbb{Z}$ ;  $f(x_0, y_0) = -1$
- Saddle point:  $(0, 1)$ ;  $f(0,1) = \sin(0,1) = 0$ . It is important to observe that analogous saddle points occur when  $\sin(x \ln y) = 0$  or  $x \ln y = \pi k$ ,  $k \in \mathbb{Z}$ . Please refer to **Remark 3.4.2**.

### 3-6-5. Summary of $\sin(x,y)$

Definition (3.35) defines the new sine function with two variables, and we demonstrate that this definition is equivalent to the function  $\sin(x \ln y)$ , which has domain  $x \in (-\infty, \infty)$  with  $y > 0$ , and range  $[-1, 1]$ . The sine graph looks like the cosine graph after it has been shifted along the logarithmic scale of the  $y$ -axis. The function  $\sin(x, y)$  has local extrema similar to  $\cos(x, y)$ , but unlike  $\cos(x, y)$ , it has a saddle point at  $\sin(0, 1) = 0$ . It is clear that when  $y$  equals  $e$ , the function  $\sin(x, e)$  is equal to  $\sin(x)$ . For each slice of  $y$ , the function exhibits a constant period with respect to  $x$ . However, as  $y$  varies in  $y$ , the overall period of the function becomes non-constant. The function  $\sin(x, y)$  represents a sine-like surface in three dimensions, as it originates from the elementary sine function. It exhibits repetitive behavior in terms of  $x$  and  $y$ , though its pattern is not uniform across the entire  $x$ - $y$  domain. Along individual  $x$ -direction or  $y$ -direction, the function possess all the key characteristics and properties of the fundamental sine function in that direction. In general, the interaction between  $x$  and  $y$  could alter the overall behavior of the function  $\sin(x, y)$ , even though it retains some key characteristics of sine in the individual directions.

### 3-6-6. Application

When a multivariable function is deep understood in one standard form, we can use it as a powerful model for studying many other function of the same structural type. In below example, we use known properties of  $\sin(x \ln y)$ , for instance its periodicity and maxima in  $y$  to study  $f(x, y) = \sin(xy)$ . We can transfer this

knowledge to function  $\sin(xy)$  by recognizing how the inner expression transforms the coordinate directions. A change of variables like  $y \rightarrow e^y$  converts multiplicative behavior in the standard function into additive behavior in the new one, allowing properties such as periods, extrema, and oscillation patterns to be obtained without repeating calculus.

### Example.

Find the periodicity and maxima of  $f(x,y) = \sin(xy)$  in  $y$ .

### Solution

We use the properties of the function  $\sin(x \ln y)$  in the left column to determine the periodicity and maxima of the function  $\sin(xy)$ . The findings are presented in the right column of the table below:

<b><math>f(x, y) = \sin(x \ln y)</math></b>	<b><math>f_1(x, y) = \sin(xy)</math></b>
<b>Variables:</b> $x, y$	<b>Change variables:</b> $x, e^y$
<b>Period in y-axis:</b> $T_y(x) = \left( e^{\frac{2\pi k}{ x }} - 1 \right) y, \quad x \neq 0, k \in \mathbb{Z}$	<b>Period in y-axis:</b> If $T$ is a period of $f_1(x, y)$ , then $e^T = T_y(x)$ . It implies that $T = \ln(T_y(x))$ .
<b>Local maxima:</b> $x_0 \ln y_0 = 2\pi k, k \in \mathbb{Z}$ .	<b>Local maxima:</b> $x_0 \ln e^{y_0} = 2\pi k, k \in \mathbb{Z}$ . It implies that $x_0 y_0 = 2\pi k, k \in \mathbb{Z}$ .

**Example.** Express the following function in the standard form:

$$f(x, y) = \sin[2(1 + x^2 \cdot y^2)]$$

### Solution

Multiplying out the argument and applying the identity  $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$  to the given function yields

$$\begin{aligned} f(x, y) &= \sin(2) \cos(2x^2 y^2) + \sin(2x^2 y^2) \cos(2) \\ &= \sin(2) \cos(2x^2, y^2) + \sin(2x^2, y^2) \cos(2). \end{aligned}$$

The given example illustrates that even complicated functions can be rewritten in a

standard form, making them easier to handle. In particular, it allows us to study them more effectively as functions of  $x$  or  $y$ , for example. This implies that by establishing a standard form for our functions, we can rewrite complicated expressions in a unified and manageable way. This allows us to analyze and control their behavior using the properties of the standard form.

### Section 3-7. Foundations of the Extended Tangent Function $\tan(x,y)$

We use this section to define definition and derive the theorems and properties of the function  $\tan(x,y)$ .

#### 3-7-1. Definition and Theorem

**Definition.** For any real numbers  $x, y$ , the extended tangent function, denoted as  $\tan(x,y)$ , is defined by

$$\tan(x, y) \equiv \frac{1}{i} \left( \frac{y^{ix} - y^{-ix}}{y^{ix} + y^{-ix}} \right), \quad (3.60)$$

where  $y^{ix} + y^{-ix} \neq 0$ , and  $y > 0$ .

**Theorem 21.** The extended tangent function satisfies the relation

$$\tan(x, y) \equiv \frac{1}{i} \left( \frac{y^{ix} - y^{-ix}}{y^{ix} + y^{-ix}} \right) = \tan(x \ln y), \quad (3.61)$$

where  $x, y$  are real with  $y > 0$ , and  $x \ln(y) \neq \pi/2 + k\pi \mid k \in \mathbb{Z}$ .

#### Proof

Definition (3.2) gives

$$\tan(x \ln y) = \frac{\sin(x \ln y)}{\cos(x \ln y)} = \frac{\sin(x, y)}{\cos(x, y)}$$

$$\begin{aligned}
&= \frac{1}{2i} (y^{ix} - y^{-ix}) \\
&= \frac{1}{2} \frac{(y^{ix} - y^{-ix})}{(y^{ix} + y^{-ix})} \\
&= \frac{1}{i} \frac{(y^{ix} - y^{-ix})}{(y^{ix} + y^{-ix})} \\
&\equiv \tan(x, y).
\end{aligned}$$

The proof is complete.

### Corollary 22

The function  $\tan(x, y)$  is defined as

$$\tan(x, y) = \frac{\sin(x, y)}{\cos(x, y)} \text{ for } x, y \in \mathbb{R} \text{ with } y > 0 \text{ and } x \ln y \neq 0. \quad (3.62)$$

### Proof

It results from the proof of Theorem 21.

### 3-7-2. Basic Properties of $\tan(x, y)$

**1. Domain:**  $x, y \in \mathbb{R}, y > 0, x \ln(y) \neq \pi/2 + k\pi \mid k \in \mathbb{Z}$ .

**2. Range:**  $-\infty < \tan(x, y) \equiv \tan(x \ln y) < \infty$ .

**3. Function type:** Symmetry in  $x$ : The function is odd in  $x$  for all  $y > 0$ ; symmetry in  $y$ : The function is neither even nor odd in  $y$ .

### Proof

**3. Function type:** We consider two cases:

a. Symmetry in  $x$ :  $\tan(-x, y) = \tan(-x \ln y) = -\tan(x \ln y) = -\tan(x, y)$ . Therefore, the function is odd in  $x$ .

b. Symmetry in  $y$ : Since  $\ln(y)$  only defined for  $y > 0$ , the function is neither even or odd in  $y$ .

## 4. Zeros of $\tan(x, y)$ with Respect to $x$ and $y$

### 4A. Zeros of $\tan(x, y)$ in $x$

**Theorem 23.** For fixed  $y > 0$ , the zeros of  $\tan(x, y)$  in  $x$  are given by

$$x_k = \frac{\pi k}{|\ln y|}, \quad y > 0, y \neq 1, x_k \neq \frac{\pi}{|\ln y|} \left( \frac{1}{2} + k \right), k \in \mathbb{Z}. \quad (3.63)$$

#### Proof

The zeros of  $\tan(x, y)$  in  $x$  occur when  $\tan(x, y) = \tan(x \ln y) = 0$ . Because  $\tan(x \ln y + \pi k) = \tan(x \ln y)$  for  $y > 0$  and  $k \in \mathbb{Z}$ , it follows that

$$x \ln y = \pi k, \quad y > 0, k \in \mathbb{Z}$$

Solving for  $x$  in terms of  $k$  and  $y$  gives

$$x = \frac{\pi k}{\ln y}, \quad y > 0, y \neq 1, k \in \mathbb{Z},$$

from which we complete the proof. Notice that we can use the absolute value of  $\ln(y)$  for convenience, as it helps distinguish the value of  $x$ , whether positive or negative, when  $0 < y < 1$ , while  $k$  is a positive integer in the set of  $\mathbb{Z}$ .

$$x = \frac{\pi k}{|\ln y|}, \quad y > 0, y \neq 1, x \neq \frac{\pi}{|\ln y|} \left( \frac{1}{2} + k \right), k \in \mathbb{Z}.$$

### 4B. Zeros of $\tan(x, y)$ in $y$

**Theorem 24.** For fixed  $x > 0$ , the zeros of  $\tan(x, y)$  in  $y$  are given by

$$y_k = e^{\frac{\pi k}{x}}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.64)$$

#### Proof

Using the proof of Theorem 23, we follow the same step-by-step procedure as before, but instead of solving for  $x$ , we now solve for  $y$  in terms of  $x$ . This completes the proof.

## 5. Periodicity of $\tan(x, y)$ in $x$ and $y$

### 5A. Periodicity in $x$ (for fixed $y > 0$ )

**Theorem 25.** For fixed  $y > 0$ , the period of  $\tan(x, y)$  is given by  $T_x(y) = \pi k / |\ln y|$  for  $y \neq 1$ , and  $k \in \mathbb{Z}$ .

#### Proof

The function  $\tan(x, y)$  has a period in  $x$  if there exists a period  $T_x(y) \equiv T_x$  such that

$$\tan(x + T_x, y) = \tan(x, y),$$

from which it follows that

$$\begin{aligned} \tan((x + T_x) \ln y) &= \tan(x \ln y), \quad y > 0, \\ \Rightarrow (x + T_x) \ln y &= x \ln y + \pi k, \quad y > 0, \quad k \in \mathbb{Z}, \\ \Rightarrow T_x \ln y &= \pi k, \quad y > 0, \quad k \in \mathbb{Z}. \quad (*) \end{aligned}$$

Solving for  $T_x$  from equation (\*) gives

$$T_x = \frac{\pi k}{|\ln y|}, \quad y > 0, \quad y \neq 1, \quad k \in \mathbb{Z}.$$

Since  $T_x(y)$  is a constant in  $x$  for each fixed  $y$ , but it is not a constant as a function of  $y$ . We conclude that  $\tan(x, y)$  is periodic in  $x$ , but the period depends on  $y$ .

### 5B. Periodicity in $y$ (for fixed $x$ )

**Theorem 26.** For fixed  $x$ , the period of  $\tan(x, y)$  is given by

$$T_y(x, y) = y(e^{\pi k/x} - 1), \quad \text{for } x \neq 0 \text{ and } k \in \mathbb{Z}.$$

#### Proof

The function  $\tan(x, y)$  has a period in  $y$  if there exists a period  $T_y(x) \equiv T_y$  such that

$$\tan(x, y + T_y) = \tan(x, y),$$

from which we have

$$\begin{aligned}
\tan(x \ln(y + T_y)) &= \tan(x \ln y) \\
\Rightarrow x \ln(y + T_y) &= x \ln y + \pi k, \quad k \in \mathbb{Z} \\
\Rightarrow \ln\left(\frac{y + T_y}{y}\right) &= \frac{\pi k}{x}, \quad x \neq 0, k \in \mathbb{Z} \\
\Rightarrow \frac{y + T_y}{y} &= e^{\pi k / x}, \quad x \neq 0, k \in \mathbb{Z}.
\end{aligned}$$

Solving for  $T_y$  gives

$$T_y \equiv T_y(x, y) = y(e^{\pi k / x} - 1), \quad x \neq 0, k \in \mathbb{Z},$$

from which  $T_y$  depends on  $x$ ,  $y$  and  $k$ . This leads to our conclusion that for each fixed value of  $x$ , the period of  $\tan(x, y)$  with respect to  $y$  is not constant, because it depends on  $y$ .

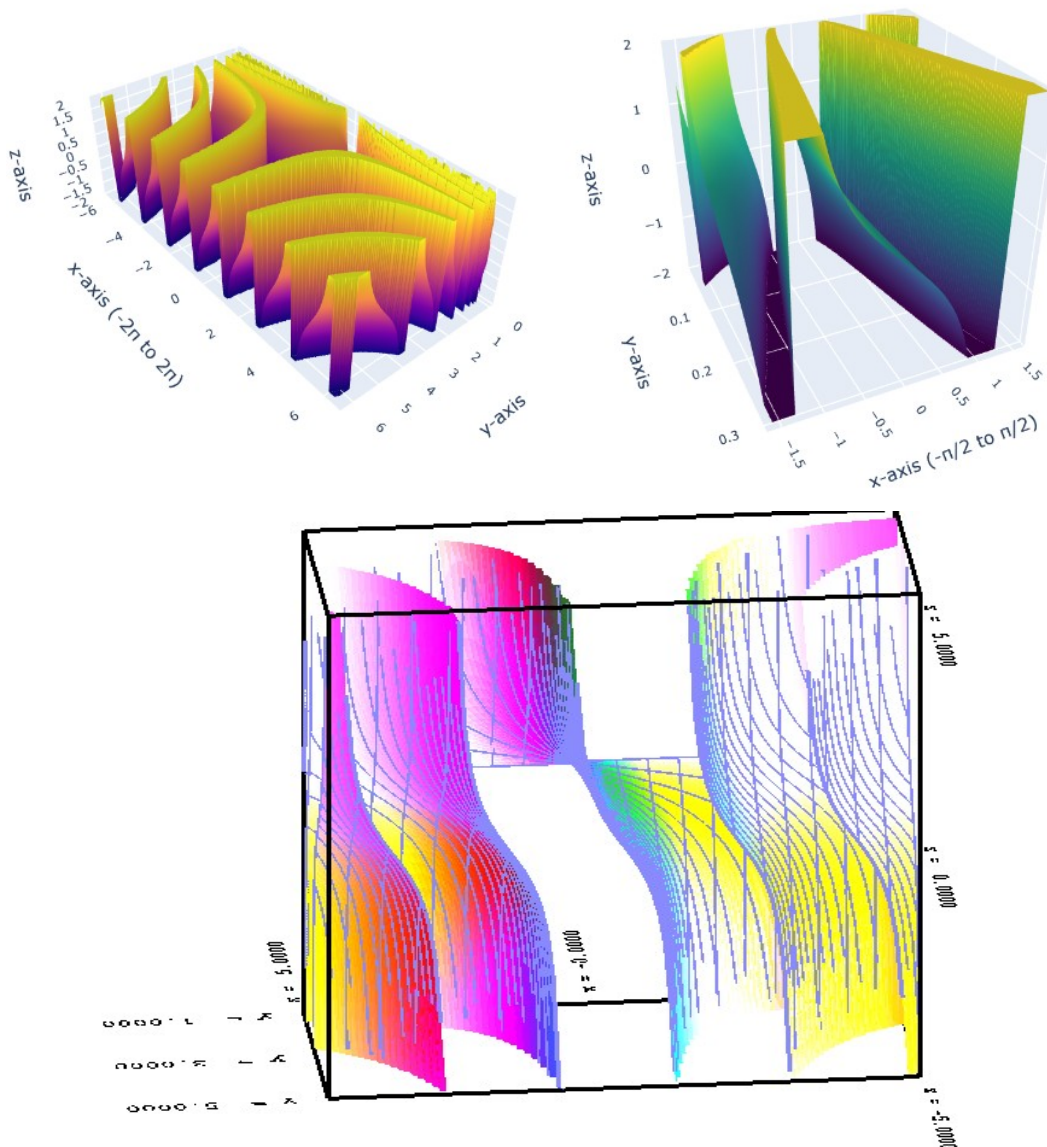
### 3-7-3. Graph of $\tan(x, y)$

The plot of  $\tan(x, y)$  is shown in Figure 3.3. Since the extended function  $\tan(x, y)$  can be expressed as  $\tan(x \ln y)$ , its graph is expected to reflect the characteristics of the circular tangent function. The graph appears to have no local maxima or minima. Further analysis of this function will be conducted in the next sections.

**3D Surface of  $z = \tan(x,y)$**

Graph:  $\tan(x,y)$  for  $x$  in  $(-2\pi, 2\pi)$

Graph:  $\tan(x,y)$  for  $x$  in  $(-\pi/2, \pi/2)$



**Figure 3.3** shows the graph of  $\tan(x,y) \equiv \frac{1}{i} \left( \frac{y^{ix} - y^{-ix}}{y^{ix} + y^{-ix}} \right)$  on different domains.

**3-7-4. Local Maxima, Minima and Saddle Points of  $\tan(x,y)$  via Calculus**

**Function:**  $f(x,y) = \tan(x,y) = \tan(x \ln y)$

**Domain:**  $x, y \in \mathbb{R}$  with  $y > 0$  and  $\cos(x \ln y) \neq 0$  or

$$x \ln y \neq \pi \left( k + \frac{1}{2} \right), k \in \mathbb{Z}.$$

### First derivatives

$$f_x = \sec^2(x \ln y) \ln y$$

$$f_y = \sec^2(x \ln y) \frac{x}{y}$$

### Critical points

Setting  $f_x = 0$  and  $f_y = 0$ : The critical points occur when

$$f_x = 0 \Rightarrow \sec^2(x \ln y) \ln y = 0 \Rightarrow y = 1$$

and

$$f_y = 0 \Rightarrow \sec^2(x \ln y) \frac{x}{y} = 0 \Rightarrow x = 0.$$

The result indicates that only  $(x_0, y_0) = (0, 1)$  is a critical point of  $\tan(x, y)$ .

### Second derivatives

$$f_{xx} = 2 \sec^2(x \ln y) \tan(x \ln y) \ln^2 y,$$

$$f_{yy} = 2 \sec^2(x \ln y) \tan(x \ln y) \frac{x^2}{y^2} - \sec^2(x \ln y) \frac{x}{y^2},$$

$$f_{xy} = 2 \sec^2(x \ln y) \tan(x \ln y) \frac{x \ln y}{y} + \sec^2(x \ln y) \frac{1}{y}.$$

At  $(x_0, y_0) = (0, 1)$ , we have  $f_{xx} = 0$ ,  $f_{yy} = 0$ ,  $f_{xy} = 1$ , therefore, the Hessian determinant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = -1 < 0,$$

from which it confirms a saddle point at  $(0,1)$ . Since  $\tan(x,y)$  is undefined as  $\cos(x \ln y) = 0$ , these curves are vertical asymptote in the  $xy$ -plane. Near these curves,  $\tan(x,y)$  takes arbitrarily large positive and negative values, so no local maxima or minima occur there.

### Summary of critical points of $\tan(x,y)$

- **Critical points:** only  $(0,1)$ .
- **Saddle point:**  $(0,1)$ .
- **Local maxima/minima:** None; the function has no local maxima or minima due to its unbounded behavior near the asymptotes  $x \ln y = \pi(k+1/2)$ .

### 3-7-5. Summary of $\tan(x,y)$

Definition (3.60) defines the new tangent function with two variables, and we demonstrate that the function  $\tan(x,y)$  is equivalent to  $\tan(x \ln y)$ . Its domain is  $x, y \in \mathbb{R}$  with  $y > 0$  and  $x \ln(y) \neq \pi/2 + k\pi$ ,  $k \in \mathbb{Z}$ , and its range is  $(-\infty, \infty)$ . It is evident that when  $y$  is equal to  $e$ ,  $\tan(x,e)$  corresponds to  $\tan(x)$ . The function exhibits a constant period for each slice of  $y$  in  $x$ . However, as  $y$  changes, the overall period of  $\tan(x,y)$  is a non-constant in both  $x$  and  $y$ . The function  $\tan(x,y)$  does not possess any local extrema; however, it features a single saddle point located at  $(0,1)$ . This function depicts a tangent-like surface in three dimensional space, as it is derived from the fundamental tangent function. It exhibits repetitive behavior, though its pattern is not uniform across the entire  $x$ - $y$  domain. Similar patterns of the function  $\sin(x,y)$  or  $\cos(x,y)$ , along individual axis, the function  $\tan(x,y)$  possess all the properties of the elementary tangent function in that direction.

## Section 3-8. Foundations of the Extended Cotangent Function $\cot(x,y)$

This section covers definition, theorems and properties of the extended cotangent function  $\cot(x,y)$ .

### 3-8-1. Definition and Theorem

**Definition.** For any real numbers  $x, y$ , the extended cotangent function, denoted

$\cot(x, y)$ , is defined by

$$\cot(x, y) \equiv i \left( \frac{y^{ix} + y^{-ix}}{y^{ix} - y^{-ix}} \right), \quad (3.65)$$

where  $y^{ix} - y^{-ix} \neq 0$ , and  $y > 0$ .

**Theorem 27.** The extended cotangent function satisfies the relation

$$\cot(x, y) \equiv i \left( \frac{y^{ix} + y^{-ix}}{y^{ix} - y^{-ix}} \right) = \cot(x \ln y), \quad (3.66)$$

where  $x, y$  are real,  $y > 0$ , and  $x \ln(y) \neq k\pi \mid k \in \mathbb{Z}$ .

### Proof

By definition (3.6), we have:

$$\begin{aligned} \cot(x \ln y) &= \frac{\cos(x \ln y)}{\sin(x \ln y)} = \frac{\cos(x, y)}{\sin(x, y)} \\ &= \frac{\frac{1}{2}(y^{ix} + y^{-ix})}{\frac{1}{2i}(y^{ix} - y^{-ix})} \\ &= i \frac{(y^{ix} - y^{-ix})}{(y^{ix} + y^{-ix})} \\ &\equiv \cot(x, y), \end{aligned}$$

from which Theorem 27 follows.

### Corollary 28

The function  $\cot(x, y)$  is defined as

$$\cot(x, y) = \frac{\cos(x, y)}{\sin(x, y)}. \quad (3.67)$$

**Proof**

It results from the proof of Theorem 27.

**3-8-2. Basic Properties of  $\cot(x,y)$** 

**1. Domain:**  $x, y \in \mathbb{R}, y > 0, x \ln(y) \neq k\pi \mid k \in \mathbb{Z}$ .

**2. Range:**  $-\infty < \cot(x,y) \equiv \cot(x \ln y) < \infty$ .

**3. Function type:**  $\cot(x,y)$  is even for  $0 < y \leq 1$ , and odd for  $y > 1$ .

**Proof**

**3.** We consider two cases:

**a.** Symmetry in  $x$ :  $\cot(-x,y) = \cot(-x \ln y) = -\cot(x \ln y) = -\cot(x,y)$ . Therefore,  $\cot(x,y)$  is odd in  $x$ .

**b.** Symmetry in  $y$ : Since  $\ln(y)$  only defined for  $y > 0$ ,  $\cot(x,y)$  is neither even or odd in  $y$ .

**4. Zeros of  $\cot(x,y)$  with Respect to  $x$  and  $y$** **4A. Zeros of  $\cot(x,y)$  in  $x$** 

**Theorem 29.** For fixed  $y$ , the zeros of  $\cot(x,y)$  in  $x$  are given by

$$x = \frac{\pi k}{|\ln y|}, \quad y > 0, y \neq 1, k \in \mathbb{Z}. \quad (3.68)$$

**Proof**

The zeros of  $\cot(x,y)$  in  $x$  occur when  $\cot(x,y) = \cot(x \ln y) = 0$ . Because  $\cos(x \ln y + \pi k) = \cos(x \ln y)$  and  $k \in \mathbb{Z}$ , it follows that

$$x \ln y = \pi k, \quad \text{for } y > 0, \text{ and } k \in \mathbb{Z}$$

Solving for  $x$  in terms of  $y$  gives

$$x = \frac{\pi k}{\ln y}, \quad y > 0, y \neq 1, k \in \mathbb{Z},$$

which we complete the proofs. Notice that we can use the absolute value of  $\ln(y)$  for convenience, as it helps distinguish the value of  $x$ , whether positive or negative, when  $0 < y < 1$ , while  $k$  is a positive integer in the set of  $\mathbb{Z}$ .

$$x = \frac{\pi k}{|\ln y|}, \quad y > 0, y \neq 1, k \in \mathbb{Z}.$$

#### 4B. Zeros of $\cot(x,y)$ in $y$

**Theorem 30.** For fixed  $x$ , the zeros of  $\cot(x,y)$  in  $y$  are given by

$$y = e^{\frac{\pi k}{|x|}}, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.69)$$

#### Proof

Using the proof of Theorem 29, we follow the same step-by-step procedure as before, but instead of solving for  $x$ , we now solve for  $y$  in terms of  $x$ . This completes the proof.

### 5. Periodicity of $\cot(x,y)$ in the X and Y Directions

#### 5A. Periodicity in $x$

**Theorem 31.** For fixed  $y > 0$ , the period of  $\cot(x,y)$  with respect to  $x$  is given by

$$T_x(y) = \frac{\pi k}{|\ln y|} \text{ for } y \neq 1 \text{ and } k \in \mathbb{Z}.$$

#### Proof

The function  $\cot(x,y)$  has a period in  $x$  if there exists a period  $T_x$  such that

$$\cot(x + T_x, y) = \cot(x, y).$$

Therefore, we have:

$$\begin{aligned} \cot((x + T_x) \ln y) &= \cot(x \ln y), \quad y > 0, \\ \Rightarrow (x + T_x) \ln y &= x \ln y + \pi k, \quad y > 0, k \in \mathbb{Z}, \\ \Rightarrow T_x \ln y &= \pi k, \quad y > 0, k \in \mathbb{Z}. \end{aligned}$$

Solving for  $T_x$  gives

$$T_x = T_x(x, y) = \frac{\pi k}{|\ln y|}, \quad y > 0, y \neq 1, k \in \mathbb{Z}.$$

### 5B. Periodicity in y

**Theorem 32.** The period of  $\cot(x, y)$  with respect to  $y$  is given by

$$T_y(x, y) = y(e^{\pi k/x} - 1), \quad x \neq 0, k \in \mathbb{Z}.$$

#### Proof

The function  $\cot(x, y)$  has a period in  $y$  if there exists a period  $T_y(x) \equiv T_y$  such that

$$\cot(x, y + T_y) = \cot(x, y).$$

This implies that

$$\begin{aligned} \cot(x \ln(y + T_y)) &= \cot(x \ln y) \\ \Rightarrow x \ln(y + T_y) &= x \ln y + \pi k, \quad k \in \mathbb{Z} \\ \Rightarrow \ln\left(\frac{y + T_y}{y}\right) &= \frac{\pi k}{x}, \quad x \neq 0, k \in \mathbb{Z} \\ \Rightarrow \frac{y + T_y}{y} &= e^{\pi k/x}, \quad x \neq 0, k \in \mathbb{Z}. \quad (*) \end{aligned}$$

Solving for  $T_y$  from equation (\*) gives

$$T_y \equiv T_y(x) = y(e^{\pi k/x} - 1), \quad x \neq 0, k \in \mathbb{Z}.$$

### 6. Reciprocal Tangent Function

**Theorem 33.** The extended cotangent function is the reciprocal of the extended tangent function and is expressed as

$$\cot(x, y) = \frac{1}{\tan(x, y)}, \quad (3.70)$$

where  $\tan(x,y)$  is defined and nonzero.

### Proof

From definition (3.67) and  $\sin(x,y) \neq 0$ , we have

$$\cot(x,y) = \frac{\cos(x,y)}{\sin(x,y)} = \frac{1}{\frac{\sin(x,y)}{\cos(x,y)}} = \frac{1}{\tan(x,y)},$$

where  $\tan(x,y) \neq 0$ .

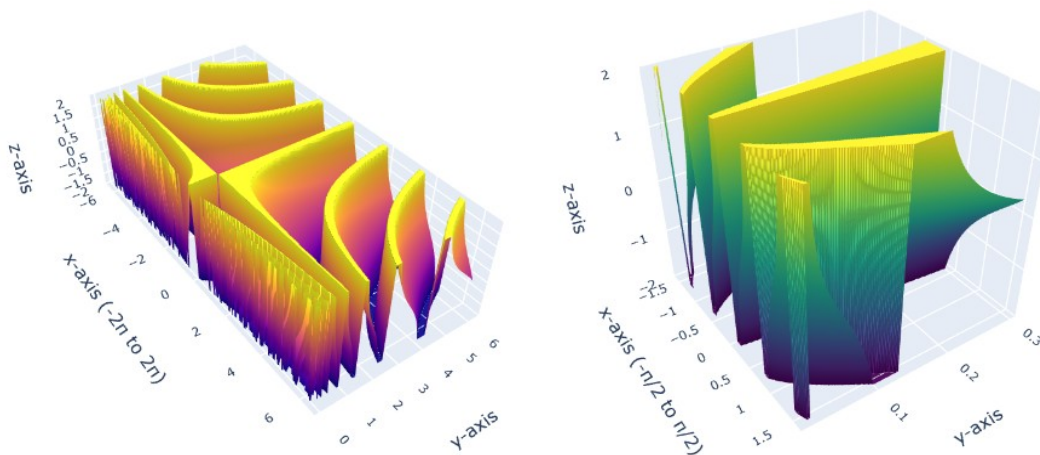
### 3-8-3. Graph of $\cot(x,y)$

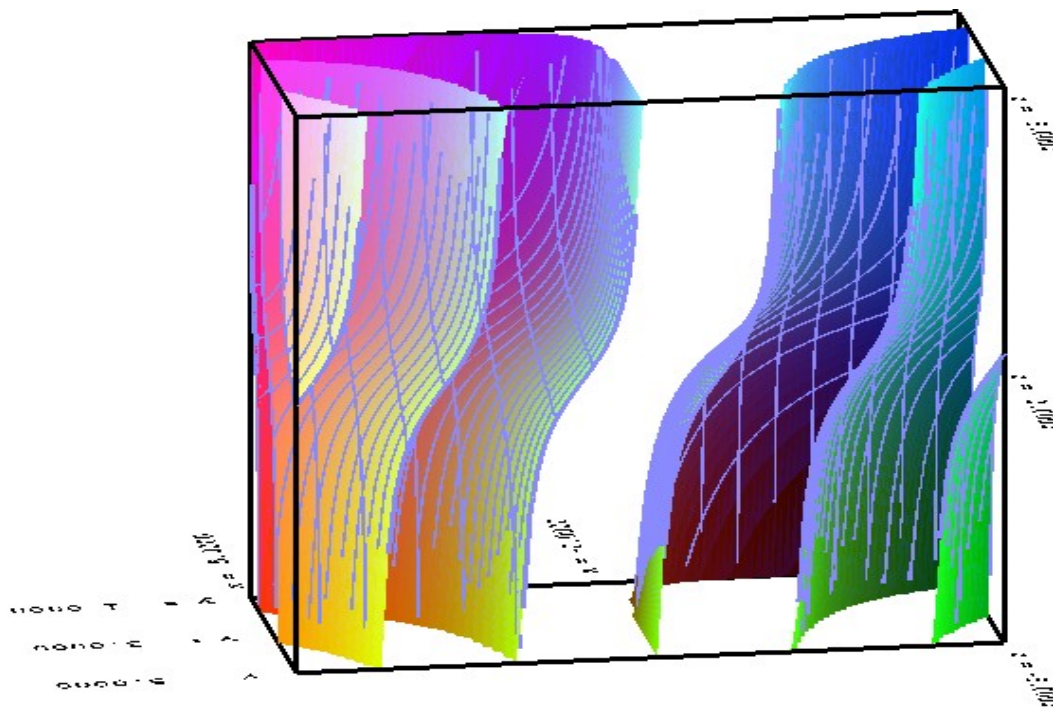
The 3D plot of  $\cot(x,y)$  is illustrated in Figure 3.4. Since the extended cotangent function  $\cot(x,y)$  can be expressed as  $\cot(x|ny)$ , its behavior follows that of the cotangent function.

#### 3D Surface of $z = \cot(x,y)$

Graph:  $\cot(x,y)$  for  $x$  in  $(-2\pi, 2\pi)$

Graph:  $\cot(x,y)$  for  $x$  in  $(-\pi/2, \pi/2)$





**Figure 3.4** shows the graph of  $\cot(x,y) \equiv \frac{1}{i} \left( \frac{y^{ix} + y^{-ix}}{y^{ix} - y^{-ix}} \right)$  on different domains.

### 3-8-4. Summary of $\cot(x,y)$

The definition provided in (3.65) introduces the extended cotangent function with two variables. The function  $\cot(x,y)$  is equivalent to  $\cot(x \ln y)$ , and serves as the reciprocal of the tangent function  $\tan(x,y)$ . Its domain is  $x \in (-\infty, \infty)$  with  $y > 0$  and  $x \ln y \neq k\pi, k \in \mathbb{Z}$ , and its range is  $(-\infty, \infty)$ . When  $y$  equals  $e$ , the function  $\cot(x, e)$  is equal to  $\cot(x)$ . The function maintains a constant period for each slice of  $y$  with respect to  $x$ , while exhibiting a non-constant period with respect to  $y$ . The function  $\cot(x,y)$  has no local extrema or saddle points and can be visualized as a cotangent-like surface in three dimensional space, as it originates from the fundamental cotangent function. Although it demonstrates repetitive behavior, its pattern is not uniform throughout the entire  $x$ - $y$  domain. Along each individual axis,  $\cot(x,y)$  retains the characteristics and properties of the elementary cotangent function in that specific direction. However, in both  $x$  and  $y$  directions, the interplay between  $x$  and  $y$  via  $x \ln y$  can influence the overall behavior of  $\cot(x,y)$ , even as it retains some key characteristics of the cotangent in each individual direction.

## Section 3-9. Overviews of the Extended Secant and Cosecant Functions

### 3-9-1. Extended Secant Function – $\sec(x,y)$

The extended secant function, denoted as  $\sec(x,y)$ , is the reciprocal of the extended cosine function. It's expressed as

$$\sec(x, y) = \frac{1}{\cos(x, y)}, \quad (3.71)$$

for real  $x, y$  and  $y > 0$  and  $\cos(x,y) \neq 0$ .

The alternative form of  $\sec(x,y)$  is

$$\sec(x, y) = \frac{2}{y^{ix} + y^{-ix}}. \quad (3.72)$$

The extended secant function has a similar characteristics and properties of the extended cosine function  $\cos(x,y)$  but it is undefined at points where  $\cos(x,y)=0$ . We do not go into its details and other properties here and leave that to the readers to explore further.

### 3-9-2. Extended Cosecant Function – $\csc(x,y)$

The extended cosecant function, denoted as  $\csc(x,y)$ , is the reciprocal of the extended sine function  $\sin(x,y)$ . It is defined as

$$\csc(x, y) = \frac{1}{\sin(x, y)}, \quad (3.73)$$

for real  $x, y$  with  $y > 0$  and  $\sin(x,y) \neq 0$ .

The alternative form of  $\csc(x,y)$  is expressed as

$$\csc(x, y) = \frac{2i}{y^{ix} - y^{-ix}}. \quad (3.74)$$

The extended cosecant function has a similar characteristics and properties of the extended sine function  $\sin(x,y)$  but it is undefined at points where  $\sin(x,y)=0$ . We do not go into its details and other properties here and leave that to the readers to explore further.

***In the next sections, we proceed to establish the hyperbolic functions of two variables through a series of definitions, theorems and properties.***

To define  $\cosh(x,y)$ ,  $\sinh(x,y)$ ,  $\tanh(x,y)$ ,  $\coth(x,y)$ ,  $\operatorname{sech}(x,y)$  and  $\operatorname{csch}(h)$ , we begin with the following identities, valid for real numbers  $x$ ,  $y$  and  $y > 0$ :

$$y^x = \cosh(x \ln y) - \sinh(x \ln y) \quad (3.80)$$

and

$$y^{-x} = \cosh(x \ln y) + \sinh(x \ln y). \quad (3.81)$$

Formula (3.80) is derived by substituting  $x$  with  $x \ln(y)$  in (3.7) and using  $e^{x \ln y} = y^x$ , while formula (3.81) results from substituting  $x$  with  $-x$  in (3.80).

## Section 3-10. Foundations of the Extended Hyperbolic Cosine Function $\cosh(x,y)$

### 3-10-1. Definition and Theorem

**Definition.** The extended hyperbolic cosine function, denoted as  $\cosh(x,y)$ , is defined by

$$\cosh(x, y) \equiv \frac{1}{2}(y^x + y^{-x}) \text{ for } x, y \in \mathbb{R} \text{ and } y > 0. \quad (3.82)$$

**Theorem 34.** The extended hyperbolic cosine function satisfies the relation

$$\cosh(x, y) \equiv \frac{1}{2}(y^x + y^{-x}) = \cosh(x \ln y) \text{ for real } x, y \text{ and } y > 0. \quad (3.83)$$

**Proof**

By adding (3.80) and (3.81), we have

$$y^x + y^{-x} = 2 \cosh(x \ln y)$$

Solve for  $\cosh()$  from the equation, we obtain

$$\cosh(x \ln y) = \frac{1}{2}(y^x + y^{-x}).$$

Combining the result with definition (3.82) gives

$$\cosh(x, y) \equiv \frac{1}{2}(y^x + y^{-x}) = \cosh(x \ln y).$$

**3-10-2. Basic Properties of  $\cosh(x, y)$** 

**1. Domain:**  $x \in (-\infty, \infty)$ ,  $y > 0$ .

**2. Range:**  $1 \leq \cosh(x, y) < \infty$ .

**3. Function type:**  $\cosh(x, y)$  is an even in  $x$ ; the function is neither even nor odd in  $y$ .

**Proof**

3. We have  $\cosh(-x, y) = \cosh(-x \ln y) = \cosh(x \ln y) = \cosh(x, y)$ . Therefore, the function  $\cosh(x, y)$  is even in  $x$ .

**4. Maxima, Minima and Zeros of  $\cosh(x, y)$  with Respect to  $x$** 

**Theorem 35.** The maxima, minima and zeros of  $\cosh(x, y)$  with respect to  $x$  can be determined as follows:

**(a) Maxima:** None

**(b) Minima:**  $(x, y) = (0, y_0)$

**(c) Zeros:** None

**Proof**

To find the critical points and determine if there are any extrema, we use the first and second derivatives as follows:

### i. First derivative

$$\frac{d(\cosh(x, y))}{dx} = \frac{d(\cosh(x \ln y))}{dx} = \ln(y) \sinh(x \ln y)$$

For  $y > 0$  and  $y \neq 1$ ,

$$\sinh(x \ln y) = 0 \Rightarrow x = 0,$$

from which it indicates that  $\cosh(x, y)$  has a critical point at  $x_0 = 0$  and  $\cosh(0, y_0) = 1$ .

### ii. Second derivative

$$\frac{d^2(\cosh(x, y))}{dx^2} = [\ln(y) \sinh(x \ln y)]' = \ln^2(y) \cosh(x \ln y).$$

Since  $\ln^2(y) \cosh(x \ln y) > 0$  for all  $x, y > 0$  and  $y \neq 1$ , this shows that  $\cosh(x, y)$  is always concave up. Thus, we conclude that

**(a) Minima:**  $\cosh(x, y)$  has a local minimum at  $(x, y) = (0, y)$  for  $y > 0$ , and

**(b) Maxima:**  $\cosh(x, y)$  has no local maximum with respect to  $x$ .

### (c) Zeros of $\cosh(x, y)$ in $x$

Since  $\cosh(x \ln(y)) \geq 1$  for all  $x$  and  $y > 0$ , there is no zeros with respect to  $x$ .

## 5. Periodicity in $x$

**Theorem 36.** For each fixed  $y > 0$  with  $y \neq 1$ ,  $\cosh(x, y)$  has no real period but exhibits a non-constant complex period in  $x$ , given by

$$T_x(y) = \frac{2\pi i k}{\ln(y)}, \quad (3.84)$$

### Proof

Let  $T_x \equiv T_x(y)$ . Assume  $T_x$  is the period of  $\cosh(x, y)$  in  $x$ . Then  $T_x$  must satisfies the

equation

$$\begin{aligned}
 \cosh(x + T_x, y) &= \cosh(x, y) \\
 \Rightarrow \cosh((x + T_x) \ln(y)) &= \cosh(x \ln y) \\
 \Rightarrow \cosh((x + T_x) \ln(y)) &= \cosh(x \ln y + 2\pi i k), \quad \text{where } i \text{ is complex unit, } k \in \mathbb{Z}. \\
 \Rightarrow (x + T_x) \ln(y) &= x \ln y + 2\pi i k \\
 \Rightarrow T_x \ln(y) &= 2\pi i k \quad (*)
 \end{aligned}$$

Solving for  $T_x$  from equation (\*) in terms of  $y$  and  $k$  gives

$$T_x = \frac{2\pi i k}{\ln(y)}, \quad k \in \mathbb{Z}.$$

We conclude that  $T_x$  has no real period but exhibits a non-constant period in the complex plane.

Let's verify that  $T_x = i2\pi k/\ln(y)$ :

$$\cosh(x + T_x, y) = \cosh((x + T_x) \ln(y))$$

By substituting the value of  $T_x$ , we get

$$\begin{aligned}
 &= \cosh\left(\left(x + \frac{2\pi i k}{\ln(y)}\right) \ln(y)\right) \\
 &= \cosh(x \ln(y) + 2\pi i k) \\
 &= \cosh(x \ln(y)) \\
 &= \cosh(x, y).
 \end{aligned}$$

Thus, we demonstrate  $\cosh(x + T_x, y) = \cosh(x, y)$ , where  $T_x$  is (3.84).

We conclude that  $\cosh(x,y)$  has no real period but exhibits a non-constant complex period in  $x$ .

**Remark 3.4.8.** For clarity and correctness, periodicity should be analyzed only on the real domain; complex-domain periodicity lies outside the intended scope. We include the complex-domain formula for algebraic completeness or theoretical study, but we do not analyze or interpret periodicity in the complex plane.

## 6. Extrema and Zeros of $\cosh(x,y)$ with Respect to $y$

### A. Extrema of $\cosh(x,y)$ in $y$

To find the critical points and determine if there are any extrema, we use the first and second derivatives along the  $y$ -axis as follows:

#### i. First derivative

$$\frac{d(\cosh(x,y))}{dy} = \frac{d(\cosh(x \ln y))}{dy} = \frac{x}{y} \sinh(x \ln y)$$

For  $y > 0$  with fixed  $x$ , we set

$$\frac{x}{y} \sinh(x \ln y) = 0.$$

Solving for  $y$  from the equation gives

$$y = 1.$$

Thus,  $y = 1$  is the critical point for real  $x$ . For fixed  $x$ , the derivative  $d/dy(\cosh(x,y))$  changes sign from negative to positive as  $y$  passes through  $y = 1$ . Therefore,  $(x, 1)$  is a local minimum of  $\cosh(x,y)$ .

### B. Zeros of $\cosh(x,y)$ with Respect to $Y$

**Theorem 37.** The function  $\cosh(x,y)$  has no real zeros in  $y$  but exhibits complex zeros, given by

$$y = e^{\left(k + \frac{1}{2}\right)i\pi/x} \quad \text{for } k \in \mathbb{Z}.$$

**Proof**

**a.** We observe that  $\cosh(x, y) = \cosh(x \ln(y)) > 0$  for all  $x$  and  $y > 0$ . Therefore, the function  $\cosh(x, y)$  has no real zeros.

**b.** The equation  $\cosh(x, y) = 0$  has complex roots in  $y$ .

**First method**

By definition (3.82), we have:

$$\begin{aligned} \frac{1}{2}(y^x + y^{-x}) &= 0 \\ y^{2x} + 1 &= 0 \\ y^{2x} &= -1 = e^{-i\pi} \\ &= e^{i(\pi + 2\pi k)} \quad \text{for } k \in \mathbb{Z}. \quad (*) \end{aligned}$$

Solving for  $y$  from equation (\*) gives

$$y = e^{\left(k + \frac{1}{2}\right)i\pi/x} \quad \text{for } k \in \mathbb{Z}.$$

**Second method**

$$\begin{aligned} \cosh(x \ln(y)) &= 0 \\ &= \cosh\left(\frac{i\pi}{2} + \pi i k\right) \quad \text{for } k \in \mathbb{Z} \\ \Rightarrow x \ln(y) &= \frac{\pi i}{2} + \pi i k \\ \Rightarrow \ln(y) &= \frac{\pi i}{x} \left(k + \frac{1}{2}\right) \\ \Rightarrow y &= e^{\left(k + \frac{1}{2}\right)\pi i/x}, \quad k \in \mathbb{Z} \end{aligned}$$

**7. Periodicity in  $y$** 

**Theorem 38.** The function  $\cosh(x, y)$  has no real period but exhibits a non-constant period in complex form in  $y$ , given by

$$T_y(x, y) = \left( e^{\frac{i2\pi k}{|x|}} - 1 \right) y, \quad x \neq 0, k \in \mathbb{Z}, \quad (3.85)$$

### Proof

Let  $T_y = T_y(x, y)$ . Assume  $T_y$  is the period of  $\cosh(x, y)$  in  $y$ . Then  $T_y$  must satisfy the equation

$$\begin{aligned} \cosh(x, y + T_y) &= \cosh(x, y) \\ \Rightarrow \cosh(x \ln(y + T_y)) &= \cosh(x \ln y) \\ \Rightarrow x \ln(y + T_y) &= x \ln y + i2\pi k \quad \text{for some integer } k. \\ \Rightarrow x \ln\left(\frac{y + T_y}{y}\right) &= i2\pi k \\ \Rightarrow \ln\left(\frac{y + T_y}{y}\right) &= \frac{i2\pi k}{|x|} \\ \Rightarrow \frac{y + T_y}{y} &= e^{i2\pi k/|x|} \quad (*) \end{aligned}$$

Solving for  $T_y$  from equation (\*) in terms of  $|x|$ ,  $y$  and  $k$  gives

$$T_y = T_y(x, y) = y \left( e^{i2\pi k/|x|} - 1 \right), \quad k \in \mathbb{Z}.$$

Since  $T_y$  from (3.85) depends on  $i$ ,  $x$ ,  $y$  and  $k$  along the  $y$ -axis, we conclude that  $\cosh(x, y)$  has no real period but exhibits a non-constant period in complex form in  $y$ . Please refer to **Remark 3.4.8** to stay within the intended scope of our analysis.

### 3-10-3. Graph of $\cosh(x, y)$

The 3D plot of  $\cosh(x, y)$  is shown in Figure 3.5. The function  $\cosh(x, y)$  retains the same behavior as  $\cosh(x)$  for each value of  $y$ , its surface exhibits the same curvature along the  $x$ -axis as the standard hyperbolic cosine. In essence, the shape is non-negative and is locally flat along the two intersecting lines of absolute minima.

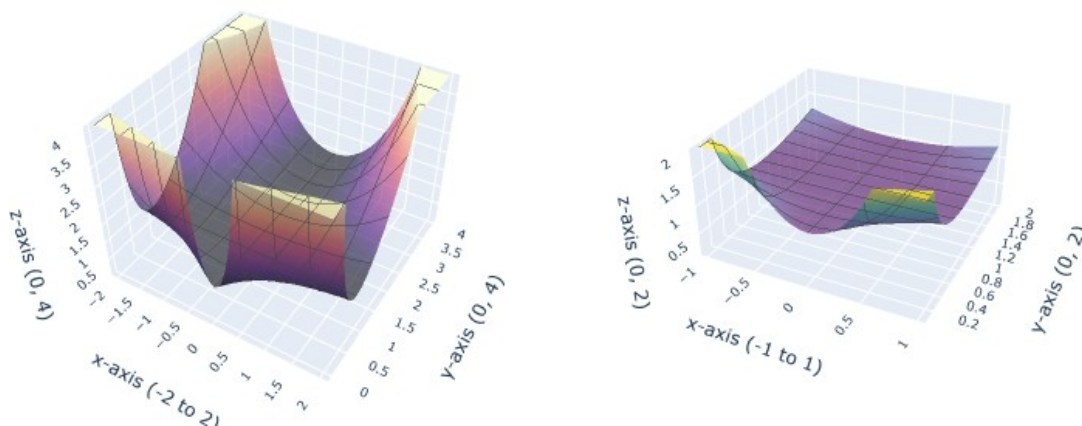
3D Surface of  $z = \cosh(x,y)$ Graph:  $\cosh(x,y)$  for  $x$  in  $(-2,2)$ Graph:  $\cosh(x,y)$  for  $x$  in  $(-1,1)$ 

Figure 3.5 shows the graph of  $\cosh(x,y) \equiv \frac{1}{2}(y^x + y^{-x})$ .

### 3-10-4. Local Maxima, Minima and Saddle Points of $\cosh(x,y)$ via Calculus

#### First partial derivatives

$$f_x = \frac{d}{dx} \cosh(x \ln y) = \sinh(x \ln y) \cdot \ln y$$

$$f_y = \frac{d}{dy} \cosh(x \ln y) = \frac{x}{y} \sinh(x \ln y)$$

#### Critical points

Setting  $f_x = 0$  and  $f_y = 0$ :

$$f_x = 0 \Rightarrow \sinh(x \ln y) \cdot \ln y = 0 \Rightarrow \begin{cases} \sinh(x \ln y) = 0 & \Rightarrow (a) \quad x \ln y = i\pi k, \text{ for } k \in \mathbb{Z} \\ \ln y = 0 & \Rightarrow (b) \quad y = 1 \end{cases}$$

$$f_y = 0 \Rightarrow \frac{x}{y} \sinh(x \ln y) = 0 \Rightarrow x \sinh(x \ln y) = 0 \Rightarrow \begin{cases} (c) \quad x = 0 \\ \sinh(x \ln y) = 0 \Rightarrow (d) \quad x \ln y = i\pi k, \text{ for } k \in \mathbb{Z} \end{cases}$$

The solutions from (a), (b), (c) and (d) can be summarized in three cases:

- Case 1: At  $x = 0$  for any  $y > 0$ , it gives  $f_x = 0$  and  $f_y = 0$ , and therefore  $(0, y_0)$  is a set of the critical points along the positive y-axis or the line  $x_0=0$ .
- Case 2: At  $y = 1$  for any  $x \in \mathbb{R}$ , it gives  $f_x = 0$  and  $f_y = 0$ , and therefore  $(x_0, 1)$  is a set of the critical point along the line  $y_0 = 1$ .
- Case 3: At points  $(x_0, y_0)$  satisfying  $\sinh(x_0 \ln y_0) = 0$ , we obtain the complete set of critical points. This set is given by the family of the curves  $x_0 \ln y_0 = i\pi k$ , for  $k \in \mathbb{Z}$ .

**Remark 3.4.7.** This set also covers in Case 1 and Case 2 when  $k = 0$ . The equation yields only complex form when  $k \neq 0$ . Since complex points do not correspond to real critical points of a real-valued function, this case has no maxima or minima. We continue the analysis only for algebraic completeness or theoretical study.

## Second order partial derivatives

$$f_{xx} = \cosh(x \ln y) \cdot \ln^2 y$$

$$f_{yy} = \frac{x^2 \cosh(x \ln y) - x \sinh(x \ln y)}{y^2}$$

$$f_{xy} = \frac{x}{y} \cosh(x \ln y) \ln(y) + \frac{1}{y} \sinh(x \ln y)$$

## Calculate Discriminant D

- It is sufficient to consider Case 3, as the other Case 1 and Case 2 are special instances of it. At critical curves defined by  $x \ln y = i\pi k$ , for  $k \in \mathbb{Z}$  and  $y > 0$ , we have:

$$\begin{aligned} D(x_0, y_0) &= (f_{xx})(f_{yy}) - (f_{xy})^2 \\ &= (\cosh(x_0 \ln y_0) \cdot \ln^2 y_0) \cdot \left( \frac{x_0^2 \cosh(x_0 \ln y_0) - x_0 \sinh(x_0 \ln y_0)}{y_0^2} \right) - \\ &\quad \left( \frac{x_0}{y_0} \cosh(x_0 \ln y_0) \ln(y_0) + \frac{1}{y_0} \sinh(x_0 \ln y_0) \right)^2 \end{aligned}$$

Substituting  $\sinh(i\pi k) = 0$  into the expression gives

$$\begin{aligned} D(x_0, y_0) &= \left( \cosh(x_0 \ln y_0) \cdot \ln^2 y_0 \right) \cdot \left( \frac{x_0^2 \cosh(x_0 \ln y_0)}{y_0^2} \right) - \left( \frac{x_0^2}{y_0^2} \ln^2 y_0 \cosh^2(x_0 \ln y_0) \right) \\ &= \frac{x_0^2}{y_0^2} \ln^2(y_0) \cosh^2(x_0 \ln y_0) - \frac{x_0^2}{y_0^2} \ln^2(y_0) \cosh^2(x_0 \ln y_0) \\ &= 0. \quad (\text{The test is inconclusive.}) \end{aligned}$$

### Behavior at infinity

- For  $y > 0$  with  $y \neq 1$ , we consider two cases:

- If  $0 < y < 1$ , then  $\ln(y) < 0$ , and we have

$$\lim_{x \rightarrow \pm\infty} \cosh(x, y) = \lim_{x \rightarrow \pm\infty} \cosh(x \ln y) = +\infty.$$

- If  $y > 1$ , then  $\ln(y) > 0$ , and we have

$$\lim_{x \rightarrow \pm\infty} \cosh(x, y) = \lim_{x \rightarrow \pm\infty} \cosh(x \ln y) = +\infty.$$

- For  $y = 1$ , then

$$\lim_{x \rightarrow \pm\infty} \cosh(x, y) = \lim_{x \rightarrow \pm\infty} \cosh(x \ln y) = 1.$$

Since  $D(x_0, y_0) = 0$ , the Second Derivative Test is inconclusive. However, at a set of the critical points,  $x_0 \ln y_0 = i\pi k$ , for  $k \in \mathbb{Z}$ , it gives  $\cosh(x_0 \ln y_0) = 1$ . Since  $\cosh(x, y) \geq 1$  everywhere on its domain, **the real critical points  $(0, y_0)$  and  $(x_0, 1)$  for  $k = 0$  are not only local minima but also global minima.** Moreover,  $\cosh(x, y)$  has no saddle points, because all critical points (where  $\cosh(x_0, y_0) = 1$ ) correspond to the local minima and none correspond to local maxima.

### Note

$$\begin{cases} \ln(y) \geq 0 & \text{for } y \geq 1 \\ \ln(y) < 0 & \text{for } 0 < y < 1 \end{cases}$$

### Summary of critical points of $\cosh(x, y)$

- **Critical points:**  $(0, y_0)$ ,  $(x_0, 1)$

- **Local maxima:** None
- **Set of local minima:**  $(0, y_0), (x_0, 1)$
- **Saddle point:** None

### 3-10-5. Summary of $\cosh(x, y)$

Formula (3.17) defines the new hyperbolic cosine function with two arguments as  $\cosh(x, y) \equiv \cosh(x \ln y)$ . Its domain is  $x \in (-\infty, \infty)$  and  $y > 0$ , and its range is  $[1, \infty)$ . The function  $\cosh(x, y)$  has no real period but exhibits a non-constant period in complex form. It is clear that when  $y$  equals  $e$ ,  $\cosh(x, e)$  is equal to  $\cosh(x)$ . The function  $\cosh(x, y)$  represents a hyperbolic cosine-like surface in 3D, characterized by a complete cross of local/global minima: the vertical line  $(0, y)$  for  $y > 0$  and the horizontal line  $(x, 1)$  for all  $x \in \mathbb{R}$ .

## Section 3-11. Foundations of the Extended Hyperbolic Sine Function $\sinh(x, y)$

### 3-11-1. Definition and Theorem

**Definition.** The extended hyperbolic sine function, denoted as  $\sinh(x, y)$ , is defined by

$$\sinh(x, y) \equiv \frac{1}{2}(y^x - y^{-x}) \quad \text{for } x, y \in \mathbb{R} \text{ and } y > 0. \quad (3.86)$$

**Theorem 39.** The extended hyperbolic sine function satisfies the relation

$$\sinh(x, y) \equiv \frac{1}{2}(y^x - y^{-x}) = \sinh(x \ln y) \quad \text{for real } x, y \text{ and } y > 0. \quad (3.87)$$

#### Proof

By subtracting (3.80) and (3.81), we have

$$y^x - y^{-x} = 2 \sinh(x \ln y),$$

from which we get

$$\sinh(x \ln y) = \frac{1}{2}(y^x - y^{-x}).$$

Combining the above result with definition (3.86) gives

$$\sinh(x, y) \equiv \sinh(x \ln y).$$

### 3-11-2. Basic Properties of $\sinh(x, y)$

**1. Domain:**  $x \in (-\infty, \infty)$ ,  $y > 0$ .

**2. Range:**  $-\infty < \sinh(x, y) < \infty$ .

**3. Function type:**  $\sinh(x, y)$  is odd in  $x$  for all  $y > 0$ .

#### Proof

3. We have:  $\sinh(-x, y) = \sinh(-x \ln y) = -\sinh(x \ln y) = -\sinh(x, y)$ . The function  $\sinh(x, y)$  is odd in  $x$  for all  $y > 0$ .

### 4. Maxima, Minima and Zeros of $\sinh(x, y)$ with Respect to $x$

**Theorem 40.** The maxima, minima and zeros of  $\sinh(x, y)$  with respect to the  $x$ -axis can be determined as follows:

**(a) Maxima:** No local maxima.

**(b) Minima:** No local minima.

**(c) Zeros:**  $(0, y)$ ,  $y > 0$ .

#### Proof

To find the critical points and determine if there are any extrema, we use the first and second derivatives as follows:

#### i. First derivative

$$\frac{d}{dx} \sinh(x, y) = \frac{d}{dx} \sinh(x \ln y) = \ln(y) \cosh(x \ln y)$$

- For  $0 < y < 1$ , then  $d/dx(\sinh(x, y)) < 0$  for all  $x$ , the function is strictly decreasing in  $x$ , no extrema occur.

- For  $y > 1$ , then  $d/dx(\sinh(x,y)) > 0$  for all  $x$ , the function is strictly increasing in  $x$ , no extrema occur.
- For  $y = 1$ , then  $d/dx(\sinh(x,y)) = 0$  for all  $x$ . So, at  $x = 0$  is an inflection point along the  $x$ -axis.

Thus, since there are no extrema, we do not need to consider the second derivative and can conclude that

**(a) Maxima in  $x$ :** None

**(b) Minima in  $x$ :** None

### **(c) Zeros of $\sinh(x,y)$ in $x$**

The zeros of  $\sinh(x,y)$  in  $x$  occur when

$$\sinh(x \ln y) = 0 \text{ for fixed } y > 0.$$

The equation has real solutions only when  $x \ln y = 0$ , which gives  $x = 0$  or  $y = 1$ . At  $x = 0$ ,  $\sinh(0,y) = 0$  for real  $y > 0$ , and at  $y = 1$ ,  $\sinh(x, 1) = 0$  for all real  $x$ . The equation also has complex solutions, namely

$$x = \frac{i\pi k}{\ln y}, y \neq 1, k \in \mathbb{Z}.$$

## **5. Behavior of $\sinh(x,y)$**

- $0 < y < 1$ :  $\lim_{x \rightarrow \pm\infty} \sinh(x, y) = \lim_{x \rightarrow \pm\infty} \sinh(x \ln y) = \mp\infty$
- $y > 1$ :  $\lim_{x \rightarrow \pm\infty} \sinh(x, y) = \lim_{x \rightarrow \pm\infty} \sinh(x \ln y) = \pm\infty$
- $y = 1$ :  $\lim_{x \rightarrow \pm\infty} \sinh(x, y) = \lim_{x \rightarrow \pm\infty} \sinh(x \ln y) = 0$
- $y = 0^+$ :  $\lim_{x \rightarrow \pm\infty} \sinh(x, y) = \lim_{x \rightarrow \pm\infty} \sinh(x \ln y) = \mp\infty$

## **6. Periodicity in $x$**

**Theorem 41.** For each fixed  $y > 0$  with  $y \neq 1$ ,  $\sinh(x,y)$  has no real period but exhibits a non-constant complex period in  $x$ , given by

$$T_x(y) = \frac{2\pi i k}{\ln(y)}, \quad (3.88)$$

**Proof**

1. Let  $T_x \equiv T_x(y)$ . Assume  $T_x$  is the period of  $\cosh(x, y)$  in  $x$ . Then  $T_x$  must satisfies the equation

$$\begin{aligned} \sinh(x + T_x, y) &= \sinh(x, y) \\ \Rightarrow \sinh((x + T_x) \ln(y)) &= \sinh(x \ln y) \\ \Rightarrow \sinh((x + T_x) \ln(y)) &= \sinh(x \ln y + 2\pi i k), \quad \text{where } i \text{ is complex unit, } k \in \mathbb{Z}. \\ \Rightarrow (x + T_x) \ln(y) &= x \ln y + 2\pi i k \\ \Rightarrow T_x \ln(y) &= 2\pi i k \end{aligned}$$

Solving for  $T_x$  in terms of  $y$  and  $k$  gives

$$T_x = T_x(y) = \frac{2\pi i k}{\ln(y)}, \quad k \in \mathbb{Z}.$$

Thus,  $\cosh(x, y)$  has no real period in  $x$  but exhibits a period in complex form.

2. Let's verify that  $T_x = i2\pi k/\ln(y)$ :

$$\sinh(x + T_x, y) = \sinh(x \ln(y))$$

By substituting the value of  $T_x$  in the above expression, we get

$$\begin{aligned} &= \sinh\left(\left(x + \frac{2\pi i k}{\ln(y)}\right) \ln(y)\right) \\ &= \sinh(x \ln(y) + 2\pi i k) \\ &= \sinh(x \ln(y)) \\ &= \sinh(x, y). \end{aligned}$$

Thus, we conclude that  $\sinh(x + T_x, y) = \sinh(x, y)$ , where  $T_x$  is  $i2\pi k/\ln(y)$ .

## 7. Zeros of $\sinh(x, y)$ in $y$

**Theorem 42.** The zeros of  $\sinh(x, y)$  in  $y$  are given by

$$y = e^{i\pi k/x}, \quad x \neq 0, k \in \mathbb{Z}.$$

### Proof

#### a. First method

By definition (3.82), it gives the equation

$$\frac{1}{2}(y^x - y^{-x}) = 0,$$

from which it follows that

$$\begin{aligned} y^{2x} - 1 &= 0 \\ \Rightarrow e^{2x \ln y} &= e^{i2\pi k} \\ \Rightarrow 2x \ln y &= i2\pi k. \end{aligned}$$

Solving for  $y$  gives

$$y = e^{i\pi k/x}, \quad x \neq 0, k \in \mathbb{Z}.$$

#### b. Second method

$$\begin{aligned} \sinh(x \ln(y)) &= 0 \\ &= \sinh(i\pi k) \text{ for } k \in \mathbb{Z}, \end{aligned}$$

from which it follows that

$$x \ln(y) = i\pi k.$$

Solving for  $y$  from the equation, we obtain

$$y = e^{i\pi k/x}, \quad x \neq 0, k \in \mathbb{Z}.$$

## 8. Periodicity in $y$

**Theorem 43.** For each fixed  $x > 0$ ,  $\sinh(x,y)$  has no real period but exhibits a non-constant complex period in  $y$ , given by

$$T_y(x, y) = \left( e^{\frac{i2\pi k}{|x|}} - 1 \right) y, \quad x \neq 0, k \in \mathbb{Z}. \quad (3.89)$$

### Proof

Let  $T_y = T_y(x, y)$ . Assume  $T_y$  is the period of  $\sinh(x,y)$  in  $y$ . Then  $T_y$  must satisfy the following equation

$$\begin{aligned} \sinh(x, y + T_y) &= \sinh(x, y) \\ \Rightarrow \sinh(x \ln(y + T_y)) &= \sinh(x \ln y) \\ \Rightarrow x \ln(y + T_y) &= x \ln y + i2\pi k \quad \text{for some integer } k. \\ \Rightarrow x \ln\left(\frac{y + T_y}{y}\right) &= i2\pi k \\ \Rightarrow \ln\left(\frac{y + T_y}{y}\right) &= \frac{i2\pi k}{|x|} \\ \Rightarrow \frac{y + T_y}{y} &= e^{i2\pi k/|x|} \quad (*) \end{aligned}$$

Solving for  $T_y$  from equation (\*) gives

$$T_y(x, y) = y \left( e^{i2\pi k/|x|} - 1 \right), \quad k \in \mathbb{Z}.$$

Since  $T_y$  from (3.89) depends on  $i$ ,  $x$ ,  $y$  and  $k$  in  $y$ , we conclude that  $\sinh(x,y)$  has no real period but exhibits a non-constant period in complex form in  $y$ .

### 3-11-3. Graph of $\sinh(x,y)$

The 3D graph of  $\sinh(x,y)$  is shown in Figure 3.6. Since the function  $\sinh(x,y)$  can be expressed as  $\sinh(x \ln y)$ , it retains the same behavior as  $\sinh(x)$  for each value of  $y$ , its surface exhibits the same curvature along the  $x$ -axis as the standard hyperbolic sine. Its 3D graph is a surface crossing the plane  $f(x,y) = 0$  along two perpendicular lines  $x = 0$  and  $y = 1$ , with a saddle at  $(0,1)$ , and exponential growth

in the four surrounding regions.

### 3D Surface of $z = \sinh(x,y)$

Graph:  $\sinh(x,y)$  for  $x$  in  $(-4,4)$

Graph:  $\sinh(x,y)$  for  $x$  in  $(-2,2)$

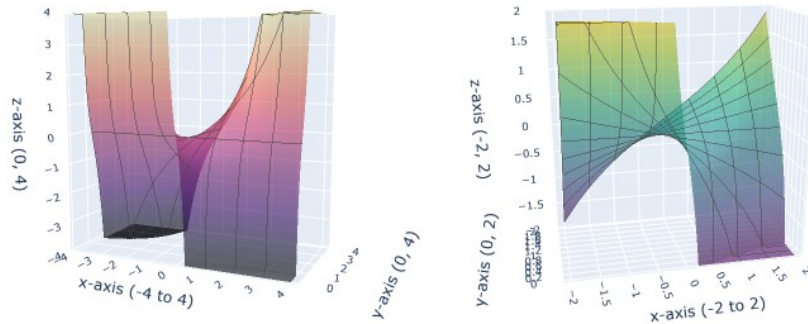


Figure 3.6 shows the graph of the function  $\sinh(x,y) \equiv \frac{1}{2}(y^x + y^{-x})$ .

### 3-11-4. Local Maxima, Minima and Saddle Points of $\sinh(x,y)$ via Calculus

#### First partial derivatives

$$f_x = \frac{d}{dx} \sinh(x \ln y) = \cosh(x \ln y) \cdot \ln y$$

$$f_y = \frac{d}{dy} \sinh(x \ln y) = \frac{x}{y} \cosh(x \ln y)$$

#### Critical points

Setting  $f_x = 0$  and  $f_y = 0$ :

$$f_x = 0 \Rightarrow \cosh(x \ln y) \cdot \ln y = 0 \Rightarrow \begin{cases} \cosh(x \ln y) = 0 & \Rightarrow (a) \ x \ln y = \frac{i\pi}{2} + i\pi k, \text{ for } k \in \mathbb{Z} \\ \ln y = 0 & \Rightarrow (b) \ y = 1 \end{cases}$$

$$f_y = 0 \Rightarrow x \cosh(x \ln y) = 0 \Rightarrow \begin{cases} (c) \ x = 0 \\ \cosh(x \ln y) = 0 \Rightarrow (d) \ x \ln y = \frac{i\pi}{2} + i\pi k, \text{ for } k \in \mathbb{Z} \end{cases}$$

The solutions from (a), (b), (c) and (d) can be summarized in four cases:

- **Case 1:** At  $x_0 = 0$  and  $y_0 = 1$ , the point satisfies  $f_x = 0$  and  $f_y = 0$ . Therefore,  $(0,1)$  is a critical point.
- **Case 2:** At  $x_0 = 0$  for any  $y_0 > 0$  and  $y_0 \neq 1$ , it gives  $f_x = 0$  and  $f_y = \ln(y_0) \neq 0$ . Hence,  $(0, y_0)$  is not a critical point (unless  $y_0 = 1$ ).
- **Case 3:** At  $y_0 = 1$  for any  $x_0 \in \mathbb{R}$  and  $x_0 \neq 0$ , it gives  $f_x = 0$  and  $f_y = x_0 \neq 0$ . Therefore,  $(x_0, 1)$  is not a critical point.
- **Case 4:** At points  $(x_0, y_0)$  satisfying  $\cosh(x_0 \ln y_0) = 0$ , we obtain the complete set of critical points in the complex form. This set is given by the family of the curves

$$x_0 \ln y_0 = \frac{i\pi}{2} + i\pi k, \text{ for } k \in \mathbb{Z}.$$

**Note:** We continue the analysis of Case 4 only for theoretical study. Please refer to **Remark 3.4.7**.

## Second order partial derivatives

$$f_{xx} = \sinh(x \ln y) \cdot \ln^2 y$$

$$f_{yy} = \frac{x^2 \sinh(x \ln y) - x \cosh(x \ln y)}{y^2}$$

$$f_{xy} = \frac{x}{y} \sinh(x \ln y) \ln(y) + \frac{1}{y} \cosh(x \ln y)$$

## Calculate Discriminant D

- At  $x = 0, y = 1$ , we have

$$f_{xx} = \sinh(0 \ln 1) \cdot \ln^2 1 = 0$$

$$D(0,1) = (f_{xx})(f_{yy}) - (f_{xy})^2$$

$$= (\sinh(0 \ln 1) \cdot \ln^2 1) \cdot \left( \frac{0^2 \sinh(0 \ln 1) - \cosh(0 \ln 1)}{1^2} \right) - \left( \frac{0}{1} \sinh(0 \ln 1) \ln(1) + \frac{1}{1} \cosh(0 \ln 1) \right)^2$$

$$= -1$$

Since  $D(0,1) = -1 < 0$ , we conclude  $(0,1)$  is a saddle point of  $\cosh(x,y)$ .

- Please refer to **Remark 3.4.7**. At critical curves defined by  $x_0 \ln y_0 = i\pi/2 + i\pi k$ , for  $k \in \mathbb{Z}$  and  $y_0 > 0$ , we have:

$$\begin{aligned} D(x_0, y_0) &= (f_{xx})(f_{yy}) - (f_{xy})^2 \\ &= (\sinh(x_0 \ln y_0) \cdot \ln^2 y_0) \cdot \left( \frac{x_0^2 \sinh(x_0 \ln y_0) - \cosh(x_0 \ln y_0)}{y_0^2} \right) - \\ &\quad \left( \frac{x_0}{y_0} \sinh(x_0 \ln y_0) \ln(y_0) + \frac{1}{y_0} \cosh(x_0 \ln y_0) \right)^2 \end{aligned}$$

Substituting  $\cosh(i\pi/2 + i\pi k) = 0$  into the expression gives

$$\begin{aligned} D(x_0, y_0) &= (\sinh(x_0 \ln y_0) \cdot \ln^2 y_0) \cdot \left( \frac{x_0^2 \sinh(x_0 \ln y_0)}{y_0^2} \right) - \left( \frac{x_0^2}{y_0^2} \ln^2 y_0 \sinh^2(x_0 \ln y_0) \right) \\ &= \frac{x_0^2}{y_0^2} \ln^2(y_0) \sinh^2(x_0 \ln y_0) - \frac{x_0^2}{y_0^2} \ln^2(y_0) \sinh^2(x_0 \ln y_0) \\ &= 0. \end{aligned}$$

Since  $D(x_0, y_0) = 0$ , the Second Derivative Test is inconclusive.

### Summary of critical points of $\sinh(x,y)$

- Critical point:  $(0,1)$
- Real local maxima: None
- Real local minima: None
- Saddle point:  $(0,1)$

### 3-11-5. Summary of $\sinh(x,y)$

Formula (3.86) defines the extended hyperbolic sine function with two arguments as  $\sinh(x,y) \equiv \sinh(x \ln y)$ . The function is defined on the domain  $x \in (-\infty, \infty)$  and  $y$

$> 0$ , with a range of  $(-\infty, \infty)$ . It has no real period but exhibits non-constant complex periods. The function  $\sinh(x,y)$  represents a hyperbolic sine-like surface in 3D. It is an odd in  $x$  and has no local extrema, possessing only a saddle point at  $(0,1)$ . It is clear that when  $y$  equals  $e$ , the function  $\sinh(x, e)$  is equal to  $\sinh(x)$ .

## Section 3-12. Foundations of the Extended Hyperbolic Tangent Function $\tanh(x,y)$

### 3-12-1. Definition and Theorem

**Definition.** The extended hyperbolic tangent function, denoted as  $\tanh(x,y)$ , is defined by

$$\tanh(x, y) \equiv \frac{y^x - y^{-x}}{y^x + y^{-x}} \quad \text{for } x, y \in \mathbb{R} \text{ and } y > 0. \quad (3.90)$$

**Theorem 44.** The extended hyperbolic tangent function satisfies the relation

$$\tanh(x, y) \equiv \frac{y^x - y^{-x}}{y^x + y^{-x}} = \tanh(x \ln y) \quad \text{for real } x, y \text{ and } y > 0. \quad (3.91)$$

#### Proof

By adding (3.80) and (3.81), we have

$$y^x + y^{-x} = 2 \cosh(x \ln y)$$

By subtracting (3.81) from (3.80), we have

$$y^x - y^{-x} = 2 \sinh(x \ln y)$$

Taking the ratio of the two expressions:

$$\frac{y^x - y^{-x}}{y^x + y^{-x}} = \frac{2 \sinh(x \ln y)}{2 \cosh(x \ln y)} = \tanh(x \ln y)$$

Combining the above result with definition (3.90) gives

$$\tanh(x, y) \equiv \tanh(x \ln y).$$

### 3-10-2. Basic Properties of $\tanh(x, y)$

**1. Domain:**  $x \in (-\infty, \infty)$ ,  $y > 0$ .

**2. Range:**  $-1 < \tanh(x, y) < 1$ .

#### Proof

Let  $t = x \ln y$ , we have

$$\lim_{t \rightarrow +\infty} \tanh(t) = \lim_{t \rightarrow +\infty} \frac{e^t - e^{-t}}{e^t + e^{-t}} = \lim_{t \rightarrow +\infty} \frac{e^t}{e^t} = +1. \quad \text{and} \quad \lim_{t \rightarrow -\infty} \tanh(t) = \lim_{t \rightarrow -\infty} \frac{e^t - e^{-t}}{e^t + e^{-t}} = \lim_{t \rightarrow -\infty} -\frac{e^{-t}}{e^{-t}} = -1.$$

**3. Function type:**  $\tanh(x, y)$  is an odd in  $x$ ; the function is neither even nor odd in  $y$ .

#### Proof

We have  $\tanh(-x, y) = \tanh(-x \ln y) = -\tanh(x \ln y) = -\tanh(x, y)$ . Therefore, the function  $\tanh(x, y)$  is odd in  $x$ . Since  $\ln(y)$  only defined for  $y > 0$ , this does not create even/odd symmetry in  $y$ .

### 4. Maxima, Minima and Zeros of $\tanh(x, y)$ with Respect to $x$

**Theorem 45.** The maxima, minima and zeros of  $\tanh(x, y)$  with respect to  $x$  can be determined as follows:

- (a) **Maxima:** None
- (b) **Minima:** None
- (c) **Zeros:**  $x = 0$  for all  $y > 0$

#### Proof

To find the critical points and determine if there are any extrema, we use the first and second derivatives as follows:

#### First derivative

$$\frac{d}{dx} \tanh(x, y) = \frac{d}{dx} \tanh(x \ln y) = \frac{\ln y}{\cosh^2(x \ln y)}$$

### Analysis for fixed $y$

- For fixed  $y \neq 1$ :
  - If  $0 < y < 1$ , then  $\ln(y) < 0$ , and since  $\cosh^2(x \ln y) > 0$ , we have  $d/dx[\tanh(x,y)] < 0$  for all  $x$ . The function is strictly decreasing in  $x$ , no extrema occur.
  - If  $y > 1$ , then  $\ln(y) > 0$ , and again  $\cosh^2(x \ln y) > 0$ , so  $d/dx[\tanh(x,y)] > 0$  for all  $x$ . The function is strictly increasing in  $x$ , no extrema occur.
- For  $y = 1$ , then  $\ln(y) = 0$ , therefore  $d/dx[\tanh(x,1)] = 0$  for all  $x$ . In this case,  $\tanh(x,1) = 0$ , so every point  $(x,1)$  is a critical point, but the function is constant along the entire line  $y = 1$  and does not attain a maximum or minimum.

**Remark 3.4.9.** At this point, we observe two facts: 1. In the full two variable world,  $(0,1)$  is the only critical point, and it is saddle point. 2. Along the one dimensional slice  $y = 1$ ,  $(x,1)$  is a critical point for every  $x$ , because the function becomes constant there. Clearly, this is one of those moments where multivariable calculus shows its richer structure namely the behavior of a function along a slice can be dramatically different from its behavior in the full space.

### Summary

**(a) Maxima:** None

**(b) Minima:** None.

**(c) Zeros of  $\tanh(x,y)$  in  $x$**  occur when  $\tanh(x,y) = 0$ , which is equivalent to  $\tanh(x \ln y) = 0$  or  $x \ln(y) = 0$ . The equation gives two possibilities:

- $x = 0$
- $\ln y = 0 \Rightarrow y = 1$

As we examine  $\tanh(x,y)$  in  $x$ , we determine that the sole zero occurs at  $x = 0$  for any  $y > 0$ .

## 5. Periodicity in $x$

**Theorem 46.** For each fixed  $y > 0$  with  $y \neq 1$ ,  $\tanh(x,y)$  has no real period but exhibits a non-constant complex period, given by

$$T_x(y) = \frac{i\pi k}{\ln(y)}, \quad (3.92)$$

### Proof

Let  $T_x \equiv T_x(y)$ . Assume  $T_x$  is the period of  $\tanh(x,y)$  in  $x$ . Then  $T_x$  must satisfy the equation

$$\begin{aligned} \tanh(x + T_x, y) &= \tanh(x, y) \\ \Rightarrow \tanh((x + T_x) \ln(y)) &= \tanh(x \ln y) \\ \Rightarrow \tanh((x + T_x) \ln(y)) &= \tanh(x \ln y + i\pi k), \quad \text{where } i \text{ is complex unit, } k \in \mathbb{Z}. \\ \Rightarrow (x + T_x) \ln(y) &= x \ln y + i\pi k \\ \Rightarrow T_x \ln(y) &= i\pi k \end{aligned}$$

Solving for  $T_x$  in terms of  $y$  and  $k$  gives

$$T_x = \frac{i\pi k}{\ln(y)}, \quad k \in \mathbb{Z}.$$

We conclude that  $\tanh(x,y)$  has no real period in  $x$  but contains a non-constant period in complex form.

## 6. Extrema and Zeros of $\tanh(x,y)$ with Respect to $y$

### 6A. Extrema of $\tanh(x,y)$ in $y$

#### First derivative

$$\frac{d}{dy} \tanh(x, y) = \frac{d}{dy} \tanh(x \ln y) = \frac{x}{y \cosh^2(x \ln y)}$$

#### Analysis for fixed $x$

- For  $y > 0$  with  $x \neq 0$ , then

$$\frac{x}{y \cosh^2(x \ln y)} > 0.$$

We conclude  $\tanh(x,y)$  is strictly increasing in  $y$  when  $x > 0$  and is decreasing when  $x < 0$ . Therefore, no extrema occur in this case.

- For  $y > 0$  with  $x = 0$ , then  $d/dy(\tanh(0,y)) = 0$  for all  $y > 0$ . In this case,  $\tanh(0,y) = 0$ , so every point  $(0,y)$  is a critical point, but the function is constant along the entire line  $x = 0$  and does not attain a maximum or minimum.

### Summary of extrema of $\tanh(x,y)$ in $y$

**(a) Maxima:** None

**(b) Minima:** None

### 6B. Zeros of $\tanh(x,y)$ in $y$

The zeros of  $\tanh(x,y)$  occur when  $\tanh(x,y) = 0$ , from which it follows that  $\tanh(x \ln y) = 0$  or  $x \ln(y) = 0$ . The equation gives two possibilities:

- $x = 0$
- $\ln y = 0 \Rightarrow y = 1$

As we examine  $\tanh(x,y)$  in  $y$ , we determine that the sole zero occurs at  $y = 1$  for all  $x$ .

### 7. Periodicity in $y$

**Theorem 47.** For fixed  $x$ , the function  $\tanh(x,y)$  has no real period but exhibits a non-constant complex period, given by

$$T_y(x, y) = \left( e^{\frac{i\pi k}{|x|}} - 1 \right) y, \quad x \neq 0, k \in \mathbb{Z}, \quad (3.93)$$

#### Proof

Let  $T_y = T_y(x, y)$ . Assume  $T_y$  is the period of  $\tanh(x,y)$  in  $y$ . Then  $T_y$  must satisfy the equation

$$\tanh(x, y + T_y) = \tanh(x, y)$$

$$\begin{aligned}
\Rightarrow \tanh(x \ln(y + T_y)) &= \tanh(x \ln y) \\
\Rightarrow x \ln(y + T_y) &= x \ln y + i\pi k \quad \text{for some integer } k. \\
\Rightarrow x \ln\left(\frac{y + T_y}{y}\right) &= i\pi k \\
\Rightarrow \ln\left(\frac{y + T_y}{y}\right) &= \frac{i\pi k}{|x|} \\
\Rightarrow \frac{y + T_y}{y} &= e^{i\pi k/|x|}
\end{aligned}$$

Solving for  $T_y$  in terms of  $|x|$ ,  $y$  and  $k$  gives

$$T_y = T_y(x, y) = y(e^{i\pi k/|x|} - 1), \quad k \in \mathbb{Z}.$$

Since  $T_y(x, y)$  from (3.93) depends on  $i$ ,  $x$ ,  $y$  and  $k$  in  $y$ , we conclude that  $\tanh(x, y)$  has no real period in  $y$  but exhibits a non-constant period in complex form.

### 3-12-3. Graph of $\tanh(x, y)$

The 3D graph of  $\tanh(x, y)$  is shown in Figure 3.7. The function  $\tanh(x, y)$  retains the same behavior as  $\tanh(x)$  for each value of  $y$ , its surface exhibits the same curvature along the  $x$ -axis as the standard hyperbolic tangent. There are no cliffs, no singularities, no vertical walls. The surface rises one side and sinks in other side.

### 3D Surface of $z = \tanh(x,y)$

Graph:  $\tanh(x,y)$  for  $x$  in  $(-2,2)$

Graph:  $\tanh(x,y)$  for  $x$  in  $(-1,1)$

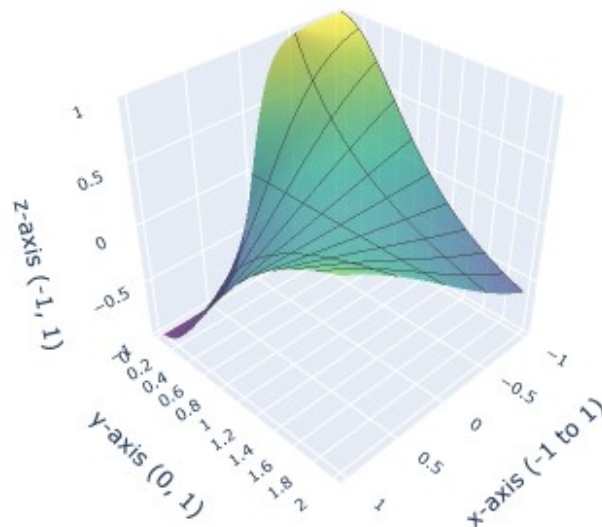
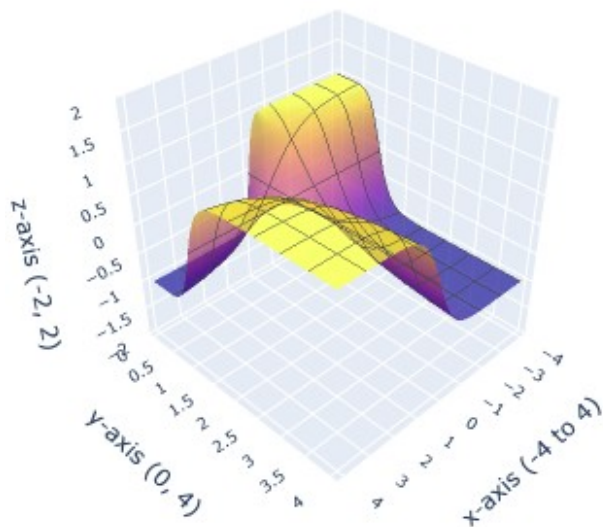


Figure 3.7 shows the graph of  $\tanh(x,y) \equiv \frac{y^x - y^{-x}}{y^x + y^{-x}}$ .

#### 3-12-4. Local Maxima, Minima and Saddle Points of $\tanh(x,y)$ via Calculus

##### First partial derivatives

$$f_x = \frac{d}{dx} \tanh(x \ln y) = \operatorname{sech}^2(x \ln y) \cdot \ln y$$

$$f_y = \frac{d}{dy} \tanh(x \ln y) = \frac{x}{y} \operatorname{sech}^2(x \ln y)$$

##### Critical points

Setting  $f_x = 0$  and  $f_y = 0$ : Since  $\operatorname{sech}(x \ln y) \geq 1$  for  $y > 0$  and all  $x$ , we have

$$f_x = 0 \Rightarrow \ln y \operatorname{sech}^2(x \ln y) \Rightarrow \ln y = 0 \Rightarrow y = 1$$

and

$$f_y = 0 \Rightarrow \frac{x}{y} \operatorname{sech}^2(x \ln y) \ln y = 0 \Rightarrow x = 0 \text{ or } y = 1$$

The solutions can be summarized in two cases:

- Case 1: At  $x = 0$  for  $y > 0$  with  $y \neq 1$ , it gives  $f_x \neq 0$  and  $f_y = 0$ , and therefore  $(0, y_0)$  with  $y_0 \neq 1$  is not a set of the critical points along the positive  $y$ -axis.
- Case 2: At  $y = 1$  and  $x = 0$ , it gives  $f_x = 0$  and  $f_y = 0$ , and therefore  $(0, 1)$  is a critical point.

Thus, only  $(0, 1)$  is a critical point of  $\tanh(x, y)$ .

### Second order partial derivatives

$$f_{xx} = -2 \operatorname{sech}^2(x \ln y) \tanh(x \ln y) \ln^2(y) \Rightarrow f_{xx}(0, 1) = 0$$

$$f_{yy} = -\frac{x}{y^2} \operatorname{sech}^2(x \ln y) - \frac{2x^2}{y^2} \operatorname{sech}^2(x \ln y) \tanh(x \ln y) \Rightarrow f_{yy}(0, 1) = 0$$

$$f_{xy} = \frac{1}{y} \operatorname{sech}^2(x \ln y) - \frac{2x \ln y}{y} \operatorname{sech}^2(x \ln y) \tanh(x \ln y) \Rightarrow f_{xy}(0, 1) = 1$$

### Calculate Discriminant $D(0, 1)$

$$\begin{aligned} D(0, 1) &= (f_{xx})(f_{yy}) - (f_{xy})^2 \\ &= (0)(0) - 1^2 \\ &= -1 \end{aligned}$$

Since  $D(0, 1) = -1 < 0$ , the critical point  $(0, 1)$  is a saddle point of  $\tanh(x, y)$ .

### Behavior of $\tanh(x, y)$

- For  $y > 0$  with  $y \neq 1$ , we consider two cases:
  - If  $0 < y < 1$ , then  $\ln(y) < 0$ , and we have

$$\lim_{x \rightarrow \pm\infty} \tanh(x, y) = \lim_{x \rightarrow \pm\infty} \tanh(-x |\ln y|) = \lim_{x \rightarrow \pm\infty} \{-\tanh(x |\ln y|)\} = \mp 1$$

- If  $y > 1$ , then  $\ln(y) > 0$ , and we have

$$\lim_{x \rightarrow \pm\infty} \tanh(x, y) = \lim_{x \rightarrow \pm\infty} \tanh(x |\ln y|) = \pm 1$$

- For  $y = 1$ , then

$$\lim_{x \rightarrow \pm\infty} \tanh(x, 1) = \lim_{x \rightarrow \pm\infty} \tanh(x |\ln 1|) = 0.$$

### Summary of critical points of $\tanh(x, y)$

- **Local maxima:** None
- **Local minima:** None
- **Saddle point:**  $(0, 1)$

#### 3-10-5. Summary of $\tanh(x, y)$

Formula (3.90) defines the new hyperbolic tangent function with two arguments as  $\tanh(x, y) \equiv \tanh(x \ln y)$ . The function is defined on the domain  $x \in (-\infty, \infty)$  and  $y > 0$ , with a range of  $(-1, 1)$ . It has no real period, though its complex extension exhibits a non-constant complex period behavior. In three dimensions, the function represents a hyperbolic tangent-like surface. It is an odd in  $x$  and has no local extrema, possessing only a saddle point at  $(0, 1)$ . The graph has no cliffs, singularities, and vertical walls. It is clear that when  $y$  equals  $e$ , the function  $\tanh(x, e)$  is equal to  $\tanh(x)$ .

### Section 3-13. Foundations of the Extended Hyperbolic Cotangent Function $\coth(x, y)$

#### 3-13-1. Definition and Theorem

**Definition.** The extended hyperbolic cotangent function, denoted as  $\coth(x, y)$ , is defined by

$$\coth(x, y) \equiv \frac{y^x + y^{-x}}{y^x - y^{-x}} \tag{3.94}$$

for  $x, y \in \mathbb{R}$ ,  $x \ln y \neq 0$ , and  $y > 0$ .

**Theorem 48.** The extended hyperbolic cotangent function satisfies the relation

$$\coth(x, y) \equiv \frac{y^x + y^{-x}}{y^x - y^{-x}} = \coth(x \ln y) \text{ for real } x, y \text{ with } x \ln y \neq 0 \text{ and } y > 0. \quad (3.95)$$

### Proof

By subtracting (3.81) from (3.80), we have

$$y^x - y^{-x} = 2 \cosh(x \ln y)$$

By adding (3.80) and (3.81), we have

$$y^x + y^{-x} = 2 \sinh(x \ln y)$$

Taking the ratio of the two expressions:

$$\frac{y^x + y^{-x}}{y^x - y^{-x}} = \frac{2 \cosh(x \ln y)}{2 \sinh(x \ln y)} = \coth(x \ln y)$$

Combining the result with definition (3.94) gives

$$\coth(x, y) \equiv \coth(x \ln y).$$

### 3-13-2. Basic Properties of $\coth(x, y)$

**1. Domain:**  $x \in (-\infty, \infty)$ ,  $y > 0$ ,  $x \ln(y) \neq 0$ .

**2. Range:**  $-1 < \coth(x, y) < 1$ .

### Proof

Let  $u = x \ln y$ , we have

$$\lim_{u \rightarrow +\infty} \coth(u) = \lim_{u \rightarrow +\infty} \frac{e^u + e^{-u}}{e^u - e^{-u}} = \lim_{u \rightarrow +\infty} \frac{e^u}{e^u} = +1 \quad \text{and} \quad \lim_{u \rightarrow -\infty} \coth(u) = \lim_{u \rightarrow -\infty} \frac{e^u + e^{-u}}{e^u - e^{-u}} = \lim_{u \rightarrow -\infty} -\frac{e^{-u}}{e^{-u}} = -1$$

**3. Function type:**  $\coth(x, y)$  is an odd in  $x$ ; the function is neither even nor odd in  $y$ .

### Proof

We have  $\coth(-x,y) = \coth(-x\ln y) = -\coth(x\ln y) = -\coth(x,y)$ . Therefore, the function  $\coth(x,y)$  is odd in  $x$ . Since  $\ln(y)$  only defined for  $y > 0$ , the function is neither even nor odd in  $y$ .

#### 4. Maxima, Minima and Zeros of $\coth(x,y)$ with Respect to $x$

**Theorem 45.** The maxima, minima and zeros of  $\coth(x,y)$  with respect to  $x$  can be determined as follows:

(a) **Maxima:** None

(b) **Minima:** None

(c) **Zeros:**  $x = 0$

#### Proof

We examine the followings:

#### First derivative

$$\frac{d}{dx} \coth(x, y) = \frac{d}{dx} \coth(x \ln y) = -\frac{\ln y}{\sinh^2(x \ln y)}$$

#### Analysis for fixed $y$

- For fixed  $y$  with  $y \neq 1$  and  $x \neq 0$ :
  - If  $0 < y < 1$ , then  $\ln(y) < 0$ , and since  $\sinh^2(x\ln y) > 0$ , we have  $d/dx[\coth(x,y)] > 0$  for all  $x$ . The function is strictly increasing in  $x$ , no extrema occur.
  - If  $y > 1$ , then  $\ln(y) > 0$ , and again  $\sinh^2(x\ln y) > 0$ , so  $d/dx[\coth(x,y)] < 0$  for all  $x$ . The function is strictly decreasing in  $x$ , no extrema occur.
- For  $y = 1$  or  $x = 0$ ,  $\coth(x,y)$  is undefined, no extrema occur in this case.

#### Summary of extrema of $\coth(x,y)$ in $x$

- a) **Maxima:** None
- b) **Minima:** None.

- c) **Zeros of  $\coth(x,y)$  in  $x$ :** Since  $\coth(x \ln y)$  is not defined at any point with  $x \ln y = 0$  and  $\cosh(x \ln y)$  is always positive, there are no zeros in  $x$ .

## 5. Periodicity in $x$

**Theorem 46.** For each fixed  $y > 0$  with  $y \neq 1$ ,  $\coth(x,y)$  has no real period but exhibits a non-constant complex period, given by

$$T_x(y) = \frac{i\pi k}{\ln(y)}, \quad (3.96)$$

### Proof

Let  $T_x \equiv T_x(y)$ . Assume  $T_x$  is the period of  $\coth(x,y)$  in  $x$ . Then  $T_x$  must satisfy the equation

$$\begin{aligned} \coth(x+T_x, y) &= \coth(x, y) \\ \Rightarrow \coth((x+T_x) \ln(y)) &= \coth(x \ln y) \\ \Rightarrow \coth((x+T_x) \ln(y)) &= \coth(x \ln y + i\pi k), \quad \text{where } i \text{ is complex unit, } k \in \mathbb{Z}. \\ \Rightarrow (x+T_x) \ln(y) &= x \ln y + i\pi k \\ \Rightarrow T_x \ln(y) &= i\pi k \end{aligned}$$

Solving for  $T_x$  in terms of  $y$  and  $k$  gives

$$T_x = \frac{i\pi k}{\ln(y)}, \quad k \in \mathbb{Z}.$$

We conclude that  $\coth(x,y)$  has no real period in  $x$  but exhibits a non-constant complex period.

## 6. Extrema and Zeros of $\coth(x,y)$ with Respect to $y$

### 6A. Extrema of $\coth(x,y)$ in $y$

#### First derivative

$$\frac{d}{dy} \coth(x, y) = \frac{d}{dy} \coth(x \ln y) = -\frac{x}{y \sinh^2(x \ln y)}$$

#### Analysis for fixed $x$

- For  $y > 0$  with  $x \neq 0$  and  $y \neq 1$ :

- Since  $\sinh^2(x \ln y) > 0$ , we have  $d/dy[\coth(x,y)] > 0$  for all  $x < 0$ . The function is strictly increasing in  $x$ , no extrema occur.
- $d/dx[\coth(x,y)] < 0$  for all  $x > 0$ . The function is strictly decreasing in  $x$ , no extrema occur.
- For  $y = 1$  or  $x = 0$ ,  $\coth(x,y)$  is undefined, no extrema occur in this case.

### Summary

**(a) Maxima:** None

**(b) Minima:** None

### 6B. Zeros of $\coth(x,y)$ in $y$

Since  $\coth(x \ln y)$  is not defined at any point with  $x \ln y = 0$  and  $\cosh(x \ln y)$  is always positive, there are no zeros in  $y$ .

### 7. Periodicity in $y$

**Theorem 47.** The function  $\coth(x,y)$  has no real period but exhibits a non-constant complex period in  $y$ , given by

$$T_y(x, y) = \left( e^{\frac{i\pi k}{|x|}} - 1 \right) y, \quad x \neq 0, k \in \mathbb{Z}, \quad (3.97)$$

### Proof

Let  $T_y = T_y(x,y)$ . Assume  $T_y$  is the period of  $\coth(x,y)$  in  $y$ . Then  $T_y$  must satisfy the equation

$$\begin{aligned} \coth(x, y + T_y) &= \coth(x, y) \\ \Rightarrow \coth(x \ln(y + T_y)) &= \coth(x \ln y) \\ \Rightarrow x \ln(y + T_y) &= x \ln y + i\pi k \quad \text{for some integer } k. \\ \Rightarrow x \ln\left(\frac{y + T_y}{y}\right) &= i\pi k \\ \Rightarrow \ln\left(\frac{y + T_y}{y}\right) &= \frac{i\pi k}{|x|} \\ \Rightarrow \frac{y + T_y}{y} &= e^{i\pi k/|x|} \end{aligned}$$

Solving for  $T_y$  in terms of  $|x|$ ,  $y$  and  $k$  gives

$$T_y(x, y) = y(e^{i\pi k/|x|} - 1), \quad k \in \mathbb{Z}.$$

### 3-13-3. Graph of $\coth(x, y)$

The 3D graph of  $\coth(x, y)$  is shown in Figure 3.8. Since the function  $\coth(x, y)$  retains the same behavior as  $\coth(x)$  for each value of  $y$ , its surface exhibits the same curvature along the  $x$ -axis as the standard hyperbolic cotangent. The graph features a vertical chasm at  $y = 1$ , flanked by two mirror-like sheets on either side. Each sheet ascends to  $+\infty$  on one side of the chasm while descending to  $-\infty$  on the opposite side. Additionally, away from the chasm, the sheets level out into expansive horizontal plains at heights of 1 and  $-1$ . The level curves depict exponential arcs ( $x \ln y = c \rightarrow y = e^{c/x}$ , constant  $c$ ) that radiate outward from the  $x$ -axis.

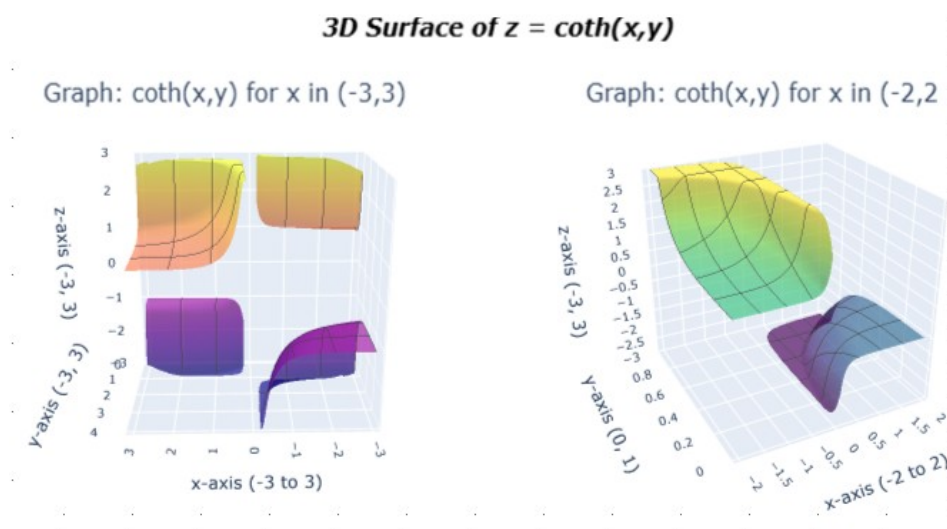


Figure 3.8 shows the graph of the function  $\coth(x, y) \equiv \frac{y^x + y^{-x}}{y^x - y^{-x}}$ .

### 3-13-4. Local Maxima, Minima and Saddle Points of $\coth(x, y)$ via Calculus

#### First partial derivatives

For  $x \ln y \neq 0$  and  $y > 0$ , we have

$$f_x = \frac{d}{dx} \coth(x, y) = \frac{d}{dx} \coth(x \ln y) = -\frac{\ln y}{\sinh^2(x \ln y)}$$

$$f_y = \frac{d}{dy} \coth(x, y) = \frac{d}{dy} \coth(x \ln y) = -\frac{x}{y \sinh^2(x \ln y)}$$

### Critical points

Setting  $f_x = 0$  and  $f_y = 0$ : As  $x \ln y \neq 0$  and  $y > 0$  for all  $x$ , we have

$$f_x = 0 \Rightarrow \ln y = 0 \Rightarrow y = 1 \text{ for all } x \neq 0$$

and

$$f_y = 0 \Rightarrow x = 0 \text{ for all } y > 0.$$

Notice that  $\cosh(x, y)$  is undefined at  $y = 1$ .

The solutions can be summarized in two cases:

- **Case 1:** At  $x_0 = 0$  for  $y_0 > 0$ , it gives  $f_x = \infty$  and  $f_y = \infty$ , and therefore  $(0, y_0)$  with  $y_0 \neq 1$  is not a set of the critical points.
- **Case 2:** At  $y = 1$ , it gives  $f_x = \infty$  and  $f_y = \infty$ , and therefore  $(x_0, 1)$  is not a set of the critical points.

Since  $\coth(x, y)$  has no critical points, the second-derivative test is unnecessary, and the function has no local extrema or saddle points.

### Summary of critical points of $\coth(x, y)$

1. **Critical points:** None
2. **Maxima:** None
3. **Minima:** None
4. **Saddle:** None

### Behavior of $\coth(x, y)$

- If  $0 < y < 1$ , then  $\ln(y) < 0$ , and we have

$$\lim_{x \rightarrow \pm\infty} \coth(x, y) = \lim_{x \rightarrow \pm\infty} \coth(-x |\ln y|) = \lim_{x \rightarrow \pm\infty} \{-\coth(x |\ln y|)\} = \mp 1$$

- If  $y > 1$ , then  $\ln(y) > 0$ , and we have

$$\lim_{x \rightarrow \pm\infty} \coth(x, y) = \lim_{x \rightarrow \pm\infty} \coth(x |\ln y|) = \pm 1$$

- For  $y = 1$ , then

$$\lim_{y \rightarrow 1} \coth(x, y) = \lim_{y \rightarrow 1} \coth(x \ln y) = \frac{1}{\infty} = 0$$

- For  $x = 0$ , then

$$\lim_{x \rightarrow 0} \coth(x, y) = \lim_{x \rightarrow 0} \coth(x \ln y) = \frac{1}{\infty} = 0$$

### 3-13-5. Summary of $\coth(x, y)$

Formula (3.94) defines the new hyperbolic cotangent function with two arguments as  $\coth(x, y) \equiv \coth(x \ln y)$ . Its domain is  $x \in (-\infty, \infty)$  and  $y > 0$  with  $x \ln y \neq 0$ , and its range is  $(-1, 1)$ . It has no period in real domain but exhibits a non-constant period in complex form. The function represents a hyperbolic cotangent-like surface in 3D. It is an odd in  $x$  and has no local extrema or saddle points. It is clear that when  $y$  equals  $e$ , the function  $\coth(x, e)$  is equal to  $\coth(x)$ .

## Section 3-14. Overviews of the Extended Hyperbolic Secant and Cosecant Functions

### 3-14-1. Extended Hyperbolic Secant Function – $\operatorname{sech}(x, y)$

The extended hyperbolic secant function, denoted as  $\operatorname{sech}(x, y)$ , is the reciprocal of the extended hyperbolic cosine function. It's expressed as

$$\operatorname{sech}(x, y) = \frac{1}{\cosh(x, y)}, \quad \text{for real } x, y \text{ and } y > 0. \quad (3.98)$$

The alternative form of  $\operatorname{sech}(x, y)$  is

$$\operatorname{sech}(x, y) = \frac{2}{y^x + y^{-x}}. \quad (3.99)$$

The function  $\operatorname{sech}(x, y)$  has a similar characteristics and properties of  $\cosh(x, y)$ . We do not go into its details and other properties here and leave that to the readers to explore further.

### 3-14-2. Extended Hyperbolic Cosecant Function – $\operatorname{csch}(x,y)$

The extended hyperbolic cosecant function, denoted as  $\operatorname{csch}(x,y)$ , is the reciprocal of the extended hyperbolic sine function. It is defined as

$$\operatorname{csch}(x,y) = \frac{1}{\sinh(x,y)}, \quad (3.100)$$

for real  $x, y$  with  $y > 0$  and  $\sinh(x,y) \neq 0$  or  $x \ln y \neq 0$ .

The alternative form of  $\operatorname{csch}(x,y)$  is expressed as

$$\operatorname{csch}(x,y) = \frac{2}{y^x - y^{-x}}. \quad (3.101)$$

The extended hyperbolic function has a similar characteristics and properties of  $\sinh(x,y)$  but it is undefined at points where  $\sinh(x,y)=0$  or  $x \ln y = 0$ . We do not go into its details and other properties here and leave that to the readers to explore further.

## Section 3-15. Relationships between Extended Trigonometric Functions and Elementary Trigonometric Functions

We introduced a family of extended trigonometric functions with two variables constructed from the trigonometric functions with one variable in the previous sections. Each extended trigonometric function is defined as a composition of two simpler functions. For example, we may take an outer (wrapper) function such as  $\cos(x,y)$  and an inner function such as  $x \ln(y)$  so that the resulting extended function is  $\cos(x \ln y)$ . These extended trigonometric functions inherit many of the familiar properties of the trigonometric functions. In particular, their identities and relationships behave analogously, although we focus in this section only on the identities that are essential for our development.

### 3-15-1. Goal and Introduction of Basic Identities

Our next goal is to develop the inverse functions of those extended trigonometric functions. Before doing this task, we need to transform the inner function  $\ln(y)$  into the hyperbolic or trigonometric function so that it helps us to do conversion

between trigonometric functions and hyperbolic functions later. We know  $\ln(y)$  relates to two inverse hyperbolic functions:  $\sinh^{-1}$  or arcsinh and  $\cosh^{-1}$  or arccosh. The inverse hyperbolic sine and cosh are defined:

$$\sinh^{-1}(u) = \ln(u + \sqrt{u^2 + 1}), \quad -\infty < u < \infty \quad (3.102)$$

and

$$\cosh^{-1}(u) = \ln(u + \sqrt{u^2 - 1}), \quad 1 \leq u < \infty \quad (3.103)$$

Let  $y$  be real and if we set  $y = u + \sqrt{u^2 + 1}$  for all  $u$ , then solve for  $u$  in terms of  $y$  gives

$$u = \frac{y^2 - 1}{2y}. \quad (3.104)$$

Similarly, by setting  $y = u + \sqrt{u^2 - 1}$  for  $|u| \geq 1$ , we obtain

$$u = \frac{y^2 + 1}{2y}. \quad (3.105)$$

Therefore, the above analysis gives us the following identities:

$$\ln y = \sinh^{-1} \frac{y^2 - 1}{2y}, \quad y \geq 0 \quad \text{OR} \quad \ln y = \begin{cases} \cosh^{-1} \frac{y^2 + 1}{2y} & \text{for } y \geq 1 \\ -\cosh^{-1} \frac{y^2 + 1}{2y} & \text{for } 0 < y < 1 \end{cases} \quad (3.106)$$

and

$$\sinh^{-1} \frac{y^2 - 1}{2y} = \begin{cases} -\cosh^{-1} \frac{y^2 + 1}{2y} & -1 < y < 0 \\ \cosh^{-1} \frac{y^2 + 1}{2y} & y \geq 1 \end{cases} \quad \text{or} \quad \sinh^{-1} \frac{y^2 - 1}{2y} = \begin{cases} \cosh^{-1} \left( -\frac{y^2 + 1}{2y} \right) & 0 < y < 1 \\ -\cosh^{-1} \left( -\frac{y^2 + 1}{2y} \right) & y \leq -1 \end{cases} \quad (3.107)$$

Note that in case we consider the negative  $y$ , we rewrite the inverse of the hyperbolic sine function of (3.106) as

$$\sinh^{-1}\left(\frac{y^2-1}{2y}\right) = \begin{cases} \ln y & y > 0 \\ -\ln(-y) & y < 0 \end{cases} \quad (3.108)$$

### 3-15-2. Relationships and Identities

By integrating the findings from the preceding analysis with the definitions of the newly introduced functions, we establish several fundamental relationships between the extended trigonometric functions and the basic trigonometric functions as outlined below:

$$1. \quad \sin(x, y) = \frac{1}{2i}(y^{ix} - y^{-ix}) = \sin\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \sin\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \sin(x \ln y) \quad (3.109)$$

$$2. \quad \cos(x, y) = \frac{1}{2}(y^{ix} + y^{-ix}) = \cos\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \cos\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \cos(x \ln y) \quad (3.110)$$

$$3. \quad \tan(x, y) = \frac{1}{i}\left(\frac{y^{ix} - y^{-ix}}{y^{ix} + y^{-ix}}\right) = \tan\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \tan\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \tan(x \ln y) \quad (3.111)$$

$$4. \quad \cot(x, y) = i\left(\frac{y^{ix} + y^{-ix}}{y^{ix} - y^{-ix}}\right) = \cot\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \cot\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \cot(x \ln y). \quad (3.112)$$

Similarly, the relationships between extended hyperbolic functions and hyperbolic functions are:

$$5. \quad \sinh(x, y) = \frac{1}{2}y^x - \frac{1}{2}y^{-x} = \sinh\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \sinh\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \sinh(x \ln y) \quad (3.113)$$

$$6. \quad \cosh(x, y) = \frac{1}{2}y^x + \frac{1}{2}y^{-x} = \cosh\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \cosh\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \cosh(x \ln y) \quad (3.114)$$

$$7. \quad \tanh(x, y) = \frac{y^x - y^{-x}}{y^x + y^{-x}} = \tanh\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \tanh\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \tanh(x \ln y) \quad (3.115)$$

$$8. \quad \coth(x, y) = \frac{y^x + y^{-x}}{y^x - y^{-x}} = \coth\left(x \sinh^{-1}\frac{y^2-1}{2y}\right) = \coth\left(x \cosh^{-1}\frac{y^2+1}{2y}\right) = \coth(x \ln y). \quad (3.116)$$

### 3-15-3. Relationship via Power and Inverse

By adding formulas (3.113) and (3.114) and solve for  $y^x$ , we derive the identity:

$$y^x = \cosh\left(x \sinh^{-1} \frac{y^2-1}{2y}\right) + \sinh\left(x \sinh^{-1} \frac{y^2-1}{2y}\right), \text{ for } y > 0 \text{ and } x \in \mathbb{R}. \quad (3.117)$$

or

$$y^x = \cosh\left(x \cosh^{-1} \frac{y^2+1}{2y}\right) + \sinh\left(x \cosh^{-1} \frac{y^2+1}{2y}\right), \text{ for } y \geq 1 \text{ and } x \in \mathbb{R}. \quad (3.118)$$

(See formula (3.107) for  $y \geq 1$ .)

We notice that  $y$  raised to the power of  $x$  can be expressed in terms of both as a sinh function and a cosh function, which are the periodic functions since the period of sinh or cosh is  $2\pi i$ . This realization leads us to infer that any number can be expressed in hyperbolic form, implying that **every number has a complex period!**

### 3-15-4. Euler's Identity in Two Variables

By substituting  $\sinh^{-1}(y^2 - 1)/(2y) = \ln(y)$  into (3.117), we get

$$\begin{aligned} y^x &= \cosh(x \ln y) + \sinh(x \ln y) \quad \text{for } y > 0 \text{ and } x \in \mathbb{R} \\ &= \cosh(x, y) + \sinh(x, y) \quad \text{for } y > 0 \text{ and } x \in \mathbb{R}. \end{aligned} \quad (3.117a)$$

Substituting  $x$  for  $ix$  in (3.117a), and using the relations  $\sinh(ix) = i\sin(x)$  &  $\cosh(ix) = \cos(x)$ , yields a two variables of Euler's identity:

$$y^{ix} = \cos(x \ln y) + i \sin(x \ln y) \quad \text{for } y > 0 \text{ and } x \in \mathbb{R}. \quad (3.117b)$$

An alternative method to quickly obtain formula (3.117b) involves substituting  $x$  with  $x \ln y$  in (3.1) and applying  $e^{\ln f} = f$ . Furthermore, this formula can be expressed using the newly extended trigonometric functions:

$$y^{ix} = \cos(x, y) + i \sin(x, y) \quad \text{for } y > 0 \text{ and } x \in \mathbb{R}. \quad (3.117c)$$

**Example.** Show that

$$i^{i^i} = \cos\left(\frac{\pi e^{-\pi/2}}{2}\right) + i \sin\left(\frac{\pi e^{-\pi/2}}{2}\right)$$

**Solution**

Substituting  $y = i$  in (3.117a) gives

$$\begin{aligned} i^x &= \cosh\left(x \cdot \frac{i\pi}{2}\right) + \sinh\left(x \cdot \frac{i\pi}{2}\right) \\ &= \cos\left(\frac{x\pi}{2}\right) + i \sin\left(\frac{x\pi}{2}\right) \end{aligned} \quad (3.117d)$$

Substituting  $x$  by  $i^x$  in the above expression yields

$$i^{i^x} = \cos\left(\frac{\pi i^x}{2}\right) + i \sin\left(\frac{\pi i^x}{2}\right) \quad (3.117e)$$

Let  $x = i$ , and by applying  $i^i = e^{-\pi/2}$  [please refer to **Remark 3.4.2.**], we obtain

$$i^{i^i} = \cos\left(\frac{\pi e^{-\pi/2}}{2}\right) + i \sin\left(\frac{\pi e^{-\pi/2}}{2}\right).$$

**Corollaries of Euler's Identity in Two Variables**

We introduce some beautiful formulas that are derived from Euler's identity (3.117a).

**Corollary 29.** For  $x, y, a$  and  $b$  are real with  $a, b > 0$ ,

$$a^x + b^y = a^{x/2} b^{y/2} \cosh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right), \quad (3.117f)$$

$$a^x - b^y = a^{x/2} b^{y/2} \sinh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right), \quad (3.117g)$$

and

$$\tanh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right) = \frac{a^x - b^y}{a^x + b^y}. \quad (3.117h)$$

**Proof**

1. Substituting  $y$  by  $a$  in (3.117a) gives

$$a^x = \cosh(x \ln a) + \sinh(x \ln a)$$

By replacing  $y$  with  $b$  and  $x$  with  $y$  in (3.117a), we obtain

$$b^y = \cosh(y \ln b) + \sinh(y \ln b)$$

Adding two expressions together gives

$$\begin{aligned} a^x + b^y &= \cosh(x \ln a) + \cosh(y \ln b) + \sinh(x \ln a) + \sinh(y \ln b) \\ &= 2 \cosh\left(\frac{x \ln a + y \ln b}{2}\right) \cosh\left(\frac{x \ln a - y \ln b}{2}\right) + 2 \sinh\left(\frac{x \ln a + y \ln b}{2}\right) \cosh\left(\frac{x \ln a - y \ln b}{2}\right) \\ &= 2 \cosh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right) \left[ \cosh\left(\frac{x}{2} \ln a + \frac{y}{2} \ln b\right) + \sinh\left(\frac{x}{2} \ln a + \frac{y}{2} \ln b\right) \right] \\ &= 2 \cosh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right) \exp\left(\frac{x}{2} \ln a + \frac{y}{2} \ln b\right) \\ &= 2 \cosh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right) \exp(\ln(a^{x/2} b^{y/2})) \\ &= 2 \cosh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right) \left(\frac{a^{x/2} b^{y/2}}{2}\right) \\ &= a^{x/2} b^{y/2} \cosh\left(\frac{x}{2} \ln a - \frac{y}{2} \ln b\right). \end{aligned}$$

2. We follow a similar procedure to obtain formula (3.117g).

3. Dividing expression (3.117f) by expression (3.117g) gives formula (3.117h).

### Note

(i). Base on Fermat's Last Theorem, we state that the equation

$$a^n + b^n = a^{n/2} b^{n/2} \cosh\left(\frac{n}{2} \ln \frac{a}{b}\right) \quad (3.117i)$$

has no solutions in positive integers  $a, b, c$  when  $n > 2$ , where

$$c = a^{n/2} b^{n/2} \cosh\left(\frac{n}{2} \ln \frac{a}{b}\right).$$

(ii). Is it possible to use formula (3.117i) as a short way to prove Fermat's Last Theorem?

**Corollary 30.** For  $x$ ,  $a$  and  $b$  are real numbers with  $a, b > 0$ ,

$$\tanh^{-1}\left(\frac{a^x - b^x}{a^x + b^x}\right) = x \tanh^{-1}\left(\frac{a - b}{a + b}\right) \quad \text{for } a \neq -b. \quad (3.117k)$$

### Proof

For  $a, b > 0$  and by substituting  $y = x$  in (3.117h), we have

$$\frac{a^x - b^x}{a^x + b^x} = \tanh\left(\frac{x}{2} \ln a - \frac{x}{2} \ln b\right) = \tanh\left(\frac{x}{2} \ln\left(\frac{a}{b}\right)\right) \quad (i)$$

Recall that the inverse hyperbolic tangent is defined as

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad \text{for } -1 < x < 1.$$

Let  $u = (1+x)/(1-x)$ , solving for  $x$  in terms of  $u$  yields

$$x = \frac{u-1}{u+1}.$$

from which it follows that

$$\tanh^{-1}\left(\frac{u-1}{u+1}\right) = \frac{1}{2} \ln u \quad \text{for } u > 0.$$

Replacing  $u$  with  $a/b$  gives

$$\tanh^{-1}\left(\frac{a-b}{a+b}\right) = \frac{1}{2} \ln\left(\frac{a}{b}\right) \quad \text{for } a \neq -b. \quad (ii)$$

Combining result (i) with result (ii) gives

$$\frac{a^x - b^x}{a^x + b^x} = \tanh\left(x \tanh^{-1}\left(\frac{a-b}{a+b}\right)\right) \text{ for } a \neq -b. \quad (\text{iii})$$

Taking artanh of both sides of (iii) gives a beautiful identity,

$$\tanh^{-1}\left(\frac{a^x - b^x}{a^x + b^x}\right) = x \tanh^{-1}\left(\frac{a-b}{a+b}\right), \text{ for } a \neq -b.$$

### 3-15-5. Imaginary Angle (ix) Formulas

The relationships between extended trigonometric functions and extended hyperbolic functions through imaginary angle  $ix$  are present as follows:

$$1. \sin(ix, y) = \frac{i}{2}(y^x - y^{-x}) = i \sinh(x, y) \quad (3.119)$$

$$2. \cos(ix, y) = \frac{1}{2}(y^x + y^{-x}) = \cosh(x, y) \quad (3.120)$$

$$3. \tan(ix, y) = i \left( \frac{y^x - y^{-x}}{y^x + y^{-x}} \right) = i \tanh\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) = i \tanh(x, y) \quad (3.121)$$

$$4. \cot(ix, y) = -i \left( \frac{y^x + y^{-x}}{y^x - y^{-x}} \right) = -i \coth\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) = -i \coth(x, y) \quad (3.122)$$

$$5. \sinh(ix, y) = i \sin(x, y) \quad (3.123)$$

$$6. \cosh(ix, y) = i \cos(x, y) \quad (3.124)$$

$$7. \tanh(ix, y) = i \tan(x, y) \quad (3.125)$$

$$8. \coth(ix, y) = -i \cot(x, y) \quad (3.126)$$

### Proof

It is straightforward to prove the above identities by applying the identities:  $\sin(ix) = i \sinh(x)$ ,  $\cos(ix) = \cosh(x)$ ,  $\tan(ix) = i \tanh(x)$  and  $\cot(ix) = -i \coth(x)$ .

### 3-15-6. Imaginary Angle ( $iy$ ) Formulas

For  $y > 0$ , the relationships between extended trigonometric functions and extended hyperbolic functions through imaginary angle  $iy$  are:

$$1. \sin(x, iy) = \frac{1}{2i} (y^{ix} e^{-\pi x/2} - y^{-ix} e^{\pi x/2}) = \cosh\left(\frac{\pi x}{2}\right) \sin(x, y) + i \sinh\left(\frac{\pi x}{2}\right) \cos(x, y) \quad (3.127)$$

$$= \cosh\left(\frac{\pi x}{2}\right) \sin\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) + i \sinh\left(\frac{\pi x}{2}\right) \cos\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) \quad (3.128)$$

$$2. \cos(x, iy) = \frac{1}{2} (y^{ix} e^{-\pi x/2} + y^{-ix} e^{\pi x/2}) = \cosh\left(\frac{\pi x}{2}\right) \cos(x, y) - i \sinh\left(\frac{\pi x}{2}\right) \sin(x, y) \quad (3.129)$$

$$= \cosh\left(\frac{\pi x}{2}\right) \cos\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) - i \sinh\left(\frac{\pi x}{2}\right) \sin\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) \quad (3.130)$$

$$3. \tan(x, iy) = \frac{1}{i} \left( \frac{y^{ix} e^{-\pi x/2} - y^{-ix} e^{\pi x/2}}{y^{ix} e^{-\pi x/2} + y^{-ix} e^{\pi x/2}} \right) = \frac{\tan(x, y) + i \tanh\left(\frac{\pi x}{2}\right)}{1 - i \tan(x, y) \tanh\left(\frac{\pi x}{2}\right)} \quad (3.131)$$

$$= \frac{\tan\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) + i \tanh\left(\frac{\pi x}{2}\right)}{1 - i \tan\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) \tanh\left(\frac{\pi x}{2}\right)} \quad (3.132)$$

$$4. \cot(x, iy) = i \left( \frac{y^{ix} e^{-\pi x/2} + y^{-ix} e^{\pi x/2}}{y^{ix} e^{-\pi x/2} - y^{-ix} e^{\pi x/2}} \right) = \frac{\cot(x, y) \coth\left(\frac{\pi x}{2}\right) - i}{\coth\left(\frac{\pi x}{2}\right) + i \cot(x, y)} \quad (3.133)$$

$$= \frac{\cot\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right) \coth\left(\frac{\pi x}{2}\right) - i}{\coth\left(\frac{\pi x}{2}\right) + i \cot\left(x \sinh^{-1} \frac{y^2 - 1}{2y}\right)} \quad (3.134)$$

$$5. \sinh(x, iy) = \cos\left(\frac{\pi x}{2}\right) \sinh(x, y) + i \sin\left(\frac{\pi x}{2}\right) \cosh(x, y) \quad (3.135)$$

$$6. \cosh(x, iy) = \cos\left(\frac{\pi x}{2}\right) \cosh(x, y) + i \sin\left(\frac{\pi x}{2}\right) \sinh(x, y) \quad (3.136)$$

$$7. \tanh(x, iy) = \frac{i \tan\left(\frac{\pi x}{2}\right) + \tanh(x, y)}{1 + i \tan\left(\frac{\pi x}{2}\right) \tanh(x, y)} \quad (3.137)$$

$$8. \coth(x, iy) = \frac{1 - i \cot\left(\frac{\pi x}{2}\right) \coth(x, y)}{\coth(x, y) - i \cot\left(\frac{\pi x}{2}\right)} \quad (3.138)$$

**Proof**

**1. Formulas (3.127)/(3.128):** By replacing  $y$  with  $iy$  into (3.35), we have:

$$\begin{aligned} \sin(x, iy) &= \frac{1}{2i} \left( (iy)^{ix} - (iy)^{-ix} \right) \\ &= \frac{1}{2i} \left( y^{ix} i^{ix} - y^{-ix} i^{-ix} \right) \\ &= \frac{1}{2i} \left( y^{ix} e^{-\pi x/2} - y^{-ix} e^{\pi x/2} \right) \quad (\text{because } i^i = e^{-\pi/2} ) \end{aligned}$$

On other hand, expressing  $\sin(x, iy)$  in terms of  $x \ln y$  gives

$$\begin{aligned} \sin(x, iy) &= \sin(x \ln(iy)) \text{ for } y > 0. \\ &= \sin\left(x \left( \frac{i\pi}{2} + \ln y \right)\right) \quad (\text{because } \ln(i) = \frac{i\pi}{2} ) \\ &= \sin\left(\frac{i\pi x}{2} + x \ln y\right) \\ &= \sin\left(\frac{i\pi x}{2}\right) \cos(x \ln y) + \sin(x \ln y) \cos\left(\frac{i\pi x}{2}\right) \\ &\quad (\text{Applying } \sin(a+b) = \sin a \cos b + \cos a \sin b, \text{ and} \\ &\quad \sin(ix) = i \sinh(x); \cos(ix) = \cosh(x)) \\ &= i \sinh\left(\frac{\pi x}{2}\right) \cos(x, y) + \cosh\left(\frac{\pi x}{2}\right) \sin(x, y), \end{aligned}$$

which leads to formula (3.127). By applying (3.117), the results are derived in (3.128), thus concluding the proof.

**Example.** Find  $\sin(1/2, 3i)$

**Solution.** By substituting  $x = 1/2$  and  $y = 3$  in formula (3.127), we obtain

$$\sin\left(\frac{1}{2}, 3i\right) = \frac{1}{2i} \left( (3i)^{i/2} - (3i)^{-i/2} \right) \approx 0.69157 + 0.74087i,$$

and

$$\sin\left(\frac{1}{2}, 3i\right) = \cosh\left(\frac{\pi}{4}\right) \sin\left(\frac{1}{2} \ln 3\right) + i \sinh\left(\frac{\pi}{4}\right) \cos\left(\frac{1}{2} \ln 3\right) \approx 0.69157 + 0.74087i.$$

Both results indicate that

$$\frac{1}{2i} \left( (3i)^{i/2} - (3i)^{-i/2} \right) = \cosh\left(\frac{\pi}{4}\right) \sin\left(\frac{1}{2} \ln 3\right) + i \sinh\left(\frac{\pi}{4}\right) \cos\left(\frac{1}{2} \ln 3\right),$$

where the real and imaginary parts can be computed explicitly.

In the similar manner used in #1, we derive other formulas:

- **2. Formulas (3.129)/(3.130):** Applying  $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$  gives the results (3.129) and (3.130).
- **3. Formulas (3.131)/(3.132):** Applying  $\tan(a+b) = [\tan(a)+\tan(b)] / [1+\tan(a)\tan(b)]$  and  $\tan(ix) = i \tanh(x)$  give results (3.131) and (3.132).
- **4. Formulas (3.133)/(3.134):** By applying  $\cot(a+b) = [\cot(a)\cot(b)-1] / [\cot(a)+\cot(b)]$  and  $\cot(ix) = -i \coth(x)$ , which lead results (3.133) and (3.134).

**5. Formula (3.135):** By replacing  $y$  with  $iy$  in (3.86), we have:

$$\sinh(x, iy) = \sinh(x \ln(iy))$$

$$\begin{aligned}
&= \sinh\left(\frac{i\pi x}{2} + x \ln y\right) \\
&= \sinh\left(\frac{i\pi x}{2}\right) \cosh(x \ln y) + \sinh(x \ln y) \cosh\left(\frac{i\pi x}{2}\right) \\
&= \cos\left(\frac{\pi x}{2}\right) \sinh(x, y) + i \sin\left(\frac{\pi x}{2}\right) \cosh(x, y)
\end{aligned}$$

**6. Formula (3.136):** By replacing  $y$  with  $iy$  in (3.82), we have:

$$\begin{aligned}
\cosh(x, iy) &= \cosh(x \ln(iy)) \\
&= \cosh\left(\frac{i\pi x}{2} + x \ln y\right) \\
&= \cosh\left(\frac{i\pi x}{2}\right) \cosh(x \ln y) + \sinh(x \ln y) \sinh\left(\frac{i\pi x}{2}\right) \\
&= \cos\left(\frac{\pi x}{2}\right) \cosh(x, y) + i \sin\left(\frac{\pi x}{2}\right) \sinh(x, y)
\end{aligned}$$

**7. Formula (3.137):** By replacing  $y$  with  $iy$  into (3.90), we have:

$$\begin{aligned}
\tanh(x, iy) &= \tanh(x \ln(iy)) \\
&= \tanh\left(\frac{i\pi x}{2} + x \ln y\right) \\
&= \frac{\tanh\left(\frac{i\pi x}{2}\right) + \tanh(x \ln y)}{1 + \tanh\left(\frac{i\pi x}{2}\right) \tanh(x \ln y)} \\
&= \frac{i \tan\left(\frac{\pi x}{2}\right) + \tanh(x, y)}{1 + i \tan\left(\frac{\pi x}{2}\right) \tanh(x, y)}
\end{aligned}$$

**8. Formula (3.138):** By replacing  $y$  with  $iy$  in (3.94), we have:

$$\coth(x, iy) = \coth(x \ln(iy))$$

$$\begin{aligned}
&= \coth\left(\frac{i\pi x}{2} + x \ln y\right) \\
&= \frac{\coth\left(\frac{i\pi x}{2}\right) \coth(x \ln y) + 1}{\coth\left(\frac{i\pi x}{2}\right) + \coth(x \ln y)} \\
&= \frac{1 - i \cot\left(\frac{\pi x}{2}\right) \coth(x, y)}{\coth(x, y) - i \cot\left(\frac{\pi x}{2}\right)}
\end{aligned}$$

**Remark 3.4.10.**

**a.** Find  $i^i = e^{-\pi/2}$

By substituting  $x = \pi/2$  in the Euler's identity,  $e^{ix} = \cos x + i \sin x$ , we obtain

$$e^{i\pi/2} = i$$

$$\Leftrightarrow (e^{i\pi/2 + i2\pi k}) = i, \quad k \in \mathbb{Z}.$$

Raising both sides to the power  $i$  gives

$$i^i = (e^{i\pi/2 + i2\pi k})^i = e^{-\pi/2 - 2\pi k}, \quad k \in \mathbb{Z}.$$

For the principal value, take  $k = 0$ ,

$$i^i = e^{-\pi/2}.$$

**b.** The right-hand expressions of the formulas from (3.131) to (3.134), (3.137) and (3.138) can be expressed in terms of the real part and the imaginary part, respectively.

### 3-15-6. Imaginary Angles (ix) and (iy) Formulas

The relationships between extended trigonometric functions and extended hyperbolic functions through imaginary angle  $ix$  and  $iy$  are:

$$1. \sin(ix, iy) = \frac{i}{2} [(iy)^x - (iy)^{-x}] = -\sin\left(\frac{\pi x}{2}\right) \cosh(x, y) + i \cos\left(\frac{\pi x}{2}\right) \sinh(x, y) \quad (3.139)$$

$$2. \cos(ix, iy) = \frac{1}{2} ((iy)^x + (iy)^{-x}) = \cos\left(\frac{\pi x}{2}\right) \cosh(x, y) + i \sin\left(\frac{\pi x}{2}\right) \sinh(x, y) \quad (3.140)$$

$$3. \tan(ix, iy) = i \frac{(iy)^x - (iy)^{-x}}{(iy)^x + (iy)^{-x}} = \frac{i \tanh(x, y) - \tan\left(\frac{\pi x}{2}\right)}{1 + i \tanh(x, y) \tan\left(\frac{\pi x}{2}\right)} \quad (3.141)$$

$$4. \cot(ix, iy) = \frac{1}{i} \frac{(iy)^x + (iy)^{-x}}{(iy)^x - (iy)^{-x}} = \frac{i + \coth(x, y) \cot\left(\frac{\pi x}{2}\right)}{i \cot\left(\frac{\pi x}{2}\right) - \coth(x, y)} \quad (3.142)$$

$$5. \sinh(ix, iy) = -\sinh\left(\frac{\pi x}{2}\right) \cos(x, y) + i \cosh\left(\frac{\pi x}{2}\right) \sin(x, y) \quad (3.143)$$

$$6. \cosh(ix, iy) = \cosh\left(\frac{\pi x}{2}\right) \cos(x, y) - i \sinh\left(\frac{\pi x}{2}\right) \sin(x, y) \quad (3.144)$$

$$7. \tanh(ix, iy) = \frac{i \tan(x, y) - \tanh\left(\frac{\pi x}{2}\right)}{1 - i \tan(x, y) \tanh\left(\frac{\pi x}{2}\right)} \quad (3.145)$$

$$8. \coth(ix, iy) = \frac{i \cot(x, y) \coth\left(\frac{\pi x}{2}\right) + 1}{i \cot(x, y) + \coth\left(\frac{\pi x}{2}\right)} \quad (3.146)$$

### Proof

**Formula (3.139):** Substituting  $x$  and  $y$  with  $ix$  and  $iy$  in (3.35) gives

$$\begin{aligned} \sin(ix, iy) &= \frac{1}{2i} [(ie)^{-x} - (iy)^x] \\ &= -\frac{1}{2i} [(ie)^x - (iy)^{-x}] \\ &= \frac{i}{2} [(ie)^x - (iy)^{-x}] \end{aligned}$$

On the other hand, we have  $\sin(x,y) = \sin(x \ln y)$ . Replacing  $x$  and  $y$  with  $ix$  and  $iy$  into this expression, we have:

$$\begin{aligned}
 \sin(ix, iy) &= \sin[ix \ln(iy)] \\
 &= \sin[ix(\ln(i) + y)] \\
 &= \sin\left[ix\left(\frac{i\pi}{2} + \ln y\right)\right] \\
 &= \sin\left(ix \ln y - \frac{\pi x}{2}\right) \\
 &= \sin(ix \ln y) \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right) \cos(ix \ln y) \\
 &= i \sinh(x \ln y) \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right) \cosh(x \ln y) \\
 &= -\sin\left(\frac{\pi x}{2}\right) \cosh(x, y) + i \cos\left(\frac{\pi x}{2}\right) \sinh(x, y)
 \end{aligned}$$

**Formula (3.140):** Applying the substitutions  $x = ix$  and  $y = iy$  in (3.16) yields

$$\cos(ix, iy) = \frac{1}{2}[(ie)^{-x} + (iy)^x].$$

On the other hand, we have  $\cos(x,y) = \cos(x \ln y)$ . Replacing  $x$  and  $y$  with  $ix$  and  $iy$  into this expression, we have:

$$\begin{aligned}
 \cos(ix, iy) &= \cos[ix \ln(iy)] \\
 &= \cos[ix(\ln(i) + y)] \\
 &= \cos\left[ix\left(\frac{i\pi}{2} + \ln y\right)\right] \\
 &= \cos\left(ix \ln y - \frac{\pi x}{2}\right) \\
 &= \cos(ix \ln y) \cos\left(\frac{\pi x}{2}\right) + \sin\left(\frac{\pi x}{2}\right) \sin(ix \ln y)
 \end{aligned}$$

$$\begin{aligned}
&= \cosh(x \ln y) \cos\left(\frac{\pi x}{2}\right) + i \sin\left(\frac{\pi x}{2}\right) \sinh(x \ln y) \\
&= \cos\left(\frac{\pi x}{2}\right) \cosh(x, y) + i \sin\left(\frac{\pi x}{2}\right) \sinh(x, y)
\end{aligned}$$

**Formula (3.141):** Substituting  $x$  and  $y$  with  $ix$  and  $iy$  in (3.60) gives

$$\tan(ix, iy) = \frac{1}{i} \frac{(iy)^{-x} - (iy)^x}{(iy)^{-x} + (iy)^x}.$$

By simplifying further, we deduce that

$$\tan(ix, iy) = i \frac{(iy)^x - (iy)^{-x}}{(iy)^x + (iy)^{-x}}.$$

On the other hand, we also have:

$$\begin{aligned}
\tan(ix, iy) &= \tan(ix \ln(iy)) \\
&= \tan\left(ix \left[\frac{i\pi}{2} + \ln y\right]\right) \\
&= \tan\left(ix \ln y - \frac{\pi x}{2}\right) \\
&= \frac{\tan(ix \ln y) - \tan\left(\frac{\pi x}{2}\right)}{1 + \tan(ix \ln y) \tan\left(\frac{\pi x}{2}\right)} \\
&= \frac{i \tanh(x, y) - \tan\left(\frac{\pi x}{2}\right)}{1 + i \tanh(x, y) \tan\left(\frac{\pi x}{2}\right)},
\end{aligned}$$

which we complete the proof. Furthermore, by multiplying both the numerator and denominator of the right-hand expression of (3.133) by  $(1 + i \tanh(x, y) \tan(\pi x/2))$ , we can express the formula in terms of its real and imaginary parts as follows:

$$\begin{aligned}\tan(ix, iy) &= \frac{\left(i \tanh(x \ln y) - \tan\left(\frac{\pi x}{2}\right)\right) \left(1 - i \tanh(x \ln y) \tan\left(\frac{\pi x}{2}\right)\right)}{\tanh^2(x \ln y) \tan^2\left(\frac{\pi x}{2}\right) + 1} \\ &= \frac{\tan\left(\frac{\pi x}{2}\right) (\tanh^2(x \ln y) - 1) + i \tanh(x \ln y) (\tan^2\left(\frac{\pi x}{2}\right) + 1)}{\tanh^2(x \ln y) \tan^2\left(\frac{\pi x}{2}\right) + 1}\end{aligned}\quad (3.141a)$$

$$= \frac{\tan\left(\frac{\pi x}{2}\right) (\tanh^2(x, y) - 1) + i \tanh(x, y) (\tan^2\left(\frac{\pi x}{2}\right) + 1)}{\tanh^2(x, y) \tan^2\left(\frac{\pi x}{2}\right) + 1}\quad (3.141b)$$

**Exercise.** Applying formula (3.141a) shows that

$$\ln[\tan(ix \ln(iy))] = \frac{1}{2} \ln \left[ \frac{\tanh^2(x \ln y) + \tan^2\left(\frac{\pi x}{2}\right)}{\tanh^2(x \ln y) \tan^2\left(\frac{\pi x}{2}\right) + 1} \right] + i \left[ \pi + \tan^{-1} \left( \frac{\tanh(x \ln y) (\tan^2\left(\frac{\pi x}{2}\right) + 1)}{\tan\left(\frac{\pi x}{2}\right) (\tanh^2(x \ln y) - 1)} \right) \right]$$

for positive real  $x$  and  $y$ .

**Hint:** Applying formula (3.141a) together with the identity

$$\ln(a+ib) = \frac{1}{2} \ln(a^2+b^2) + i \tan^{-1}\left(\frac{b}{a}\right),$$

where  $a$  and  $b$  are real. Depending on the signs of  $a$  and  $b$ , the argument may need to be corrected by adding or subtract  $\pi$ . For example, since  $\tanh^2(x \ln y) - 1 < 0$  for all  $x, y > 0$ , we add  $\pi$ ; if the given exercise lets  $x < 0$ , we instead add  $-\pi$ . For a systematic approach to determining the correct argument, see the definition  $\text{Arg}(a+ib) = \text{atan2}(b, a)$ .

**Remark 3.4.11.** The mathematical meaning of the results from the above

exercise, as well as from other newly extended functions such as from (3.139) to (3.146), is that the formulas provide a powerful tool from real analysis for separating a complex function into its real and imaginary parts, thereby revealing the underlying structure of the function.

**Formula (3.142):** In the similar manner of (3.141) and by applying the identities  $\cot(ix) = -i\coth(x)$  and  $\cot(a+b) = (\cot a \cot b - 1)/(\cot a + \cot b)$ , we derive formula (3.142). Moreover, the expression on the right side of (3.142) can be written in terms of its real and imaginary parts by multiplying both the numerator and denominator by  $(i\cot(\pi x/2) + \coth(x \ln y))$  as demonstrated below:

$$\begin{aligned} \cot(ix, iy) &= \frac{i + \coth(x \ln y) \cot\left(\frac{\pi x}{2}\right)}{i \cot\left(\frac{\pi x}{2}\right) - \coth(x \ln y)} \cdot \frac{i \cot\left(\frac{\pi x}{2}\right) + \coth(x \ln y)}{i \cot\left(\frac{\pi x}{2}\right) + \coth(x \ln y)} \\ &= \frac{\cot\left(\frac{\pi x}{2}\right)(1 - \coth^2(x \ln y)) - i \coth(x \ln y) \left(1 + \cot^2\left(\frac{\pi x}{2}\right)\right)}{\cot^2(x \ln y) + \coth^2\left(\frac{\pi x}{2}\right)} \end{aligned} \quad (3.142a)$$

$$= \frac{\cot\left(\frac{\pi x}{2}\right)(1 - \coth^2(x, y)) - i \coth(x, y) \left(1 + \cot^2\left(\frac{\pi x}{2}\right)\right)}{\cot^2(x, y) + \coth^2\left(\frac{\pi x}{2}\right)} \quad (3.142b)$$

**Formula (3.143):** Applying the substitutions  $x = ix$  and  $y = iy$  in (3.87) gives

$$\begin{aligned} \sinh(ix, iy) &= \sinh[ix \ln(iy)] \\ &= \sinh\left[ix \ln(y) - \frac{\pi x}{2}\right] \\ &= \sinh(ix \ln y) \cosh\left(\frac{\pi x}{2}\right) - \sinh\left(\frac{\pi x}{2}\right) \cosh(ix \ln y) \\ &= i \sin(x \ln y) \cosh\left(\frac{\pi x}{2}\right) - \sinh\left(\frac{\pi x}{2}\right) \cos(x \ln y) \\ &= -\sinh\left(\frac{\pi x}{2}\right) \cos(x, y) + i \cosh\left(\frac{\pi x}{2}\right) \sin(x, y) \end{aligned}$$

**Formula (3.144):** Making the substitutions  $x = ix$  and  $y = iy$  in (3.83) gives

$$\begin{aligned}
 \cosh(ix, iy) &= \cosh[ix \ln(iy)] \\
 &= \cosh\left[ix \ln(y) - \frac{\pi x}{2}\right] \\
 &= \cosh(ix \ln y) \cosh\left(\frac{\pi x}{2}\right) - \sinh\left(\frac{\pi x}{2}\right) \sinh(ix \ln y) \\
 &= \cos(x \ln y) \cosh\left(\frac{\pi x}{2}\right) - i \sinh\left(\frac{\pi x}{2}\right) \sin(x \ln y) \\
 &= \cosh\left(\frac{\pi x}{2}\right) \cos(x, y) - i \sinh\left(\frac{\pi x}{2}\right) \sin(x, y)
 \end{aligned}$$

**Formula (3.145):** Applying the substitutions  $x = ix$  and  $y = iy$  in (3.91) gives

$$\begin{aligned}
 \tanh(ix, iy) &= \tanh[ix \ln(iy)] \\
 &= \tanh\left[ix \ln(y) - \frac{\pi x}{2}\right] \\
 &= \frac{\tanh(ix \ln y) - \tanh\left(\frac{\pi x}{2}\right)}{1 - \tanh(ix \ln y) \tanh\left(\frac{\pi x}{2}\right)} \\
 &= \frac{i \tan(x \ln y) - \tanh\left(\frac{\pi x}{2}\right)}{1 - i \tan(x \ln y) \tanh\left(\frac{\pi x}{2}\right)} \\
 &= \frac{i \tan(x, y) - \tanh\left(\frac{\pi x}{2}\right)}{1 - i \tan(x, y) \tanh\left(\frac{\pi x}{2}\right)}
 \end{aligned}$$

**Formula (3.146):** Substituting  $x$  and  $y$  with  $ix$  and  $iy$  in (3.95) gives

$$\coth(ix, iy) = \coth[ix \ln(iy)]$$

$$\begin{aligned}
&= \coth\left[i x \ln(y) - \frac{\pi x}{2}\right] \\
&= \frac{\coth(i x \ln y) \coth\left(\frac{\pi x}{2}\right) - 1}{\coth(i x \ln y) - \coth\left(\frac{\pi x}{2}\right)} \\
&= \frac{-i \cot(x \ln y) \coth\left(\frac{\pi x}{2}\right) - 1}{-i \cot(x \ln y) - \coth\left(\frac{\pi x}{2}\right)} \\
&= \frac{i \cot(x, y) \coth\left(\frac{\pi x}{2}\right) + 1}{i \cot(x, y) + \coth\left(\frac{\pi x}{2}\right)}
\end{aligned}$$

**Example.** Evaluate  $\cot(i/3, 4i)$

**Solution.** By substituting  $x = 1/3$  and  $y = 4$  into (3.142) and (3.142a), we obtain:

$$\cot\left(\frac{i}{3}, i4\right) = \frac{1}{i} \frac{(i4)^{1/3} + (i4)^{-1/3}}{(i4)^{1/3} - (i4)^{-1/3}} \approx -0.903666 - i1.107633$$

and

$$\cot\left(\frac{i}{3}, i4\right) = \frac{\cot\left(\frac{\pi}{6}\right)\left(1 - \coth^2\left(\frac{2}{3} \ln 2\right)\right) - i \coth\left(\frac{2}{3} \ln 2\right)\left(1 + \cot^2\left(\frac{\pi}{6}\right)\right)}{\cot^2\left(\frac{2}{3} \ln 2\right) + \coth^2\left(\frac{\pi}{6}\right)} \approx -0.903666 - i1.107633$$

It follows that

$$\frac{1}{i} \frac{(i4)^{1/3} + (i4)^{-1/3}}{(i4)^{1/3} - (i4)^{-1/3}} = \frac{\cot\left(\frac{\pi}{6}\right)\left(1 - \coth^2\left(\frac{2}{3} \ln 2\right)\right) - i \coth\left(\frac{2}{3} \ln 2\right)\left(1 + \cot^2\left(\frac{\pi}{6}\right)\right)}{\cot^2\left(\frac{\pi}{6}\right) + \coth^2\left(\frac{2}{3} \ln 2\right)},$$

$$= -\frac{2 \cdot 3^{1/2} 2^{1/3}}{4^{4/3} - 2^{4/3} + 1} - i \frac{(4^{4/3} - 1)}{4^{4/3} - 2^{4/3} + 1},$$

where the real and imaginary parts can be evaluated explicitly in closed form.

## Section 3-16. Fundamental Inverse Functions

In this section, we delve into discovering the inverse functions of the extended trigonometric and hyperbolic functions. The notations for these inverse functions are defined in Table 2 below.

Table 2–Notations for Inverse Extended Trig & Hyperbolic Functions			
Inverse Extended Trigonometric Functions		Inverse Extended Hyperbolic Functions	
Function	Inverse Function	Function	Inverse Function
1. $\cos(x, y)$	$\arccos(x, y)$ or $\cos^{-1}(x, y)$	7. $\cosh(x, y)$	$\operatorname{arcosh}(x, y)$ or $\cosh^{-1}(x, y)$
2. $\sin(x, y)$	$\arcsin(x, y)$ or $\sin^{-1}(x, y)$	8. $\sinh(x, y)$	$\operatorname{arsinh}(x, y)$ or $\sinh^{-1}(x, y)$
3. $\tan(x, y)$	$\arctan(x, y)$ or $\tan^{-1}(x, y)$	9. $\tanh(x, y)$	$\operatorname{artanh}(x, y)$ or $\tanh^{-1}(x, y)$
4. $\cot(x, y)$	$\operatorname{arccot}(x, y)$ or $\cot^{-1}(x, y)$	10. $\coth(x, y)$	$\operatorname{arcoth}(x, y)$ or $\coth^{-1}(x, y)$
5. $\sec(x, y)$	$\operatorname{arcsec}(x, y)$ or $\sec^{-1}(x, y)$	11. $\operatorname{sech}(x, y)$	$\operatorname{arsech}(x, y)$ or $\operatorname{sech}^{-1}(x, y)$
6. $\csc(x, y)$	$\operatorname{arccsc}(x, y)$ or $\csc^{-1}(x, y)$	12. $\operatorname{csch}(x, y)$	$\operatorname{arcsch}(x, y)$ or $\operatorname{csch}^{-1}(x, y)$

In addition, the notation  $x, y$  is identical  $x \ln(y)$ , namely

$$x, y \equiv x \ln(y). \quad (*\text{notation}*)$$

We use this notation in finding the inverse functions.

### 3-16-1. Inverse Extended Hyperbolic Functions

#### A. Inverse Extended Hyperbolic Cosine Function – $\operatorname{arcosh}(x, y)$

**Definition.** The function  $\operatorname{arcosh}(x, y)$  (also written  $\cosh^{-1}(x, y)$ ) is defined as the inverse of the extended hyperbolic cosine function  $\cosh(x, y)$ . We write

$$f(x, y) \equiv \operatorname{arcosh}(x, y) = \cosh^{-1}(x, y). \quad (3.147)$$

The domain and range are:

- **Domain:**  $x, y \in \mathbb{R}$ ,  $x \ln(y) \in [1, \infty)$  and  $y > 0$ ,
- **Range:**  $\operatorname{arcosh}(x, y) \in \mathbb{R}$  or  $-\infty < \operatorname{arcosh}(x, y) < \infty$ .

**Note:** Since  $\cosh(x, y) = \cosh(x \ln y)$ , its inverse inherits the standard domain and range associated with the inverse hyperbolic cosine function.

**Theorem 48.** The inverse of the extended hyperbolic cosine function is

$$\operatorname{arcosh}(x, y) = \ln \left( x \ln(y) + \sqrt{x^2 \ln^2(y) - 1} \right) = \cosh^{-1}(x \ln y) \equiv \cosh^{-1}(x, y) \quad (3.148)$$

for  $x, y \in \mathbb{R}$ ,  $x \ln(y) \in [1, \infty)$  and  $y > 0$ .

### Proof

Definition (3.147) gives

$$f = \operatorname{arcosh}(x, y) = \cosh^{-1}(x, y),$$

from which it follows that

$$(x, y) \equiv x \ln(y) = \cosh(f) \Rightarrow x \ln(y) = \frac{1}{2}(e^f + e^{-f})$$

Rearranging terms yields a quadratic equation in  $e^f$ :

$$e^{2f} - 2x \ln y e^f + 1 = 0.$$

By solving the equation for  $e^f$ , we obtain

$$e^f = x \ln y \pm \sqrt{(x \ln y)^2 - 1} \quad \text{for } x \ln(y) \geq 1 \text{ and } y > 0.$$

By taking logarithm of both sides and choosing the positive branch, we obtain

$$f = \ln \left( x \ln(y) + \sqrt{x^2 \ln^2(y) - 1} \right).$$

By substituting  $f$  with  $\operatorname{arcosh}(x, y)$  gives

$$\operatorname{arcosh}(x, y) = \ln \left( x \ln(y) + \sqrt{x^2 \ln^2(y) - 1} \right) = \cosh^{-1}(x, y) \quad \text{for } x, y \in [1, \infty) \text{ and } y > 0.$$

**Note:** When  $y$  equals  $e$ , this reduces to usual inverse hyperbolic cosine, namely

$$\cosh^{-1}(x, e) = \cosh^{-1} x = \ln [x + \sqrt{x^2 - 1}] \quad \text{for } x \geq 1.$$

## B. Inverse Extended Hyperbolic Sine Function – $\operatorname{arsinh}(x, y)$

**Definition.** The function  $\operatorname{arsinh}(x, y)$  (also written  $\sinh^{-1}(x, y)$ ) is defined as the inverse of the extended hyperbolic sine function  $\sinh(x, y)$ . We write

$$f(x, y) \equiv \operatorname{arsinh}(x, y) = \sinh^{-1}(x, y) \quad \text{for all } x, y \in \mathbb{R} \text{ and } y > 0. \quad (3.149)$$

The domain and range are:

- Domain:  $x, y \in \mathbb{R}$  and  $y > 0$ ,
- Range:  $\operatorname{arsinh}(x, y) \in \mathbb{R}$  or  $-\infty < \operatorname{arsinh}(x, y) < \infty$ .

**Theorem 49.** The inverse of the extended hyperbolic sine function satisfies the relation

$$\operatorname{arsinh}(x, y) = \ln \left( x \ln(y) + \sqrt{x^2 \ln^2(y) + 1} \right) = \sinh^{-1}(x \ln y) = \sinh^{-1}(x, y), \quad (3.150)$$

for  $x, y \in \mathbb{R}$  and  $y > 0$ .

### Proof

Definition (3.149) gives

$$f = \operatorname{arsinh}(x, y) = \sinh^{-1}(x, y),$$

from which it follows that

$$(x, y) \equiv x \ln(y) = \sinh(f) \Rightarrow x \ln(y) = \frac{1}{2}(e^f - e^{-f})$$

Rearranging the terms yields a quadratic equation in  $e^f$ :

$$e^{2f} - 2x \ln y e^f - 1 = 0$$

Solving for  $e^f$  from the quadratic equation gives

$$e^f = x \ln y \pm \sqrt{(x \ln y)^2 + 1} \quad \text{for } x, y \in \mathbb{R} \text{ and } y > 0.$$

By taking logarithm of both sides and choosing the positive branch, we obtain

$$f = \ln(x \ln(y) + \sqrt{x^2 \ln^2(y) + 1}).$$

Substituting  $f$  with  $\operatorname{arsinh}(x, y)$  gives

$$\operatorname{arsinh}(x, y) = \ln(x \ln(y) + \sqrt{x^2 \ln^2(y) + 1}) = \sinh^{-1}(x, y) \quad \text{for } x, y \in \mathbb{R} \text{ and } y > 0.$$

**Note:** When  $y$  equals  $e$ , it gives

$$\operatorname{arsinh}(x, e) = \operatorname{arcsinh}(x) = \sinh^{-1}(x) = \ln[x + \sqrt{x^2 + 1}] \quad \text{for } x \in \mathbb{R}.$$

### C. Inverse Extended Hyperbolic Tangent Function – $\operatorname{artanh}(x, y)$

**Definition.** The function  $\operatorname{artanh}(x, y)$  (also written  $\tanh^{-1}(x, y)$ ) is defined as the inverse of the extended hyperbolic  $\tanh$  function. We write

$$f(x, y) \equiv \operatorname{artanh}(x, y) = \tanh^{-1}(x, y) \quad \text{for all } x \ln(y) \in (-1, 1) \text{ and } y > 0. \quad (3.151)$$

The domain and range are:

- Domain:  $x \ln(y) \in (-1, 1)$  and  $y > 0$ ,
- Range:  $\operatorname{artanh}(x, y) \in \mathbb{R}$  or  $-\infty < \operatorname{artanh}(x, y) < \infty$ .

**Note:** Since  $\tanh(x, y) = \tanh(x \ln y)$ , its inverse inherits the standard domain and range associated with the inverse hyperbolic tangent function.

**Theorem 49.** The inverse of the extended hyperbolic tangent function satisfies the relation

$$\operatorname{artanh}(x,y) = \frac{1}{2} \ln \left( \frac{1+x \ln y}{1-x \ln y} \right) = \tanh^{-1}(x \ln y) \equiv \tanh^{-1}(x, y) \quad (3.151)$$

for  $-1 < x \ln(y) < 1$  and  $y > 0$ .

### Proof

Definition (3.149) gives

$$f = \operatorname{artanh}(x, y) = \tanh^{-1}(x \ln(y)) \Rightarrow x \ln(y) = \tanh(f),$$

from which it follows that

$$x \ln(y) = \frac{e^f - e^{-f}}{e^f + e^{-f}} = \frac{e^{2f} - 1}{e^{2f} + 1} \quad \Rightarrow \quad x \ln(y) e^{2f} + x \ln(y) = e^{2f} - 1.$$

Solving for  $e^{2f}$  from the equation gives

$$e^{2f} = \frac{1+x \ln y}{1-x \ln y}$$

Taking logarithm of both sides and solving for  $f$ :

$$f = \frac{1}{2} \ln \left( \frac{1+x \ln y}{1-x \ln y} \right), \quad x \ln y \in (-1, 1) \text{ and } y > 0.$$

By substituting  $f$  with  $\operatorname{artanh}(x,y)$ , we obtain

$$\operatorname{artanh}(x,y) = \frac{1}{2} \ln \left( \frac{1+x \ln y}{1-x \ln y} \right) = \tanh^{-1}(x, y), \quad x \ln y \in (-1, 1) \text{ and } y > 0.$$

**Remark 3.4.12.** Through the analysis of the functions  $\cosh(x,y)$ ,  $\sinh(x,y)$  and  $\tanh(x,y)$ , we see that their inverse functions retain the standard properties of the corresponding hyperbolic functions, differing only in their domains and ranges.

Similarly, any additional extended hyperbolic functions will exhibit the same fundamental characteristics as the classical hyperbolic functions.

**Next, we only outline a summary of the theorems and formulas for each inverse extended hyperbolic function, along with related trigonometric functions.**

#### D. Inverse Extended Hyperbolic Cotangent Function – $\operatorname{arcoth}(x,y)$

**Theorem 50.** The inverse of the extended hyperbolic cotangent function is

$$\operatorname{arcoth}(x,y) = \frac{1}{2} \ln \left( \frac{x \ln(y) + 1}{x \ln(y) - 1} \right) = \coth^{-1}(x \ln y) \quad (3.152)$$

for  $y > 0$  with  $-\infty < x \ln(y) < -1$  or  $1 < x \ln(y) < \infty$ .

#### Proof

The proof proceeds similarly to #4 (Theorem 49).

#### E. Inverse Extended Hyperbolic Secant Function – $\operatorname{arsech}(x,y)$

**Theorem 51.** The inverse of the extended hyperbolic secant function is

$$\operatorname{arsech}(x,y) = \ln \left( \frac{1 + \sqrt{1 - x^2 \ln^2(y)}}{x \ln y} \right) = \operatorname{sech}^{-1}(x \ln(y)) = \operatorname{sech}^{-1}(x,y), \quad (3.153)$$

for  $0 < x \ln(y) \leq 1$ ,  $y > 0$ ,  $x \neq 0$  or  $y \neq 1$ .

#### Proof

Let  $f$  be the inverse hyperbolic secant function. Then we have

$$f = \operatorname{sech}^{-1}(x,y) = \operatorname{sech}^{-1}(x \ln y) \Rightarrow x \ln(y) = \operatorname{sech}(f) = \frac{2}{e^f + e^{-f}} = \frac{2e^f}{e^{2f} + 1},$$

from which we obtain the quadratic equation in  $f$ :

$$x \ln(y) e^{2f} - 2e^f + x \ln(y) = 0$$

Solving for  $e^f$  from the equation and then taking logarithm of both sides gives

$$f = \ln\left(\frac{1 + \sqrt{1 - x^2 \ln^2(y)}}{x \ln(y)}\right)$$

Replacing  $f$  with  $\operatorname{sech}^{-1}(x, y)$  gives

$$\operatorname{sech}^{-1}(x, y) = \ln\left(\frac{1 + \sqrt{1 - x^2 \ln^2(y)}}{x \ln(y)}\right),$$

where

$$\begin{cases} y > 0, x \ln(y) \neq 0 \\ 1 - x^2 \ln^2(y) \geq 0 \end{cases} \Rightarrow \begin{cases} y > 0, x \neq 0 \text{ or } y \neq 1 \\ |x \ln(y)| \leq 1 \end{cases}.$$

## F. Inverse Extended Hyperbolic Cosecant Function – $\operatorname{arcsch}(x, y)$

**Theorem 52.** The inverse of the extended hyperbolic cosecant function is

$$\operatorname{arcsch}(x, y) = \ln\left(\frac{1}{x \ln y} + \sqrt{\frac{1}{x^2 \ln^2(y)} + 1}\right) = \operatorname{csch}^{-1}(x \ln(y)) = \operatorname{csch}^{-1}(x, y) \quad (3.154)$$

for  $x, y \in \mathbb{R}$  with  $y > 0, x \neq 0$  or  $y \neq 1$ .

### Proof

The proof proceeds similarly to #5.

## 3-16-2. Inverse Extended Trigonometric Functions

Since the extended functions with two variables are based on the classical trigonometric functions, their domains and ranges coincide with those of the corresponding classical functions. The same applies to the inverses of the extended trigonometric functions. In the sections that follow, we express these inverse functions in exponential and logarithmic form. Although written in complex notation, each formula agrees with the usual real inverse trigonometric function

when restricted to the real axis and interpreted on the principal branch of the logarithm.

### A. Inverse Extended Sine Function – arcsin(x,y)

**Theorem 53.** The inverse of the extended sine function is

$$\arcsin(x, y) = \frac{1}{i} \ln \left( i x \ln y + \sqrt{1 - x^2 \ln^2 y} \right) = \sin^{-1}(x \ln y) = \sin^{-1}(x, y) \quad (3.155)$$

for  $-1 \leq x \ln(y) \leq 1$  and  $y > 0$ . Notice that when restricted to real arguments in the classical domain, it coincides with the usual real inverse trigonometric function. This is because the logarithmic representation is interpreted on the principal branch.

#### Proof

Let  $f$  be the inverse of the extended sine function. Then we have

$$f = \sin^{-1}(x, y) = \sin^{-1}(x \ln y) \Rightarrow x \ln(y) = \sin(f) = \frac{1}{2i} (e^{if} - e^{-if}),$$

from which we obtain the quadratic equation in  $e^{if}$ :

$$e^{2if} - 2ix \ln(y) e^{if} - 1 = 0$$

Solving for  $e^{if}$  from the equation gives

$$e^{if} = ix \ln(y) \pm \sqrt{1 - x^2 \ln^2(y)}$$

Taking logarithm of both sides and choosing the positive branch (+) gives

$$f = \frac{1}{i} \ln \left( ix \ln(y) \pm \sqrt{1 - x^2 \ln^2(y)} \right),$$

where

$$\begin{cases} y > 0 \\ x^2 \ln^2(y) - 1 \geq 0 \end{cases} \Rightarrow \begin{cases} y > 0 \\ |x \ln(y)| \geq 1. \end{cases}$$

By substituting  $f$  with  $\sin^{-1}(x, y)$ , we complete the proof.

## B. Inverse Extended Cosine Function – $\arccos(x, y)$

**Theorem 54.** The inverse of the extended cosine function is

$$\arccos(x, y) = \frac{1}{i} \ln \left( x \ln y + i \sqrt{1 - x^2 \ln^2 y} \right) = \cos^{-1}(x \ln y) = \cos^{-1}(x, y) \quad (3.156)$$

for  $-1 \leq x \ln(y) \leq 1$  and  $y > 0$ .

### Proof

The proof is in a similar manner to #1.

## C. Inverse Extended Tangent Function – $\arctan(x, y)$

**Theorem 55.** The inverse of the extended tangent function is

$$\arctan(x, y) = \frac{1}{2i} \ln \left( \frac{i - x \ln y}{i + x \ln y} \right) = \tan^{-1}(x \ln y) \equiv \tan^{-1}(x, y) \quad (3.157)$$

for  $x, y \in \mathbb{R}$  with  $y > 0$ .

### Proof

Let  $f$  be the inverse of the extended sine function. Then we have

$$f = \tan^{-1}(x, y) = \tan^{-1}(x \ln y) \Rightarrow x \ln(y) = \tan(f) = \frac{e^{if} - e^{-if}}{i(e^{if} + e^{-if})} = \frac{e^{2if} - 1}{i(e^{2if} + 1)},$$

from which we obtain the quadratic equation in  $e^{if}$ :

$$i(e^{2if} + 1)x \ln(y) = e^{2if} - 1$$

Solving for  $e^{2if}$  from the equation gives

$$e^{2if} = \frac{1 + ix \ln(y)}{1 - ix \ln(y)} = \frac{i - x \ln(y)}{i + x \ln(y)}.$$

By taking logarithm of both sides, we obtain

$$f = \frac{1}{2i} \ln \left( \frac{i - x \ln(y)}{i + x \ln(y)} \right)$$

By substituting  $f$  with  $\tan^{-1}(x, y)$ , we complete the proof.

#### D. Inverse Extended Cotangent Function – $\text{arccot}(x, y)$

**Theorem 56.** The inverse of the extended cotangent function is

$$\text{arccot}(x, y) = \frac{1}{2i} \ln \left( \frac{x \ln(y) + i}{x \ln(y) - i} \right) = \cot^{-1}(x \ln y) \equiv \cot^{-1}(x, y) \quad (3.158)$$

for  $x, y \in \mathbb{R}$  with  $y > 0$ .

#### Proof

The proof is similar to that of #3.

#### E. Inverse Extended Secant Function – $\text{arcsec}(x, y)$

**Theorem 57.** The inverse of the extended secant function is given by

$$\text{arcsec}(x, y) = \frac{1}{i} \ln \left( \frac{1}{x \ln(y)} + i \cdot \sqrt{1 - \frac{1}{x^2 \ln^2(y)}} \right) = \sec^{-1}(x \ln y) \equiv \sec^{-1}(x, y) \quad (3.159)$$

for  $x, y \in \mathbb{R}$  with  $y > 0$ ,  $|x \ln(y)| \geq 1$ . Notice that  $|x \ln(y)| \geq 1$  implies both  $x \neq 0$  or  $y \neq 1$ .

#### Proof

Let  $f$  be the inverse of  $\sec(x, y)$ . Then we have

$$f = \sec^{-1}(x, y) = \sec^{-1}(x \ln y) \Rightarrow x \ln(y) = \sec(f) = \frac{2}{e^{if} + e^{-if}},$$

from which we obtain the quadratic equation in  $e^{if}$ :

$$x \ln(y) e^{2if} - 2e^{if} + x \ln(y) = 0.$$

Solving for  $e^{if}$  from the equation gives

$$e^{if} = \frac{1}{x \ln(y)} \left(1 + \sqrt{1 - x^2 \ln^2(y)}\right) = \frac{1}{x \ln(y)} + i \sqrt{1 - \frac{1}{x^2 \ln^2(y)}}$$

Taking logarithm of both sides, we obtain

$$f = \frac{1}{i} \ln \left( \frac{1}{x \ln(y)} + i \sqrt{1 - \frac{1}{x^2 \ln^2(y)}} \right).$$

Replacing  $f$  with  $\sec^{-1}(x,y)$  completes the proof.

## F. Inverse Extended Cosecant Function – arccsc(x,y)

**Theorem 58.** The inverse of the extended cosecant function is given by

$$\operatorname{arccsc}(x, y) = \frac{1}{i} \ln \left( \sqrt{1 - \frac{1}{x^2 \ln^2(y)}} + \frac{i}{x \ln y} \right) = \csc^{-1}(x, y) \equiv \csc^{-1}(x, y) \quad (3.160)$$

for  $x, y \in \mathbb{R}$  with  $y > 0$ ,  $|x \ln(y)| \geq 1$ .

### Proof

The proof is similar to that of #5.

**Remark 3.4.13.** The functions and their inverses (if they exist) exhibit symmetry about the bisector of the initial angle or the function  $f(x) = x$ . It means that the graph of  $f^{-1}$  is obtained by reflecting the graph of  $f$  across the line  $y = x$ . If both equations holds for all  $x$  in the appropriate domains, then  $f$  and  $f^{-1}$  are indeed inverses. Similarly, **for all extended trigonometric and hyperbolic functions with two variables, the symmetry extends to their inverse counterparts over the function  $f(x,y) = x \ln(y)$ .** To verify that  $f$  and  $f^{-1}$  are true inverses, we must check two fundamental identities:  $f(f^{-1}(u)) = u$  and  $f^{-1}(f(u)) = u$ . Let's take an example,  $f(x,y) = \sin(x,y)$  and  $g(x,y) = \sin^{-1}(x,y)$ . Let  $u = x \ln y$ , which is bisector. Indeed, we have:

- $f(g^{-1}(u)) = \sin(x, y) \circ \sin^{-1}(x, y) = \sin(x \ln y) \circ \sin^{-1}(x \ln y) = \sin(\sin^{-1}(x \ln y)) = x \ln y,$
- $g^{-1}(f(u)) = \sin^{-1}(x, y) \circ \sin(x, y) \equiv \sin^{-1}(x \ln y) \circ \sin(x \ln y) = \sin^{-1}(\sin(x \ln y)) = x \ln y.$

### Section 3-17. Identities of Extended Trig & Hyperbolic Functions

Since the extended functions are the trigonometric functions and hyperbolic functions, all identities of the trig & hyperbolic functions continue to hold true when applied to them. We demonstrate some basic identities of them in this section.

#### 3-17-1. Pythagorean identities in $x$

$$\cos^2(x, y) + \sin^2(x, y) = 1 \quad (3.162)$$

$$\cos(2x, y) = \cos^2(x, y) - \sin^2(x, y) = 1 - 2\sin^2(x, y) = 2\cos^2(x, y) - 1 \quad (3.163)$$

$$\cosh^2(x, y) - \sinh^2(x, y) = 1 \quad (3.164)$$

$$\cosh(2x, y) = \sinh^2(x, y) + \cosh^2(x, y) = 2\sinh^2(x, y) + 1 = 2\cosh^2(x, y) - 1 \quad (3.165)$$

#### 3-17-2. Angle sums and differences in $x$

$$\cosh(x_1 \pm x_2, y) = \cosh(x_1, y) \cosh(x_2, y) \pm \sinh(x_1, y) \sinh(x_2, y) \quad (3.166)$$

$$\sinh(x_1 \pm x_2, y) = \sinh(x_1, y) \cosh(x_2, y) \pm \sinh(x_2, y) \cosh(x_1, y) \quad (3.167)$$

All trigonometric relationships apply consistently to  $x$ . Our next exploration focuses exclusively on the relationships governed by the variable  $y$ .

#### 3-17-3. Angle sums and differences in $y$

In the following, we demonstrate some identities of the angle sums and differences in  $y$ . For  $x$ ,  $y_1$ , and  $y_2$  are real, we obtain:

$$\cosh(x, y_1) + \cosh(x, y_2) = 2 \cosh\left(\frac{x}{2}, y_1 \cdot y_2\right) \cosh\left(\frac{x}{2}, \frac{y_1}{y_2}\right) = 2 \cosh(x, \sqrt{y_1 \cdot y_2}) \cosh\left(x, \sqrt{\frac{y_1}{y_2}}\right) \quad (3.168)$$

$$\cosh(x, y_1) - \cosh(x, y_2) = 2 \sinh\left(\frac{x}{2}, y_1 \cdot y_2\right) \sinh\left(\frac{x}{2}, \frac{y_1}{y_2}\right) = 2 \sinh(x, \sqrt{y_1 \cdot y_2}) \sinh\left(x, \sqrt{\frac{y_1}{y_2}}\right) \quad (3.169)$$

$$\sinh(x, y_1) + \sinh(x, y_2) = 2 \sinh\left(\frac{x}{2}, y_1 \cdot y_2\right) \cosh\left(\frac{x}{2}, \frac{y_1}{y_2}\right) = 2 \sinh(x, \sqrt{y_1 \cdot y_2}) \cosh\left(x, \sqrt{\frac{y_1}{y_2}}\right) \quad (3.170)$$

$$\sinh(x, y_1) - \sinh(x, y_2) = 2 \cosh\left(\frac{x}{2}, y_1 \cdot y_2\right) \sinh\left(\frac{x}{2}, \frac{y_1}{y_2}\right) = 2 \cosh(x, \sqrt{y_1 \cdot y_2}) \sinh\left(x, \sqrt{\frac{y_1}{y_2}}\right) \quad (3.171)$$

$$\tanh(x, y_1) + \tanh(x, y_2) = \frac{\sinh(x, y_1 \cdot y_2)}{\cosh(x, y_1) \cosh(x, y_2)} \quad (3.172)$$

$$\tanh(x, y_1) - \tanh(x, y_2) = \frac{\sinh\left(x, \frac{y_1}{y_2}\right)}{\cosh(x, y_1) \cosh(x, y_2)} \quad (3.173)$$

$$\coth(x, y_1) + \coth(x, y_2) = \frac{\sinh(x, y_1 \cdot y_2)}{\sinh(x, y_1) \sinh(x, y_2)} \quad (3.174)$$

$$\coth(x, y_1) - \coth(x, y_2) = \frac{\sinh\left(x, \frac{y_1}{y_2}\right)}{\sinh(x, y_1) \sinh(x, y_2)} \quad (3.175)$$

## Proof

### 1. Formula (3.168)

We have:

$$\begin{aligned} \cosh(x, y_1) + \cosh(x, y_2) &= \cosh(x \ln y_1) + \cosh(x \ln y_2) \\ &= 2 \cosh\left(\frac{x \ln y_1 + x \ln y_2}{2}\right) \cosh\left(\frac{x \ln y_1 - x \ln y_2}{2}\right) \\ &= 2 \cosh\left(\frac{x}{2} \ln(y_1 \cdot y_2)\right) \cosh\left(\frac{x}{2} \ln \frac{y_1}{y_2}\right) \\ &= 2 \cosh\left(\frac{x}{2}, y_1 \cdot y_2\right) \cosh\left(\frac{x}{2}, \frac{y_1}{y_2}\right) \text{ or } y_1 \cdot y_2 > 0 \text{ and } y_2 \neq 0. \end{aligned}$$

or

$$= 2 \cosh(x, \sqrt{y_1 \cdot y_2}) \cosh\left(x, \sqrt{\frac{y_1}{y_2}}\right), \text{ for } y_1 \cdot y_2 > 0 \text{ and } y_2 \neq 0.$$

### 2. Formula (3169)

It follows the similar proof to that of formula (3.168).

**3. Formula (3.170)**

We have:

$$\begin{aligned}
 \sinh(x, y_1) + \sinh(x, y_2) &= \sinh(x \ln y_1) + \sinh(x \ln y_2) \\
 &= 2 \sinh\left(\frac{x \ln y_1 + x \ln y_2}{2}\right) \cosh\left(\frac{x \ln y_1 - x \ln y_2}{2}\right) \\
 &= 2 \sinh\left(\frac{x}{2} \ln(y_1 \cdot y_2)\right) \cosh\left(\frac{x}{2} \ln\left(\frac{y_1}{y_2}\right)\right) \\
 &= 2 \sinh\left(\frac{x}{2}, y_1 \cdot y_2\right) \cosh\left(\frac{x}{2}, \frac{y_1}{y_2}\right), \text{ for } y_1 \cdot y_2 > 0 \text{ and } y_2 \neq 0
 \end{aligned}$$

or

$$= 2 \sinh\left(x, \sqrt{y_1 \cdot y_2}\right) \cosh\left(x, \sqrt{\frac{y_1}{y_2}}\right), \text{ for } y_1 \cdot y_2 > 0 \text{ and } y_2 \neq 0.$$

**4. Formula (3.171)**

It uses a proof that is similar to the one for formula (3.170).

**5. Formula (3.172)**

We have

$$\begin{aligned}
 \tanh(x, y_1) + \tanh(x, y_2) &= \tanh(x \ln y_1) + \tanh(x \ln y_2) \\
 &= \frac{\sinh(x \ln y_1)}{\cosh(x \ln y_1)} + \frac{\sinh(x \ln y_2)}{\cosh(x \ln y_2)} \\
 &= \frac{\sinh(x \ln y_1) \cosh(x \ln y_2) + \sinh(x \ln y_2) \cosh(x \ln y_1)}{\cosh(x \ln y_1) \cosh(x \ln y_2)} \\
 &= \frac{\sinh(x \ln y_1 + x \ln y_2)}{\cosh(x \ln y_1) \cosh(x \ln y_2)} \\
 &= \frac{\sinh(x \ln(y_1 \cdot y_2))}{\cosh(x \ln y_1) \cosh(x \ln y_2)}
 \end{aligned}$$

$$= \frac{\sinh(x, y_1 \cdot y_2)}{\cosh(x, y_1) \cosh(x, y_2)}$$

## 6. Formulas (3.173), (3.174) and (3.175)

It follows the similar proof to that of formula (3.172).

Even though there are still many relations to look into, we will stop our exploration of this **Section 3-17** here. We are introducing our readers to the extended trigonometric functions that involve three variables, along with their inverse functions.

### Section 3-18. Extended trigonometric and hyperbolic functions with three variables

This section has been edited by Tracy Vu, Tiffany Vu and Trista Vu. The project presents the results obtained within a work in progress. We want to publish all the unfinished findings here. There are undoubted mistakes in this section that we have yet to identify. They may come either from mistakes made by the author during the development of the formulas or from inaccuracies introduced when converting the author's original notations.

We have devised formulas for extending trigonometric and hyperbolic functions to encompass three variables. The motivation behind this endeavor is to explore the variations in the extended trigonometric and hyperbolic functions with two variables. Below, we provide the definitions and a summary of these newly discovered formulas.

#### 3-18-1. Definitions of the extended hyperbolic functions with three variables

For real values of  $x$ ,  $y$ , and  $z$  with  $y > 0$ , the definitions of  $\sinh(x, y, z)$ ,  $\cosh(x, y, z)$ ,  $\tanh(x, y, z)$ ,  $\coth(x, y, z)$ ,  $\operatorname{csch}(x, y, z)$  and  $\operatorname{sech}(z, y, z)$  are defined as

$$\sinh(x, y, z) = \frac{1}{2} (y^x - z \cdot y^{-x}), \quad (3.176)$$

$$\cosh(x, y, z) = \frac{1}{2} (y^x + z \cdot y^{-x}), \quad (3.177)$$

$$\tanh(x, y, z) = \frac{y^x - z \cdot y^{-x}}{y^x + z \cdot y^{-x}}, \quad y^x + z \cdot y^{-x} \neq 0, \quad (3.178)$$

$$\coth(x, y, z) = \frac{y^x + z \cdot y^{-x}}{y^x - z \cdot y^{-x}}, \quad y^x - z \cdot y^{-x} \neq 0, \quad (3.179)$$

$$\operatorname{csch}(x) = \frac{2}{y^x - z \cdot y^{-x}}, \quad y^x - z \cdot y^{-x} \neq 0, \quad (3.176 a)$$

and

$$\operatorname{sech}(x) = \frac{2}{y^x + z \cdot y^{-x}}, \quad y^x + z \cdot y^{-x} \neq 0. \quad (3.177 a)$$

**Note:** The original notations for the sinh and cosh formulas used by the author are defined as

$$\sinh_{a,b}(x) = \frac{1}{2}(a^x - b \cdot a^{-x}), \quad \text{and} \quad \cosh_{a,b}(x) = \frac{1}{2}(a^x + b \cdot a^{-x}).$$

### 3-18-2. Relations of the extended functions between two variables and three variables

The extended hyperbolic functions with three variables are expressed in terms of two variables:

$$\sinh(x, y, z) = \frac{1}{2} \{ (1+z) \sinh(x, y) + (1-z) \cosh(x, y) \}, \quad (3.180)$$

$$\cosh(x, y, z) = \frac{1}{2} \{ (1+z) \cosh(x, y) + (1-z) \sinh(x, y) \}, \quad (3.181)$$

$$\tanh(x, y, z) = \frac{(1+z) \tanh(x, y) + (1-z)}{(1+z) + (z-1) \tanh(x, y)} = \frac{\left(\frac{1+z}{1-z}\right) \tanh(x, y) + 1}{\left(\frac{1+z}{1-z}\right) - \tanh(x, y)}, \quad (3.182)$$

$$\coth(x, y, z) = \frac{\coth(x, y) - \left(\frac{1-z}{1+z}\right)}{1 + \left(\frac{1-z}{1+z}\right) \coth(x, y)}, \quad (3.183)$$

$$= \frac{(1+z) \coth(x, y) + (z-1)}{(1+z) + (1-z) \coth(x, y)}, \quad (3.184)$$

$$= \frac{\left(\frac{z+1}{z-1}\right) \coth(x, y) + 1}{\left(\frac{z+1}{z-1}\right) - \coth(x, y)}. \quad (3.185)$$

$$\operatorname{csch}(x) = \frac{2}{(1+z) \sinh(x, y) + (1-z) \cosh(x, y)}, \quad (3.180a)$$

and

$$\operatorname{sech}(x) = \frac{2}{(1+z) \cosh(x, y) + (1-z) \sinh(x, y)}, \quad (3.181a)$$

### Proof

**1. Formula (3.180):** We use formula (3.117a) to prove formula (3.180) as follows:

$$\begin{aligned} \sinh(x, y, z) &= \frac{1}{2}(y^x - z y^{-x}) \\ &= \frac{1}{2}[\cosh(x, y) + \sinh(x, y) - z(\cosh(x, y) - \sinh(x, y))] \\ &= \frac{1}{2}[(1-z) \cosh(x, y) + (1+z) \sinh(x, y)] \end{aligned}$$

**2. Formula (3.181):**

$$\begin{aligned} \cosh(x, y, z) &= \frac{1}{2}(y^x + z y^{-x}) \\ &= \frac{1}{2}[\cosh(x, y) + \sinh(x, y) + z(\cosh(x, y) - \sinh(x, y))] \\ &= \frac{1}{2}[(1+z) \cosh(x, y) + (1-z) \sinh(x, y)] \end{aligned}$$

**3. Formulas (3.182) to (3.185):** Proceeding in the same manner as above yields formulas (3.182) to (3.185).

**Example.** Prove that

$$\tanh(x, y, z) = \frac{\sinh(x, y, z)}{\cosh(x, y, z)}. \quad (3.186)$$

### Solution

By definition (3.178), we have

$$\begin{aligned} \tanh(x, y, z) &= \frac{y^x - z \cdot y^{-x}}{y^x + z \cdot y^{-x}} \\ &= \frac{\frac{1}{2}(y^x - z \cdot y^{-x})}{\frac{1}{2}(y^x + z \cdot y^{-x})} \\ &= \frac{\sinh(x, y, z)}{\cosh(x, y, z)}. \end{aligned}$$

By proceeding in the same manner, we obtain  $\coth(x, y, z) = \frac{\cosh(x, y, z)}{\sinh(x, y, z)}$ , from which it follows that

$$\tanh(x, y, z) = \frac{1}{\coth(x, y, z)}. \quad (3.187)$$

**Example.** For real  $x, y > 0$  and  $z > 0$ , show that

$$\sqrt{z} \cosh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right) = \frac{1}{2}(y^x + z \cdot y^{-x}). \quad (3.188)$$

### Solution

The given expression must satisfy the following conditions:

$$\begin{cases} -\infty < x < \infty \\ -1 < \frac{1-z}{1+z} < 1 \\ y > 0 \\ z \geq 0 \end{cases} \Rightarrow \begin{cases} -\infty < x < \infty \\ \frac{2z}{1+z} > 0 \\ \frac{2}{1+z} > 0 \\ y > 0 \\ z \geq 0 \\ z \neq -1 \end{cases} \Rightarrow \begin{cases} -\infty < x < \infty \\ y > 0 \\ z > 0 \end{cases}$$

Formula (3.181) gives

$$\begin{aligned}\cosh(x, y, z) &= \frac{1}{2} \{ (z+1) \cosh(x \ln y) + (1-z) \sinh(x \ln y) \} \\ &= \frac{z+1}{2} \left\{ \cosh(x \ln y) + \left( \frac{1-z}{1+z} \right) \sinh(x \ln y) \right\}\end{aligned}\quad (3.188a)$$

Let  $\alpha$  be a real number and suppose

$$\frac{1-z}{z+1} = \tanh(\alpha), \quad (*)$$

from which it follows that

$$\alpha = \tanh^{-1} \left( \frac{1-z}{1+z} \right). \quad (**)$$

By squaring both sides of (\*), we have

$$\frac{(1-z)^2}{(1+z)^2} = \tanh^2(\alpha) = \frac{\cosh^2(\alpha) - 1}{\cosh^2(\alpha)}.$$

Solving for  $\cosh(\alpha)$  in terms of  $z$  yields

$$\cosh(\alpha) = \frac{1+z}{2\sqrt{z}} \quad (3.188b)$$

By substituting  $(1-z)/(1+z) = \tanh(\alpha)$  in (3.194a), we have:

$$\begin{aligned}\cosh(x, y, z) &= \frac{z+1}{2} \{ \cosh(x \ln y) + \tanh(\alpha) \sinh(x \ln y) \} \\ &= \frac{z+1}{2 \cosh(\alpha)} \{ \cosh(\alpha) \cosh(x \ln y) + \sinh(\alpha) \sinh(x \ln y) \} \\ &= \frac{z+1}{2 \cosh(\alpha)} \cosh(\alpha + x \ln y)\end{aligned}$$

By substituting expression  $\cosh(\alpha)$  from (3.194b) and  $\alpha$  from (\*\*) in the above expression, we obtain

$$\cosh(x, y, z) = \sqrt{z} \cosh\left(\tanh^{-1}\left(\frac{1-z}{z+1}\right) + x \ln y\right)$$

for  $x \in \mathbb{R}$ ,  $y > 0$ ,  $z > 0$ . The combination of the result with definition (3.177) completes the proof.

**Exercise.** For real  $x$ ,  $y > 0$  and  $z > 0$ , show that

$$\cosh(x, y, z) = \sqrt{z} \cosh\left(\pm \cosh^{-1}\left(\frac{1+z}{2\sqrt{z}}\right) + x \ln y\right) = \sqrt{z} \cosh\left(\cosh^{-1}\left(\frac{1+z}{2\sqrt{z}}\right) \pm x \ln y\right).$$

**Example.** For real  $x$ ,  $y > 0$  and  $z \neq 0$ , show that

$$\frac{1}{2}(z \cdot y^x + y^{-x}) = \frac{1}{z} \cosh\left(x, y, \frac{1}{z}\right)$$

**Solution**

$$\begin{aligned} \frac{1}{2}(z \cdot y^x + y^{-x}) &= \frac{1}{2z} \left( y^x + \frac{1}{z} y^{-x} \right) \\ &= \frac{1}{z} \cosh\left(x, y, \frac{1}{z}\right). \end{aligned}$$

### 3-18-3. Alternative definitions of the extended hyperbolic functions with three variables

For real values of  $x$ ,  $y$ , and  $z$  with  $y > 0$ , the alternative definitions of  $\sinh(x, y, z)$ ,  $\cosh(x, y, z)$ ,  $\tanh(x, y, z)$ ,  $\coth(x, y, z)$ ,  $\operatorname{csch}(x, y, z)$  and  $\operatorname{sech}(x, y, z)$  are defined as follows:

$$\sinh(x, y, z) = \sqrt{z} \sinh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right), \text{ for all } x, y, z \text{ with } y > 0, z > 0. \quad (3.189)$$

$$\cosh(x, y, z) = \sqrt{z} \cosh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right), \text{ for all } x, y, z \text{ with } y > 0, z > 0. \quad (3.190)$$

$$\tanh(x, y, z) = \tanh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right), \text{ for all } x, y, z \text{ with } y > 0, z > 0. \quad (3.191)$$

$$\coth(x, y, z) = \coth\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right), x, y, z \in \mathbb{R} \text{ with } y > 0, z > 0, \sinh(x, y, z) \neq 0 \quad (3.192)$$

$$\operatorname{csch}(x, y, z) = \frac{1}{\sqrt{z} \sinh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right)}, x, y, z \in \mathbb{R} \text{ with } y > 0, z > 0, \sinh(x, y, z) \neq 0 \quad (3.189a)$$

$$\operatorname{sech}(x, y, z) = \frac{1}{\sqrt{z} \cosh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right)}, x, y, z \in \mathbb{R} \text{ with } y > 0, z > 0, \quad (3.190a)$$

### Proof

#### 1. Formula (3.189):

By formula (3.180), it follows that

$$\begin{aligned} \sinh(x, y, z) &= \frac{1}{2} \{(1+z) \sinh(x \ln y) + (1-z) \cosh(x \ln y)\} \\ &= \sqrt{z} \left\{ \left( \frac{1+z}{2\sqrt{z}} \right) \sinh(x \ln y) + \left( \frac{1-z}{2\sqrt{z}} \right) \cosh(x \ln y) \right\} \quad (*) \end{aligned}$$

Let  $\alpha$  be a real number and suppose that

$$\sinh(\alpha) = \frac{1-z}{2\sqrt{z}}.$$

Applying  $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$  to solve for  $\cosh(\alpha)$ , we get

$$\cosh(\alpha) = +\frac{1+z}{2\sqrt{z}} \text{ for } z \leq 1, \text{ according to } \sinh(\alpha) \geq 0,$$

or

$$\cosh(\alpha) = -\frac{1+z}{2\sqrt{z}} \text{ for } z > 1, \text{ according to } \sinh(\alpha) < 0.$$

By using  $\tanh(\alpha) = \sinh(\alpha) / \cosh(\alpha)$ , we obtain

$$\tanh(\alpha) = \frac{1-z}{1+z}, \quad (\text{The sign of } \sinh(\alpha) \text{ is the same with } \tanh(\alpha).)$$

from which it follows that

$$\alpha = \tanh^{-1}\left(\frac{1-z}{1+z}\right).$$

a. By substituting  $(1+z)/2\sqrt{z} = \cosh \alpha$ ,  $(1-z)/2\sqrt{z} = \sinh \alpha$  and  $\alpha = \tanh^{-1}[(1-z)/(1+z)]$ , we have

$$\begin{aligned} \sinh(x, y, z) &= \sqrt{z} \{ \sinh(x \ln y) \cosh \alpha + \sinh \alpha \cosh(x \ln y) \} \\ &= \sqrt{z} \sinh(\alpha + x \ln y) \\ &= \sqrt{z} \sinh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right), \end{aligned}$$

where  $x$ ,  $y$  and  $z$  are real numbers with

$$\left\{ \begin{array}{l} -\infty < x < \infty \\ y > 0 \\ -1 < \frac{1-z}{1+z} < 1 \\ z \neq -1 \\ z \geq 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} -\infty < x < \infty \\ y > 0 \\ z > 0. \end{array} \right.$$

b. Furthermore, by replacing  $\alpha$  with  $\sinh^{-1}$ , we obtain

$$\sinh(x, y, z) = \sqrt{z} \sinh\left(\sinh^{-1}\left(\frac{1-z}{2\sqrt{z}}\right) + x \ln y\right), \quad (3.189a)$$

where  $x$ ,  $y$ ,  $z$  are real numbers with

$$\left\{ \begin{array}{l} -\infty < x < \infty \\ y > 0 \\ -\infty < \frac{1-z}{2\sqrt{z}} < \infty \\ z \neq 0 \\ z \geq 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} -\infty < x < \infty \\ y > 0 \\ z > 0. \end{array} \right.$$

c. And by replacing  $a$  with  $\cosh^{-1}$ , we get the following results:

**Case 1 ( $z \leq 1$ ):**

$$\sinh(x, y, z) = \sqrt{z} \sinh\left(+\cosh^{-1}\left(\frac{1+z}{2\sqrt{z}}\right) + x \ln y\right), \quad (3.189b)$$

where  $x, y, z$  are real numbers with

$$\begin{cases} -\infty < x < \infty \\ y > 0 \\ 1 \leq \frac{1+z}{2\sqrt{z}} < \infty \\ z \neq 0 \\ z \geq 0 \end{cases} \Rightarrow \begin{cases} -\infty < x < \infty \\ y > 0 \\ 0 < z \leq 1. \end{cases}$$

**Case 2 ( $z > 1$ ):**

$$\sinh(x, y, z) = \sqrt{z} \sinh\left(-\cosh^{-1}\left(\frac{1+z}{2\sqrt{z}}\right) + x \ln y\right), \quad (3.189c)$$

where  $x, y, z$  are real numbers with  $-\infty < x < \infty$ ,  $y > 0$ , and  $z > 1$ .

**2. Formula (3.190):**

Similarly, by formula (3.181), it follows that

$$\begin{aligned} \cosh(x, y, z) &= \frac{1}{2} \left\{ (1+z) \cosh(x, y) + (1-z) \sinh(x, y) \right\} \\ &= \sqrt{z} \left\{ \left( \frac{1+z}{2\sqrt{z}} \right) \cosh(x \ln y) + \left( \frac{1-z}{2\sqrt{z}} \right) \sinh(x \ln y) \right\} \end{aligned}$$

Let  $\alpha$  be a real number and suppose

$$\sinh(\alpha) = \frac{1-z}{2\sqrt{z}},$$

from which it follows that

$$\begin{cases} \cosh(\sigma) = +\frac{1+z}{2\sqrt{z}} & \text{for } z \leq 1, \text{ according to } \sinh(\sigma) \geq 0, \\ \cosh(\sigma) = -\frac{1+z}{2\sqrt{z}} & \text{for } z > 1, \text{ according to } \sinh(\sigma) < 0. \end{cases}$$

By using  $\tanh(\alpha) = \sinh(\alpha) / \cosh(\alpha)$ , we obtain

$$\tanh(\alpha) = \frac{1-z}{1+z},$$

from which it follows that

$$\alpha = \tanh^{-1}\left(\frac{1-z}{1+z}\right).$$

By substituting  $(1+z)/2\sqrt{z} = \cosh \alpha$ ,  $(1-z)/2\sqrt{z} = \sinh \alpha$  and  $\alpha = \tanh^{-1}[(1-z)/(1+z)]$ , we have

$$\begin{aligned} \cosh(x, y, z) &= \sqrt{z} \{ \cosh(x \ln y) \cosh \alpha + \sinh \alpha \sinh(x \ln y) \} \\ &= \sqrt{z} \cosh(\alpha + x \ln y) \\ &= \sqrt{z} \cosh\left(\tanh^{-1}\left(\frac{1-z}{1+z}\right) + x \ln y\right), \end{aligned}$$

where  $x$ ,  $y$  and  $z$  are real numbers with

$$\begin{cases} -\infty < x < \infty \\ y > 0 \\ -1 < \frac{1-z}{1+z} < 1 \\ z \neq -1 \\ z \geq 0 \end{cases} \Rightarrow \begin{cases} -\infty < x < \infty \\ y > 0 \\ z > 0. \end{cases}$$

In addition, by replacing  $\alpha$  with  $\sinh^{-1}$ , we obtain

$$\cosh(x, y, z) = \sqrt{z} \cosh\left(\sinh^{-1}\left(\frac{1-z}{2\sqrt{z}}\right) + x \ln y\right), \quad (3.189a)$$

$$= \sqrt{z} \cosh\left(\ln \frac{y^x}{\sqrt{z}}\right) \quad (3.189b)$$

where  $x, y, z$  are real numbers with

$$\begin{cases} -\infty < x < \infty \\ y > 0 \\ -\infty < \frac{1-z}{2\sqrt{z}} < \infty \\ z \neq 0 \\ z \geq 0 \end{cases} \Rightarrow \begin{cases} -\infty < x < \infty \\ y > 0 \\ z > 0. \end{cases}$$

By replacing  $a$  with  $\cosh^{-1}$ , we get the following results:

**Case 1 ( $z \leq 1$ ):**

$$\cosh(x, y, z) = \sqrt{z} \cosh\left(+\cosh^{-1}\left(\frac{1+z}{2\sqrt{z}}\right) + x \ln y\right), \quad (3.189b)$$

where  $x, y, z$  are real numbers with

$$\begin{cases} -\infty < x < \infty \\ y > 0 \\ 1 \leq \frac{1+z}{2\sqrt{z}} < \infty \\ z \neq 0 \\ z \geq 0 \end{cases} \Rightarrow \begin{cases} -\infty < x < \infty \\ y > 0 \\ z > 0 \\ z \leq 1. \end{cases}$$

**Case 2 ( $z > 1$ ):**

$$\cosh(x, y, z) = \sqrt{z} \cosh\left(-\cosh^{-1}\left(\frac{1+z}{2\sqrt{z}}\right) + x \ln y\right), \quad (3.189c)$$

where  $x, y, z$  are real numbers with  $-\infty < x < \infty$ ,  $y > 0$ , and  $z > 1$ .

**3. Other formulas:** By applying formulas (3.186) and (3.187), we easily derive formulas (3.191) and (3.192). Use definitions (3.176a) and (3.177a), we prove that

$$\operatorname{csch}(x, y, z) = \frac{1}{\sinh(x, y, z)} \quad \text{and} \quad \operatorname{sech}(x, y, z) = \frac{1}{\cosh(x, y, z)},$$

from which it follows that formulas (3.189a) and (3.190a) can be derived from

formulas (3.189) and (3.190).

**Exercise.** Verify that

$$\bullet \frac{1}{\sqrt{z} \sinh \left( \tanh^{-1} \left( \frac{1-z}{1+z} \right) + x \ln y \right)} = \frac{2}{e^x - z e^{-x}}, \quad (3.189 d)$$

$$\bullet \frac{1}{\sqrt{z} \cosh \left( \tanh^{-1} \left( \frac{1-z}{1+z} \right) + x \ln y \right)} = \frac{2}{e^x + z e^{-x}}, \quad (3.189 e)$$

where  $x$ ,  $y$ , and  $z$  are real numbers with  $y, z > 0$  and  $z \neq -1$ .

### 3-18-4. Inverse extended hyperbolic functions with three variables

The inverse functions of  $\sinh(x,y,z)$ ,  $\cosh(x,y,z)$ ,  $\tanh(x,y,z)$  and  $\coth(x,y,z)$  are defined as:

$$\sinh^{-1}(x, y, z) = \ln \left( \operatorname{arcsinh} \left( \frac{y^x - z y^{-x}}{2} \right) + \sqrt{\operatorname{arcsinh}^2 \left( \frac{y^x - z y^{-x}}{2} \right) + 1} \right) \quad (3.193)$$

$$= \sinh^{-1} \left( \sinh^{-1} \left( \frac{y^x - z y^{-x}}{2} \right) \right) \quad (3.193 a)$$

$$\cosh^{-1}(x, y, z) = \ln \left( \operatorname{arccosh} \left( \frac{z y^x + y^{-x}}{2} \right) + \sqrt{\operatorname{arccosh}^2 \left( \frac{z y^x + y^{-x}}{2} \right) - 1} \right) \quad (3.194)$$

$$= \cosh^{-1} \left( \cosh^{-1} \left( \frac{y^x + z y^{-x}}{2} \right) \right) \quad (3.194 a)$$

$$\tanh^{-1}(x, y, z) = \frac{1}{2} \ln \left( \frac{1 + \tanh^{-1} \left( \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)} \right)}{1 - \tanh^{-1} \left( \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)} \right)} \right) \quad (3.195)$$

$$= \tanh^{-1} \left( \tanh^{-1} \left( \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)} \right) \right) \quad (3.195 a)$$

$$\coth^{-1}(x, y, z) = \frac{1}{2} \ln \left( \frac{\coth^{-1} \left( \frac{\coth(x \ln y) - \left( \frac{1-z}{1+z} \right)}{1 + \left( \frac{1-z}{1+z} \right) \coth(x \ln y)} \right) + 1}{\coth^{-1} \left( \frac{\coth(x \ln y) - \left( \frac{1-z}{1+z} \right)}{1 + \left( \frac{1-z}{1+z} \right) \coth(x \ln y)} \right) - 1} \right) \quad (3.196)$$

$$= \coth^{-1} \left( \coth^{-1} \left( \frac{\coth(x \ln y) - \left( \frac{1-z}{1+z} \right)}{1 + \left( \frac{1-z}{1+z} \right) \coth(x \ln y)} \right) \right) \quad (3.196 a)$$

### Proof

#### 1. Formula (3.193):

By taking arsinh both sides of (3.176), we have

$$(x, y, z) = \operatorname{arsinh} \left[ \frac{1}{2} (y^x - z y^{-x}) \right] \quad (*)$$

Let  $f$  be the inverse of the extended hyperbolic sine function. Then we have

$$f = \operatorname{arsinh}(x, y, z) = \sinh^{-1}(x, y, z), \quad (**)$$

from which it follows that

$$(x, y, z) = \sinh(f) = \frac{1}{2} (e^f - e^{-f})$$

Solving for  $f$  in terms of  $(x, y, z)$  gives

$$e^f = (x, y, z) + \sqrt{(x, y, z)^2 - 1}$$

Taking logarithm of both sides gives

$$f = \ln[(x, y, z) + \sqrt{(x, y, z)^2 - 1}] \quad (***)$$

Substituting expression (\*) in (\*\*\*) then combining the result with (\*) gives

$$\begin{aligned} \sinh^{-1}(x, y, z) &= \ln\left(\operatorname{arsinh}\left(\frac{1}{2}(y^x - z y^{-x})\right) + \sqrt{\operatorname{arsinh}^2\left(\frac{1}{2}(y^x - z y^{-x})\right) - 1}\right) \\ &= \sinh^{-1}\left(\sinh^{-1}\left(\frac{1}{2}(y^x - z y^{-x})\right)\right). \end{aligned}$$

**2. Formula (3.194):** Proceeding in the same manner as #1 yields formula (3.194).

**3. Formula (3.195):**

Let  $f$  be the inverse of the extended hyperbolic tangent ( $\tanh(x, y, z)$ ). Then we have

$$f = \tanh^{-1}(x, y, z), \quad (*)$$

from which it follows that

$$(x, y, z) = \tanh(f) = \frac{e^f - e^{-f}}{e^f + e^{-f}} = \frac{e^{2f} - 1}{e^{2f} + 1}$$

Solving for  $e^{2f}$  for the above equation gives

$$e^{2f} = \frac{1 + (x, y, z)}{1 - (x, y, z)}$$

Taking logarithm of both sides gives

$$f = \frac{1}{2} \ln\left(\frac{1 + (x, y, z)}{1 - (x, y, z)}\right) \quad (*)$$

Rewriting formula (3.182) in terms of  $\tanh(x \ln y)$  and  $(1-z)/(1+z)$  gives

$$\tanh(x, y, z) = \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)}$$

Applying  $\operatorname{artanh}$  of both sides gives

$$(x, y, z) = \tanh^{-1} \left( \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)} \right) \quad (**)$$

Substituting expression (\*\*) in (\*) and then combining with (\*) yields

$$\tanh^{-1}(x, y, z) = \frac{1}{2} \ln \left( \frac{1 + \tanh^{-1} \left( \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)} \right)}{1 - \tanh^{-1} \left( \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)} \right)} \right) = \frac{1}{2} \ln \left( \frac{1 + \tanh^{-1} \left( \frac{\tanh(x, y) + \frac{1-z}{1+z}}{1 - \tanh(x, y) \left( \frac{1-z}{1+z} \right)} \right)}{1 - \tanh^{-1} \left( \frac{\tanh(x, y) + \frac{1-z}{1+z}}{1 - \tanh(x, y) \left( \frac{1-z}{1+z} \right)} \right)} \right)$$

Expressing the formula in terms of inverse hyperbolic tangent gives

$$\tanh^{-1}(x, y, z) = \tanh^{-1} \left( \tanh^{-1} \left( \frac{\tanh(x \ln y) + \frac{1-z}{1+z}}{1 - \tanh(x \ln y) \left( \frac{1-z}{1+z} \right)} \right) \right) = \tanh^{-1} \left( \tanh^{-1} \left( \frac{\tanh(x, y) + \frac{1-z}{1+z}}{1 - \tanh(x, y) \left( \frac{1-z}{1+z} \right)} \right) \right).$$

**4. Formula (3.195):** Proceeding similarly to #3 yields formula (3.195).

### 3-18-5. Extended trigonometric functions with three variables

We present the extended hyperbolic functions with three variables. More generally, extended hyperbolic functions can be defined for any number of variables, provided that each variable is associated with a distinct operation within the extended function.

The extended trigonometric functions with three variables are not derived in this section. Instead, their extended formulas are listed in the table below as exercises for the reader to complete this chapter.

<b>Definitions of extended trigonometric functions with three variables</b>	
$\sin(x, y, z) = \frac{1}{2i}(y^{ix} - z \cdot y^{-ix})$	$\sin(x, y, z) = \sqrt{z} \sin\left(x \ln y - i \tanh^{-1}\left(\frac{1-z}{1+z}\right)\right)$
$\cos(x, y, z) = \frac{1}{2}(y^{ix} + z \cdot y^{-ix})$	$\cos(x, y, z) = \sqrt{z} \cos\left(x \ln y - i \tanh^{-1}\left(\frac{1-z}{1+z}\right)\right)$
$\tan(x, y, z) = \frac{1}{i} \frac{y^{ix} - z \cdot y^{-ix}}{y^{ix} + z \cdot y^{-ix}}$	$\tan(x, y, z) = \tan\left(x \ln y - i \tanh^{-1}\left(\frac{1-z}{1+z}\right)\right)$
$\cot(x, y, z) = i \frac{y^{ix} + z \cdot y^{-ix}}{y^{ix} - z \cdot y^{-ix}}$	$\cot(x, y, z) = \cot\left(x \ln y - i \tanh^{-1}\left(\frac{1-z}{1+z}\right)\right)$
$\csc(x, y, z) = \frac{2i}{y^{ix} - z \cdot y^{-ix}}$	$\csc(x, y, z) = \frac{1}{\sqrt{z} \sin\left(x \ln y - i \tanh^{-1}\left(\frac{1-z}{1+z}\right)\right)}$
$\sec(x, y, z) = \frac{2}{y^{ix} + z \cdot y^{-ix}}$	$\sec(x, y, z) = \frac{1}{\sqrt{z} \cos\left(x \ln y - i \tanh^{-1}\left(\frac{1-z}{1+z}\right)\right)}$

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## Chapter 4

### ***Unraveling the Mystery of the Riemann Hypothesis: Toward a Complete Proof***

**T**his chapter is dedicated to rigorously demonstrating the Riemann Hypothesis (RH) through purely elementary algebraic methods, namely an endeavor that has become one of our most exciting undertakings. The Riemann Hypothesis, one of the most significant unsolved problems in pure mathematics, posits that all nontrivial zeros of the Riemann zeta function are complex numbers with a real part equal to  $\frac{1}{2}$ . This profound conjecture is deeply connected to the distribution of prime numbers and intersects with numerous fields, including quantum mechanics and physics. Moreover, the Riemann Hypothesis has intrinsic ties to special functions such as the Gamma, Zeta, L, and Bessel functions, making it a cornerstone of modern mathematical research.

The Riemann zeta function itself is a fundamental object in number theory and complex analysis. Initially introduced by Leonhard Euler in the 18th century in the context of real numbers, the function was later extended to the complex plane by Bernhard Riemann in his groundbreaking 1859 paper. Its zeros, particularly those on the right half of the complex plane, hold a pivotal connection to the distribution of prime numbers. These zeros are classified into two categories: trivial and nontrivial. The trivial zeros lie outside the critical strip, positioned at the negative even integers on the real axis. The nontrivial zeros, in contrast, occur within the critical strip, where the real part lies between 0 and 1. The Riemann Hypothesis conjectures that all nontrivial zeros reside precisely on the critical line, where the real part equals  $\frac{1}{2}$ . Despite the simplicity of its statement, the underlying reasons for this behavior remain a profound mystery.

The chapter is organized into four sections to systematically address the Riemann

Hypothesis and its implications. In Section 4-1, we review the definition and fundamental properties of the Riemann zeta function, establishing the groundwork for subsequent discussions. Section 4-2 presents a first proof of the Riemann Hypothesis, leveraging a rewritten form of the Riemann functional equation. Building upon this, Section 4-3 introduces a second proof using a simplified version of the rewritten Riemann functional equation. Finally, Section 4-4 compiles our findings and offers a final proof and comprehensive conclusion on the correctness of the Riemann Hypothesis. While working to prove the Riemann Hypothesis in this section, we encountered a contextual constant, such as an expression that, within this framework, behaves like a constant. We temporarily designate it as a “*saturation constant*” due to its critical role in determining the location of the last possible nontrivial zero. Based on our analysis, we confirm that no nontrivial zeros exist beyond this point.

## Section 4-1. Definition and Properties of Riemann Zeta Function

This section provides an overview of the definitions, properties and several known facts of the Riemann zeta function which are credited to the foundational contributions of various mathematicians.

### 4-1-1. Definition of Riemann Zeta Function

The Riemann zeta function is denoted  $\zeta$  and is defined for complex values of  $s$  by the infinite series<sup>[11]</sup>:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots \text{ for } \operatorname{Re}(s) > 1. \quad (4.1)$$

This sum was originally studied by Euler for real numbers. It was later extended to complex numbers by Riemann, who provided its most important properties and the functional equation in his famous 1859 paper.

### 4-1-2. Alternative Form of Riemann Zeta Function

The Riemann zeta function can be expressed as an alternative series, known as the Dirichlet eta function, for complex values of  $s$  using an alternative infinite series<sup>[12]</sup>:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} \text{ for } \operatorname{Re}(s) > 0. \quad (4.2)$$

### 4-1-3. Riemann Zeta Function with Euler Product Formula

The Riemann zeta function can be expressed in the following product which extends all over prime numbers  $p$ :

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \text{ for } \operatorname{Re}(s) > 1. \quad (4.3)$$

This product is called the Euler product of the zeta function<sup>[13]</sup>. The Euler product not only establishes a connection between the Riemann zeta function with the series of prime numbers but also implies that the function has no zeros for  $\operatorname{Re}(s) > 1$ .

### 4-1-4. Integral Form of Riemann Zeta Function

The Riemann zeta function can be expressed in the integral form<sup>[14]</sup>,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-t} dt \text{ for } \operatorname{Re} s > 1, \quad (4.4)$$

where  $\Gamma(s)$  represents the Gamma function, which is defined as

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt. \quad (4.5)$$

The integral formula extend the definition of the series for Riemann zeta function involving the Gamma function.

### 4-1-5. Riemann Zeta Functional Equation

Since the series for the zeta function converges only for  $\operatorname{Re}(s) > 1$ , Riemann utilized the integral representation in Section 4-1-4 to extend series (4.1) in the

entire complex plane  $\mathbb{C}$  by analytic continuation in which the Riemann zeta function is satisfied the functional equation<sup>[15]</sup>

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4.6)$$

for all  $s \in \mathbb{C}$  except a simple pole at  $s = 1$ .

Equation (4.6) holds significant importance in the realm of analytic continuation. Its fundamental property is the central to modern number theory. It establishes a profound symmetry within the function, connecting its values at a complex number  $s$  to those at  $1-s$ . This step is crucial as it enables mathematicians to study the Riemann zeta function across all complex values, including those within the 'critical strip' where the non-trivial zeros are located. Additionally, it serves to link the 'trivial zeros' at the negative even integers ( $-2, -4, -6, \dots$ ) to the pole of the Gamma function. Essentially, it unifies the trivial and non-trivial zeros of the function, offering a comprehensive view of its zero distribution. Moreover, it provides the means to the extension of the Riemann zeta function to the entire complex plane, unveiling the profound relationship between its values and its zeros, which are intricately connected to the distribution of prime numbers. In contemporary times, the Riemann zeta function is recognized as a meromorphic function, a concept that was significantly advanced in the early 20th century through the contributions of Finnish mathematician Rolf Nevanlinna<sup>[16]</sup>. Overall, equation (4.6) is fundamental to the study of the Riemann zeta function and is essential for understanding the distribution of its zeros, particularly concerning the Riemann Hypothesis. Conversely, the non-trivial zeros are exclusively found in the  $\zeta(s)$  and  $\zeta(1-s)$  functions, where the gamma function enables us to define the zeta function for all other complex numbers, except of a simple pole at  $s \neq 1$ .

**We notice that the other factors  $2^s \pi^{s-1} \sin(\pi s / 2)$  cannot yield non-trivial zeros within the critical strip ( $0 < \text{Re}(s) < 1$ ), which we demonstrate in the following proof.**

### **Proof**

The product factors  $2^s \pi^{s-1} \sin(\pi s / 2)$  have no nontrivial zeros:

1.  $2^s > 0$  for all  $s$ .
2.  $\pi^{s-1} > 0$  for all  $s$ .
3.  $\sin(\pi s/2)$  has trivial zeros and no nontrivial zeros in the critical strip. Indeed, we have

a. Trivial Zeros:

$$\sin\left(\frac{\pi s}{2}\right) = 0 \Rightarrow \frac{\pi s}{2} = n\pi \Rightarrow s = 2n, \quad n \in \mathbb{Z}.$$

b. No Nontrivial Zeros:

Let  $s = x + iy$ , with  $0 < x < 1$ . Then

$$\begin{aligned} \sin\left(\frac{\pi}{2}(x+iy)\right) &= \sin\left(\frac{x}{2}\right)\cos\left(\frac{i\pi y}{2}\right) + \sin\left(\frac{i\pi y}{2}\right)\cos\left(\frac{\pi x}{2}\right) \\ &= \sin\left(\frac{x}{2}\right)\cosh\left(\frac{\pi y}{2}\right) + i\sinh\left(\frac{\pi y}{2}\right)\cos\left(\frac{\pi x}{2}\right). \end{aligned}$$

We are looking for  $x$  and  $y$  such that

$$\sin\left(\frac{\pi}{2}(x+iy)\right) = 0 \text{ for } 0 < x < 1/2.$$

This happens only if both real and imaginary parts vanish:

$$\begin{cases} \sin\left(\frac{x}{2}\right)\cosh\left(\frac{\pi y}{2}\right) = 0 \\ \sinh\left(\frac{\pi y}{2}\right)\cos\left(\frac{\pi x}{2}\right) = 0 \end{cases}$$

But we have

$$* \quad \sin\left(\frac{\pi x}{2}\right) = 0 \text{ only if } x = 0 \quad \text{and} \quad \cosh\left(\frac{\pi y}{2}\right) > 0 \text{ for all } y$$

$$* \quad \sinh\left(\frac{\pi y}{2}\right) = 0 \text{ only if } y = 0 \quad \text{and} \quad \cos\left(\frac{\pi x}{2}\right) = 0 \text{ only if } x = 1.$$

Therefore, we see that both real and imaginary parts cannot be vanish because  $x$  cannot be both 0 and 1 simultaneously. Note that  $x = 0$  or  $x = 1$  is not in the interval  $(0, 1/2)$ . Thus, we conclude that none of the factors  $2^s \pi^{s-1} \sin(\pi s / 2)$  vanish within the critical strip defined by  $0 < x < 1$ .

#### 4-1-6. Properties of Riemann Zeta Function (Known Facts)

We outline the significant results and properties related to the Riemann zeta function that mathematicians have explored over the years. We then assign these properties specific names to facilitate easy referencing purpose throughout this chapter.

i. **Zeta zero property:** The zeros of the Riemann zeta function are classified into trivial zeros and nontrivial zeros. The trivial zeros are found on the left half-plane at  $s = -2n$  for positive integer  $n$  while the nontrivial zeros has been conjectured in the form of  $1/2 + it$  for real  $t$ <sup>[17]</sup>.

ii. **Zeta conjugate and symmetry property:**  $\zeta(s)$  can be extended to a meromorphic function on the entire complex plane, having a simple pole at  $s = 1$ . It satisfies the complex conjugate identity,  $\overline{\zeta(s)} = \zeta(\bar{s}) \Leftrightarrow \zeta(s) = \overline{\zeta(\bar{s})}$ , which demonstrates  $\zeta(s)$  exhibits a symmetry with respect to the conjugate pairs<sup>[18]</sup>.

This follows that if  $s$  is a nontrivial zero then so is  $\bar{s}$  [See **Appendix 4-A-2**], indicating symmetry with respect to the real axis.  $\zeta(s)$  also satisfies relational equation (4.6), which shows that  $\zeta(s)$  and  $\zeta(1-s)$  are related, indicating symmetry about the real line  $\text{Re}(s) = 1/2$ . **Indeed, if  $\zeta(s_0) = 0$  (assuming  $s_0$  exists), then  $\zeta(1-s_0) = 0$ ; established results show that nontrivial zeros occur in quadruples:**  $s_0, \bar{s}_0, 1-s_0,$  and  $1-\bar{s}_0$ .

iii. **Riemann xi function:** Edmund Landau made significant contributions to the study of the Riemann zeta function and its analytic properties. Among these contributions is his use of the Riemann xi function, a reformulation that streamlines the analysis of the zeta function's zeros and makes its inherent symmetries more transparent. The xi function is defined by

$$\xi(s) = \frac{1}{2} \pi^{-s/2} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad (4.6 a)$$

and it satisfies the functional equation  $\xi(s) = \xi(1-s)$ .

From this definition, one sees that  $\xi(s)$  has no zeros in the half-planes  $\text{Re}(s) > 1$  or  $\text{Re}(s) < 0$ , since  $\zeta(s)$  has no zeros there and the remaining factors introduce no additional zeros. Moreover, the identity  $\overline{\xi(s)} = \xi(\bar{s})$ , inherits the same symmetry structure as the Riemann zeta function. In particular, its zeros are symmetric with respect to both the real axis<sup>[19]</sup> and the critical line  $\text{Re}(s) = 1/2$ , reflecting the deep analytic symmetries underlying the zeta function itself.

- iv. **Critical strip property:** In 1896, Jacques Hadamard and Charles-Jean de la Vallée Poussin<sup>[20]</sup> independently proved that all nontrivial zeros of the Riemann zeta function lie within the critical strip where  $0 < \text{Re}(s) < 1$ , after demonstrating there are no nontrivial zeros of the Riemann zeta function for  $\text{Re}(s) = 1$ .
- v. **Critical line zero property:** In 1914, Hardy proved that there are infinitely many zeros of the Riemann zeta function on the critical line  $\text{Re}(s) = 1/2$ <sup>[21]</sup>.
- vi. **Special Values:** By (4.6), we easily derive some special values:
  - $\zeta(-1) = -\frac{1}{12}$
  - $\zeta(0) = -\frac{1}{2}$
  - $\zeta(1) = \infty$
  - Other specific values:  $\zeta(2) = \pi^2/6$  (Euler),  $\zeta(4) = \pi^4/90$ <sup>[22]</sup>.

We have outlined some essential properties of the Riemann zeta function and have inevitably left out much of the extensive history context and the significant contributions of other mathematicians. For our objectives, we focus on the essential properties and pivotal contributions that enable us to present a comprehensive understanding of the Riemann zeta function, allowing us to explore into its mysteries.

### 4-1-7. Riemann Hypothesis

The Riemann hypothesis, introduced by the German mathematician Bernhard Riemann in his paper titled *On the Number of Primes Less Than a Given Quantity*, published in August 1859, conjectures that the Riemann zeta function  $\zeta(s)$  equals zero when the real part of  $s$  is  $\frac{1}{2}$ . In other words, *all nontrivial zeros of the Riemann zeta function lie on the critical line where the real part of  $s$  is  $\frac{1}{2}$*  <sup>[23]</sup>. The Riemann Hypothesis is one of the seven "Millennium Prize Problems" established by the Clay Mathematics Institute in 2000 <sup>[24]</sup>. It remains unsolved and mystery at the time of writing this book and continues to be a central topic in mathematical research. *Even though we know that there are infinitely many nontrivial zeros of the Riemann zeta function on the critical line  $\text{Re}(s) = \frac{1}{2}$ , the Riemann Hypothesis remains an open question regarding whether all nontrivial zeros lie exactly on  $\text{Re}(s) = \frac{1}{2}$  or if there are zeros elsewhere in the critical strip.*

#### Key Points on Zeros of the Riemann Zeta Function

To summarize, the Riemann zeta function is defined for all complex numbers ( $\mathbb{C}$ ) except at  $s = 1$ . The solutions to the equation  $\zeta(s) = 0$  are categorized into trivial zeros and nontrivial zeros. The trivial zeros occur when  $s$  is a negative even integer. The nontrivial zeros are proven to lie within the critical strip  $0 < \text{Re}(s) < 1$ . All nontrivial zeros are of the form  $\sigma \pm i\tau$ , where  $\sigma$  and  $\tau$  are real. One important property is the symmetry of the nontrivial zeros, which appear in conjugate pairs. In other words,  $s$  and the complex conjugate  $\bar{s}$  are symmetrically placed about the real axis where  $\text{Im}(s) = 0$ . Hardy<sup>[21]</sup> proved that there are infinitely many nontrivial zeros on the critical line where  $\text{Re}(s) = \frac{1}{2}$ . However, it remains undetermined whether all nontrivial zeros lie on this critical line. In 2000, the Clay Mathematics Institute proposed the Riemann Hypothesis, which conjectures that all nontrivial zeros of the Riemann zeta function have a real part of  $\frac{1}{2}$ . This hypothesis still requires a formal mathematical proof.

Up to this point, we have known the key properties of the Riemann zeta function, including the well-established result that no nontrivial zeros exist outside the critical strip. In the following sections, our analysis will only concentrate on the domain  $0 < \text{Re}(s) < 1$ , where we want to prove that there are no nontrivial zeros exist anywhere else in the critical strip except the critical line  $\text{Re}(s) = \frac{1}{2}$ .

## Section 4-2. Unraveling the Mystery of Riemann Hypothesis (First Strategy)

In this section we attempt to unravel the mystery of the Riemann Hypothesis by revising the Riemann zeta functional equation to include only the zeta functions, excluding the Gamma function. We demonstrate that the nontrivial zeros with each of its real part are located on  $\text{Re}(s) = 1/2$ . We then examine the variations of the real part of  $s$  and apply the **Zeta conjugate and symmetry property** to show that a pair of the Riemann zeta functions alone cannot have nontrivial zeros in the intervals  $(0, 1/2)$  or  $(1/2, 1)$ . This leads to the proof *that all nontrivial zeros lie exactly on the critical line  $\text{Re}(s) = 1/2$ , and there are no zeros elsewhere in the critical strip.*

### 4-2-1. Rewrite the Riemann Zeta Functional Equation

Setting aside of the trivial zeros, the Riemann zeta function has a set of four symmetrical nontrivial zeros, while the relational equation (4.6) involves a pair of zeta functions, namely  $\zeta(s)$  and  $\zeta(1-s)$ . This structure indicates a contradiction with the existence of four simultaneous nontrivial zeros. Consequently, we are led to seek an alternative form of equation (4.6) so that it contains four zeta functions, or two pair of zeta functions. We need to rewrite equation (4.6) to another form without gamma function, ensuring it exclusively involved the zeta and other elementary functions. This is a key point that our aim increases the occurrence of the zeta function within the domain  $0 < \text{Re}(s) < 1$ . Replacing  $s$  with  $-s$  in (4.6) gives

$$\zeta(-s) = -2^{-s} \pi^{-s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1+s) \zeta(1+s). \quad (4.7)$$

By multiplying (4.6) with (4.7) and simplifying, which gives

$$\zeta(s) \zeta(-s) = -\pi^{-2} \sin^2\left(\frac{\pi s}{2}\right) \Gamma(1-s) \Gamma(1+s) \zeta(1+s) \zeta(1-s). \quad (4.8)$$

On the other hand, we have the Euler's reflection formula,

$$\Gamma(s) \Gamma(-s) = -\frac{\pi}{s \sin \pi s}.$$

By multiplying both its sides with  $s^2$  and applying the Gamma recurrence identity,  $\Gamma(1+s)=s\Gamma(s)$ , the Euler's reflection formula can be rewritten as

$$\Gamma(s+1)\Gamma(1-s)=\frac{s\pi}{\sin \pi s}.$$

By substituting the last expression in (4.8) and simplifying, we obtain a rewritten form of the zeta functional equation as

$$\zeta(s)\zeta(-s)=-\frac{s}{2\pi}\tan\left(\frac{\pi s}{2}\right)\zeta(1+s)\zeta(1-s). \tag{4.9}$$

We will utilize (4.9) to determine the real part of nontrivial zeros. For clarity in the subsequent subsections, we will express (4.9) as being equivalent to one of the following equations.

$$\left\{ \begin{array}{l} \zeta(s)\zeta(-s)\cos\left(\frac{\pi s}{2}\right)=-\frac{s}{2\pi}\sin\left(\frac{\pi s}{2}\right)\zeta(1+s)\zeta(1-s) \tag{4.9a} \\ \text{or} \\ \zeta(s)\zeta(-s)\cos\left(\frac{\pi s}{2}\right)+\frac{s}{2\pi}\sin\left(\frac{\pi s}{2}\right)\zeta(1+s)\zeta(1-s)=0 \tag{4.9b} \end{array} \right.$$

Equation (4.9a) or (4.9b) is the analytic continuation in the entire complex plane  $\mathbb{C}$ , and it has simple poles  $-1, 0, 1$ , and simple zeros (known as trivial zeros) for all integers. To summarize these known results, we utilize Table I below to analyze the trivial zeros, nontrivial zeros and poles of all the functions in equation (4.9b) as follows.

<b>Table I</b>			
<b>Function</b>	<b>Trivial zeros</b>	<b>Nontrivial zeros</b>	<b>Poles</b>
$\zeta(s)$	$s = -2, -4, -6, \dots$ (negative even integers)	Yes	$s = 1$
$\zeta(-s)$	$s = 2, 4, 6, \dots$ (positive even integers)	Yes	$s = -1$
$\cos\left(\frac{s\pi}{2}\right)$	Odd integers	No	No

s	s = 0	No	No
$\sin\left(\frac{s\pi}{2}\right)$	Even integers	No	No
$\zeta(1+s)$	s = -1, -3, -5, ... (negative odd integers)	Yes	s = 0
$\zeta(1-s)$	s = 1, 3, 5, ... (positive odd integers)	Yes	s = 0

In particular, we note that at  $s = -1, 0, 1$ , and by applying  $\zeta(-1) = -1/12$ ,  $\zeta(0) = -1/2$ ,  $\zeta(2) = \pi^2/6$ , equation (4.9a) gives some special limits of the following expressions:

$$(a) \lim_{s \rightarrow -1} \left\{ \zeta(s) \zeta(-s) \cos\left(\frac{\pi s}{2}\right) \right\} = -\frac{s}{2\pi} \sin\left(\frac{\pi s}{2}\right) \zeta(1+s) \zeta(1-s) = -\left(-\frac{1}{2\pi}\right) \left(-\sin\left(\frac{\pi}{2}\right)\right) \zeta(0) \zeta(2) = \frac{\pi}{24}$$

$$(b) \lim_{s \rightarrow 0} \left\{ s \sin\left(\frac{\pi s}{2}\right) \zeta(1+s) \zeta(1-s) \right\} = -2\pi \zeta(s) \zeta(-s) \cos\left(\frac{\pi s}{2}\right) = -2\pi \zeta(0) \zeta(0) \cos(0) = -\frac{\pi}{2}$$

$$(c) \lim_{s \rightarrow 1} \left\{ \zeta(s) \zeta(-s) \cos\left(\frac{\pi s}{2}\right) \right\} = -\frac{s}{2\pi} \sin\left(\frac{\pi s}{2}\right) \zeta(1+s) \zeta(1-s) = -\left(\frac{1}{2\pi}\right) \left(\sin\left(\frac{\pi}{2}\right)\right) \zeta(2) \zeta(0) = \frac{\pi}{24}$$

(d) Substitute  $\zeta(-1) = -1/12$  in (a) and (c) above gives

$$\lim_{s \rightarrow -1} \left\{ \zeta(-s) \cos\left(\frac{s\pi}{2}\right) \right\} = \lim_{s \rightarrow 1} \left\{ \zeta(s) \cos\left(\frac{s\pi}{2}\right) \right\} = -\frac{\pi}{2}.$$

#### 4-2-2. Riemann Zeta Functional Equation in Complex Form

Let  $\sigma$  and  $\tau$  be real numbers. Substituting  $s = \sigma + i\tau$  in (4.9) gives

$$\zeta(\sigma + i\tau) \zeta(-\sigma - i\tau) = -\frac{1}{2\pi} (\sigma + i\tau) \tan\left(\frac{(\sigma + i\tau)\pi}{2}\right) \zeta(1 + \sigma + i\tau) \zeta(1 - \sigma - i\tau). \quad (4.10a)$$

Equation (4.10a) can be expressed in terms of the cosine and sine functions

$$\zeta(\sigma+i\tau)\zeta(-\sigma-i\tau)\cos\left(\frac{(\sigma+i\tau)\pi}{2}\right)=-\frac{\sigma+i\tau}{2\pi}\sin\left(\frac{(\sigma+i\tau)\pi}{2}\right)\zeta(1+\sigma+i\tau)\zeta(1-\sigma-i\tau). \quad (4.10b)$$

Equation (4.10a) or (4.10b) unveils an intriguing aspect regarding the locations of the nontrivial zeros of the zeta functions when we analyze the variations of  $\text{Re}(s)$  of  $s$ . In the following sections, we use this functional equation to provide significant understanding of the impact on  $\text{Re}(s)$  of  $s$  of each zeta function and its corresponding factor it holds in the critical strip. It is sufficient to analyze three separate scenarios of  $\text{Re}(s)$ :

- i.  $\text{Re}(s) = \sigma = 1/2$  – In this scenario, we illustrate that the Riemann zeta function possesses nontrivial zeros at this value.
- ii.  $0 < \text{Re}(s) = \sigma < 1/2$  – In this scenario, we indicate that no zeta functions within (4.10a) or (4.10b) exhibit nontrivial zeros.
- iii.  $1/2 < \text{Re}(s) = \sigma < 1$  – Similar to (ii), no zeta functions display nontrivial zeros.

These three scenarios demonstrate the positions of the nontrivial zeros of the zeta functions in (4.10a) or (4.10b). As a result, the conclusions drawn from this examination can also be used to support the Riemann Hypothesis.

### 4-2-3. Nontrivial Zeros on the Critical Line $\text{Re}(s) = 1/2$

#### First Method

We seek to determine  $\sigma$  in equation (4.10a) such that the expression is symmetric with respect to real axis and is consistent with conjugate symmetry. There are two possible cases to consider:

**(1) Case 1:** The real part of  $s = \sigma + i\tau$  appearing in  $\zeta(\sigma + i\tau)$  on the left-hand side of (4.10a) is equal to the real part of  $1-s$  appearing in  $\zeta(1-\sigma-i\tau)$  on the right-hand side. This follows that

$$\sigma = 1 - \sigma \quad \Rightarrow \quad \sigma = \frac{1}{2}$$

**(2) Case 2:** The real part of  $-s$  appearing in  $\zeta(-\sigma-i\tau)$  on the left-hand side of

(4.10) is equal to the real part of  $1+s$  appearing in  $\zeta(1+\sigma+i\tau)$  on the right-hand side. This condition yields

$$-\sigma = 1 + \sigma \quad \Rightarrow \quad \sigma = \frac{1}{2}.$$

Both cases lead to the same conclusion:  $\sigma = 1/2$ . Substituting this value in (4.10a) gives

$$\zeta\left(\frac{1}{2}+i\tau\right)\zeta\left(-\frac{1}{2}-i\tau\right)\cos\left(\left(\frac{1}{2}+i\tau\right)\frac{\pi}{2}\right) = -\frac{1}{2\pi}\left(\frac{1}{2}+i\tau\right)\sin\left(\left(\frac{1}{2}+i\tau\right)\frac{\pi}{2}\right)\zeta\left(\frac{3}{2}+i\tau\right)\zeta\left(\frac{1}{2}-i\tau\right) \quad (4.11)$$

which is equivalent to

$$\zeta\left(\frac{1}{2}+i\tau\right)\zeta\left(-\frac{1}{2}-i\tau\right)\cos\left(\left(\frac{1}{2}+i\tau\right)\frac{\pi}{2}\right) + \frac{1}{2\pi}\left(\frac{1}{2}+i\tau\right)\sin\left(\left(\frac{1}{2}+i\tau\right)\frac{\pi}{2}\right)\zeta\left(\frac{3}{2}+i\tau\right)\zeta\left(\frac{1}{2}-i\tau\right) = 0 \quad (4.12)$$

We emphasize that equation (4.11) states that both sides are equal for all values of  $\tau$ ; equivalently, equation (4.12) is always equal to zero for all values of  $\tau$ . This situation can arise only under two possible scenarios:

- i. None of the specific functions in (4.12) are equal to zero. In this case, it is always true because functional equation (4.11) is true for all real  $\tau$  in the entire complex plane  $\mathbb{C}$  by analytic continuation, but this scenario does not indicate which zeta function in (4.11) has nontrivial zeros.
- ii. At least one of the functions on the left-hand of (4.11) and one of the functions on the right-hand of (4.11) is equal to zero. Since we know  $\zeta(s)$  has infinite nontrivial zeros on  $\text{Re}(s) = 1/2$  by Property (v) in Section 4-1-6, there must exist certain real  $\tau = \tau_k$  for  $k \in \mathbb{N}$  such that  $\zeta(1/2+i\tau_k) = 0$  and  $\zeta(1/2-i\tau_k) = 0$ . To prove for this assertion, we provide the analysis possible zeros of each member function in (4.11), as shown in the following table:

<b>Table II</b>	
<b>Left-hand of equation (4.11)</b>	<b>Right-hand of equation (4.11)</b>

<b>Function:</b>	$\zeta\left(\frac{1}{2}+i\tau_k\right)$	$\zeta\left(-\frac{1}{2}-i\tau_k\right)$	$\cos\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right)$	$\frac{1}{2}+i\tau_k$	$\sin\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right)$	$\zeta\left(\frac{3}{2}+i\tau_k\right)$	$\zeta\left(\frac{1}{2}-i\tau_k\right)$
<b>Nontrivial zeros (1*)</b>	<b>Yes</b> because $\text{Re}(s) \in (0,1)$	<b>No</b> because $\text{Re}(s) \notin (0,1)$	<b>No</b>	<b>No</b>	<b>No</b>	<b>No</b> because $\text{Re}(s) \notin (0,1)$	<b>Yes</b> because $\text{Re}(s) \in (0,1)$
<b>Trivial zeros (2*)</b>	<b>No</b> because $\text{Re}(s) > -1$	<b>No</b> because $\text{Re}(s) > -1$	<b>No</b>	<b>No</b>	<b>No</b>	<b>No</b> because $\text{Re}(s) > -1$	<b>No</b> because $\text{Re}(s) > -1$

**Notes:**

**+> Nontrivial zeros (1\*):** By applying Property (iv) (Critical strip property) in Section 4-1-6, all nontrivial zeros of the Riemann zeta function lie within the critical strip where  $0 < \text{Re}(s) < 1$ .

**+> Trivial zeros (2\*):** By applying Property (i) (Zeta zero property) in Section 4-1-6, The trivial zeros of the Riemann zeta function are found on the left half-plane at  $s = -2n$  for positive integers  $n$ .

The results from Table II show that

$$\zeta\left(-\frac{1}{2}-i\tau_k\right) \neq 0,$$

$$\cos\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right) \neq 0,$$

$$-\frac{1}{2}+i\tau_k \neq 0,$$

$$\sin\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right) \neq 0,$$

and

$$\zeta\left(\frac{3}{2}+i\tau_k\right) \neq 0.$$

It follows that the left-hand side of equation (4.11) equals zero if

$$\zeta\left(\frac{1}{2}+i\tau_k\right) = 0 \text{ for certain real } t_k \text{ and } k \in \mathbb{N}. \quad (4.12a)$$

By applying Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6, it follows that

$$\zeta\left(\frac{1}{2}-i\tau_k\right)=0, \quad (4.12b)$$

from which it makes the right-hand side of equation (4.11) also equals to zero and vice versa. Thus, we conclude in this scenario (ii) that both  $s=1/2+i\tau_k$  and  $s=1/2-i\tau_k$  represent the roots of the Riemann zeta equation,  $\zeta(s) = 0$ . Note that the Property (ii) still holds since  $\text{Re}(s) = 1/2$  (or on the critical line), two pairs of zeta functions reduce to one pair because they are identical.

## Second Method

Another approach that we can derive  $\zeta\left(\frac{1}{2}+i\tau_k\right)=0$  and  $\zeta\left(\frac{1}{2}-i\tau_k\right)=0$  by setting both sides of (4.11) equal to zero at  $\tau=\tau_k$  for  $k \in \mathbb{N}$ . Then solve for  $s$ , ensuring that the value  $\tau_k$  must satisfy both sides and make them equal to zero. To proceed, we have

1. The left-hand side of equation (4.11) is

$$\zeta\left(\frac{1}{2}+i\tau_k\right)\zeta\left(-\frac{1}{2}-i\tau_k\right)\cos\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right)=0. \quad (*)$$

Since  $\zeta\left(-\frac{1}{2}-i\tau_k\right)\neq 0$  and  $\cos\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right)\neq 0$ , according to Table II, equation (\*) gives

$$\zeta\left(\frac{1}{2}+i\tau_k\right)=0.$$

2. The right-hand side of equation (4.11) is:

$$-\frac{1}{2\pi}\left(\frac{1}{2}+i\tau_k\right)\sin\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right)\zeta\left(\frac{3}{2}+i\tau_k\right)\zeta\left(\frac{1}{2}-i\tau_k\right)=0. \quad (**)$$

Again, since  $\left(\frac{1}{2}+i\tau_k\right)\neq 0$ ,  $\sin\left(\left(\frac{1}{2}+i\tau_k\right)\frac{\pi}{2}\right)\neq 0$ , and  $\zeta\left(\frac{3}{2}+i\tau_k\right)\neq 0$ , according to Table II, equation (\*\*) gives

$$\zeta\left(\frac{1}{2}-i\tau_k\right)=0.$$

Hence, we find  $\zeta\left(\frac{1}{2}+i\tau_k\right)=0$  and  $\zeta\left(\frac{1}{2}-i\tau_k\right)=0$  for real  $\tau_k$ . It follows that both  $s=\frac{1}{2}+i\tau_k$  and  $s=\frac{1}{2}-i\tau_k$  must be the roots of the Riemann zeta equation,  $\zeta(s) = 0$ , corresponding to the **Zeta conjugate and symmetry property**. *This result aligns with Property (iv) (the "critical line zero" property) in Section 4-1-6, which refers to Hardy's theorem establishing that the Riemann zeta function has infinitely many zeros on the critical line  $\text{Re}(s) = 1/2$ .*

**However, at this point, we still do not determine whether all nontrivial zeros of  $\zeta(s)$  have a real part of  $1/2$ . To finalize the demonstration, we need to examine the variations of the real part of  $s$  where  $0 < \text{Re}(s) < 1/2$  and  $1/2 < \text{Re}(s) < 1$ , and prove that all zeta functions in (4.10a) have no nontrivial zeros in these two intervals.**

We use equation (4.6) to show the nontrivial zeros on  $\text{Re}(s) = 1/2$  [See **Appendix 4-A-1**], but this approach does not provide substantial support for the proof due to the lack of a corresponding pair of zeta functions, as we will observe in the next section when examining their real parts within the intervals  $0 < \text{Re}(s) < 1/2$  or  $1/2 < \text{Re}(s) < 1$ . Now, we propose an **ideal relational equation** involving four distinct zeta functions ( $Z_1, Z_2, Z_3, Z_4$ ) or two pairs of zeta functions so that their nontrivial zeros always travel in a quartet of  $s_0, \bar{s}_0, 1-s_0$ , and  $1-\bar{s}_0$ . In this ideal relational equation, the quartet is not contained with one function, but is instead distributed across the group. Does there exist a mathematical identity<sup>(4.12c)</sup> where:

- $Z_1$  vanishes at  $s_0$ ,
- $Z_2$  vanishes at  $\bar{s}_0$ ,
- $Z_3$  vanishes at  $1-s_0$ ,
- $Z_4$  vanishes at  $1-\bar{s}_0$  ?

Next, we examine all relational identities involving the four zeta function evaluations we analyze all relational equations connecting to four values of  $\zeta(s)$ ,  $\zeta(\bar{s}_0)$ ,  $1-s_0$  and  $\zeta(1-\bar{s}_0)$  and determine whether any such identity<sup>(4.12c)</sup> can hold only when each of these functions vanishes. If such a relation exists, then its solutions necessarily correspond to nontrivial zeros of the zeta function, making it a direct criterion for identifying candidates relevant to the Riemann Hypothesis.

#### 4-2-4. Examining Zeros Off the Critical Line

##### Case 1: No Zeta Functions Within (4.10) Have Nontrivial Zeros for $0 < \sigma < 1/2$

We examine the behavior of each function in equation (4.10a) for  $\text{Re}(s)$  in the interval  $(0, 1/2)$  to determine whether it has zeros.

- (a)  $\zeta(\sigma+i\tau)$ : For  $0 < \sigma < 1/2$ ,  $\zeta(\sigma+i\tau)$  is anticipated to have nontrivial zeros. We will shortly return to this point to demonstrate that  $\zeta(\sigma+i\tau)$  alone, without having  $\zeta(\sigma-i\tau)$ , does not have nontrivial zeros in this region.
- (b)  $\zeta(-\sigma+i\tau)$ : For  $0 < \sigma < 1/2$ , the real part of  $\zeta(-\sigma+i\tau)$  lies within  $-1/2 < -\sigma < 0$ , indicating that  $\zeta(-\sigma+i\tau)$  has no nontrivial zeros.
- (c)  $\cos((\sigma + i\tau)\pi/2)$ : It has no zero for  $0 < \sigma < 1/2$ .
- (d)  $(\sigma + i\tau)$ : It has no zero for  $0 < \sigma < 1/2$ .
- (e)  $\sin((\sigma + i\tau)\pi/2)$ : It has no zero for  $0 < \sigma < 1/2$ .
- (f)  $\zeta(1+\sigma+i\tau)$ : For  $0 < \sigma < 1/2$ , the real part of  $\zeta(1+\sigma+i\tau)$  falls within  $1 < 1+\sigma < 3/2$ , indicating that  $\zeta(1+\sigma+i\tau)$  has no nontrivial zeros.
- (g)  $\zeta(1-\sigma-i\tau)$ : For  $0 < \sigma < 1/2$ , the real part of  $\zeta(1-\sigma-i\tau)$  falls within  $1/2 < 1-\sigma < 1$ , and  $\zeta(1-\sigma-i\tau)$  is anticipated to have nontrivial zeros.

By combining the results from (a) to (g), we find none of them having nontrivial zeros except (a) that  $\zeta(\sigma+i\tau)$  has nontrivial zeros for  $0 < \sigma < 1/2$ , and (g) that  $\zeta(1-\sigma-i\tau)$  has nontrivial zeros for  $1/2 < 1-\sigma < 1$ . However, no nontrivial zeros can satisfy equation (4.10a); it is limited to merely two zeta functions while suggesting four. These nontrivial zeros cannot be established for a pair of both  $\zeta(s)$  and  $\zeta(1-s)$  when we are investigating the range  $0 < \sigma < 1/2$ , as their real parts exist in separate intervals, specifically,  $0 < \sigma < 1/2$  and  $1/2 < 1-\sigma < 1$ . The simultaneous presence of all four zeros (or two pairs) cannot be reliably integrated within the framework of

equation (4.10a) due to the absence of zeros from one pair of zeta functions in the conjugate forms  $\sigma-i\tau$  and  $1-\sigma+i\tau$  **(as shown in {(4.12a) and (4.12b)} or (4.12c) where four distinct zeta functions vanish at  $s_0, \bar{s}_0, 1-s_0$  and  $1-\bar{s}_0$ , for example)**. These results do not follow Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6 in which if one nontrivial zero is located on a certain interval in the critical region, there will always be a set of nontrivial zeros that is its mirror image across the real axis and the critical line on that same interval **(see Key Note [3\*])**. Consequently, no zeta functions in the functional equation (4.10a) have nontrivial zeros for  $0 < \sigma < 1/2$ .

### Key Note [3\*]

We will come back this matter in Section 4-4 by providing a robust proof that  $\zeta(\sigma+i\tau)$  and  $\zeta(1-\sigma-i\tau)$  cannot both be zero simultaneously, whether  $0 < \sigma < 1/2$  or  $1/2 < 1-\sigma < 1$ , because of the nonexistent of such zeros. We proceed to demonstrate our approach in examining the distribution of nontrivial zeros found on four zeta functions within the context of the zeta functional equation.

### Case 2: No Zeta Functions Within (4.10) Have Nontrivial Zeros for $1/2 < \sigma < 1$

Similarly, we examine and analyze the behavior of each function in equation (4.10a) for  $\text{Re}(s)$  in the interval  $(1/2, 1)$  to determine whether it has zeros.

- ◆ **(a)**  $\zeta(\sigma+i\tau)$ : For  $1/2 < \sigma < 1$ ,  $\zeta(\sigma+i\tau)$  is expected to have nontrivial zeros.
- ◆ **(b)**  $\zeta(-\sigma+i\tau)$ : For  $1/2 < \sigma < 1$ , the real part of  $\zeta(-\sigma+i\tau)$  lies within  $-1 < -\sigma < -1/2$ , and  $\zeta(-\sigma+i\tau)$  has no nontrivial zeros.
- ◆ **(c)**  $\cos((\sigma + i\tau)\pi/2)$ : It has no zero for  $1/2 < \sigma < 1$ .
- ◆ **(d)**  $(\sigma+i\tau)$ : It has no zero for  $1/2 < \sigma < 1$ .
- ◆ **(e)**  $\sin((\sigma + i\tau)\pi/2)$ : It has no zero for  $1/2 < \sigma < 1$ .
- ◆ **(f)**  $\zeta(1+\sigma+i\tau)$ : For  $1/2 < \sigma < 1$ , the real part of  $\zeta(1+\sigma+i\tau)$  falls within  $3/2 < 1+\sigma < 2$ , indicating that  $\zeta(1+\sigma+i\tau)$  has no nontrivial zeros.
- ◆ **(g)**  $\zeta(1-\sigma-i\tau)$ : For  $1/2 < \sigma < 1$ , the real part of  $\zeta(1-\sigma-i\tau)$  lies within  $0 < 1-\sigma < 1/2$ , and  $\zeta(1-\sigma-i\tau)$  is anticipated to have nontrivial zeros.

By combining the results from (a) to (g), we expect that only  $\zeta(\sigma+i\tau)$  has nontrivial

zeros for  $\frac{1}{2} < \sigma < 1$ , and  $\zeta(1-\sigma-it)$  has nontrivial zeros for  $0 < 1-\sigma < \frac{1}{2}$ . Similar to Case 1, these nontrivial zeros cannot be established for both (a) and (g) because they do not conform to (4.12c) and Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6 due to the absence of zeros from one pair of zeta functions in the conjugate forms. **Thus, none of the zeta functions in equation (4.10a) have nontrivial zeros for  $\frac{1}{2} < \text{Re}(s) < 1$ .**

#### 4-2-5. Conclusion

The analysis results derived from both Case 1 and Case 2 indicate that none of the zeta functions in the functional equation (4.10a) have nontrivial zeros for the intervals  $0 < \text{Re}(s) < \frac{1}{2}$  or  $\frac{1}{2} < \text{Re}(s) < 1$  because they do not conform to Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6. We conclude that all nontrivial zeros occur when  $\text{Re}(s) = \frac{1}{2}$  as shown in Section 4-2-3, and no nontrivial zeros are found off the critical line.

### Section 4-3. Unraveling the Mystery of Riemann Hypothesis Using Alternative Form (Second Strategy)

We provide the second proof of Riemann Hypothesis using a simpler zeta functional equation, which does not involve the trigonometric tangent function. In this second strategy, the proof process remains similar to that in Section 4-2, but it employs a modified version of the zeta functional equation. The purpose of this strategy is to examine how nontrivial zeros influence the critical region and its surrounding areas, particularly whether there exists a value of  $s$  that satisfies all four  $\zeta$  functions defined in the alternative form of the zeta relational equation. This alternative form can be derived as follows:

- Replacing  $s$  by  $s-1$  in (4.9) gives

$$\begin{aligned}\zeta(s-1)\zeta(1-s) &= -\frac{s-1}{2\pi} \tan\left(\frac{(s-1)\pi}{2}\right) \zeta(s)\zeta(2-s) \\ &= \frac{s-1}{2\pi} \cot\left(\frac{s\pi}{2}\right) \zeta(s)\zeta(2-s)\end{aligned}\tag{4.13}$$

- Multiplying equations (4.9) and (4.13) and simplifying give the new form of

the zeta functional equation

$$\zeta(s-1)\zeta(-s) = \frac{s(1-s)}{4\pi^2} \zeta(1+s)\zeta(2-s). \quad (4.14)$$

Equation (4.14) without tangent function is similar to equation (4.9), but much simpler. It suggests that the zeta function must incorporate trigonometric function such as tangent, cosine, or sine. This inclusion is a crucial for the zeta function to have trivial zeros at negative even integers.

To proceed, substituting  $s = \sigma + i\tau$ , where  $\sigma$  and  $\tau$  are real, in (4.14) gives

$$\zeta(-1+\sigma+i\tau)\zeta(-\sigma-i\tau) = \frac{1}{4\pi^2}(\sigma+i\tau)(1-\sigma-i\tau)\zeta(1+\sigma+i\tau)\zeta(2-\sigma-i\tau). \quad (4.15)$$

Or it is equivalent to

$$\zeta(-1+\sigma+i\tau)\zeta(-\sigma-i\tau) - \frac{1}{4\pi^2}(\sigma+i\tau)(1-\sigma-i\tau)\zeta(1+\sigma+i\tau)\zeta(2-\sigma-i\tau) = 0. \quad (4.16)$$

Next, we want to determine  $\sigma$  such that the nontrivial zeros of the zeta functions in equation (4.15) or (4.16) have a symmetric about the real axis and respect to conjugate pairs. There are two possible scenarios that can be happened:

- **First scenario:** The real part of  $s$  in  $\zeta(-1+\sigma+i\tau)$  from the left-hand (4.15) is equal to the real part of  $s$  in  $\zeta(2-\sigma-i\tau)$  from the right-hand of (4.15), which implies that

$$-1+\sigma = 2-\sigma \quad \Rightarrow \quad \sigma = \frac{3}{2}$$

By substituting  $\sigma = 3/2$  in (4.16), which gives

$$\zeta\left(\frac{1}{2}+i\tau\right)\zeta\left(-\frac{3}{2}-i\tau\right) - \frac{1}{4\pi^2}\left(\frac{3}{2}+i\tau\right)\left(-\frac{1}{2}-i\tau\right)\zeta\left(\frac{5}{2}+i\tau\right)\zeta\left(\frac{1}{2}-i\tau\right) = 0 \quad (4.17)$$

- **Second scenario:** The real part of  $s$  in the  $\zeta(-\sigma-i\tau)$  from the left-hand (4.15)

is equal to the real part of  $s$  in  $\zeta(1+\sigma+i\tau)$  from the right-hand of (4.15), which implies that

$$-\sigma=1+\sigma \quad \Rightarrow \quad \sigma=-\frac{1}{2}.$$

Similarly, substituting  $\sigma = -1/2$  in (4.16) gives

$$\zeta\left(-\frac{3}{2}+i\tau\right)\zeta\left(\frac{1}{2}-i\tau\right)-\frac{1}{4\pi^2}\left(-\frac{1}{2}+i\tau\right)\left(\frac{3}{2}-i\tau\right)\zeta\left(\frac{1}{2}+i\tau\right)\zeta\left(\frac{5}{2}-i\tau\right)=0 \quad (4.18)$$

Again, the functional equations (4.17) and (4.18) provide profound understanding of how the zeta function contributes the nontrivial zeros on the critical strip. Recall that the  $\zeta(s)$  has no nontrivial zeros for  $s = 0$  or  $s = 1$ . We only consider the real part of  $s$  within the interval  $(0, 1)$ . Next, we examine the distinct cases of  $\text{Re}(s)$  in order to illustrate the nontrivial zeros of the zeta functions from functional equations (4.16), (4.17) and (4.18).

### Case 1: $\sigma = 3/2$

Equation (4.17) shows that only the functions  $\zeta(1/2+i\tau)$  and  $\zeta(1/2-i\tau)$  form conjugate pair that contributes nontrivial zeros. The equation also demonstrates that if  $\zeta(1/2+i\tau)=0$  then  $\zeta(1/2-i\tau)=0$  and vice versa at  $\tau=\tau_k$  for  $k \in \mathbb{N}$ . The functions  $\zeta(-3/2-i\tau)$  and  $\zeta(5/2+i\tau)$  have no nontrivial zeros for real  $\tau$  because their real parts lie outside the interval  $(0,1)$ . The factors  $(3/2+i\tau)$  and  $(-1/2-i\tau)$  do not have zeros for real  $\tau$ .

### Case 2: $3/2 < \sigma < 2$

We examine the nontrivial zeros of the functions and factors in (4.16) for the real part of  $s$  of the interval  $(3/2, 2)$  as shown in Table III below.

<b>Table III</b>	
<b>If <math>3/2 &lt; \sigma &lt; 2</math> then</b>	<b>Functions and Factors of Equation (4.16)</b>
$1/2 < -1+\sigma < 1 \quad \Rightarrow$	(a) $\zeta(-1+\sigma+i\tau)$ is expected to have nontrivial zeros for $\text{Re}(s) \in (0,1)$ .
$-2 < -\sigma < -3/2 \quad \Rightarrow$	(b) $\zeta(-\sigma-i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .

$3/2 < \sigma < 2$	$\Rightarrow$	(c) $(\sigma+i\tau)$ has no zeros.
$-1 < 1-\sigma < -1/2$	$\Rightarrow$	(d) $(1-\sigma-i\tau)$ has no zeros.
$5/2 < 1+\sigma < 3$	$\Rightarrow$	(e) $\zeta(1+\sigma+i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .
$0 < 2-\sigma < 1/2$	$\Rightarrow$	(f) $\zeta(2-\sigma-i\tau)$ is anticipated to have nontrivial zeros for $\text{Re}(s) \in (0,1)$ .

Similar to Case 1 of Section 4-2-4, we see that only (a) and (f) are anticipated to have nontrivial zeros due to the absence of zeros from one pair of zeta functions in the conjugate form  $-1+\sigma-i\tau$  and  $2-\sigma+i\tau$ , which do not conform to (4.12c) and Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6. Therefore, we conclude that no nontrivial zeros exist in the interval,  $3/2 < \sigma < 2$ .

### Case 3: $\sigma \geq 2$

<b>Table IV</b>		
<b>If <math>\sigma \geq 2</math> then</b>		<b>Functions and Factors of Equation (4.17)</b>
$-1+\sigma \geq 1$	$\Rightarrow$	(a) $\zeta(-1+\sigma+i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .
$-\sigma \leq -2$	$\Rightarrow$	(b) $\zeta(-\sigma-i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .
$\sigma \geq 2$	$\Rightarrow$	(c) $(\sigma+i\tau)$ has no zeros.
$1-\sigma \leq -1$	$\Rightarrow$	(d) $(1-\sigma-i\tau)$ has no zeros.
$1+\sigma \geq 3$	$\Rightarrow$	(e) $\zeta(1+\sigma+i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .
$2-\sigma \leq 0$	$\Rightarrow$	(f) $\zeta(2-\sigma-i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .

There are no nontrivial zeros to be found for  $\sigma \geq 2$ . Notice that  $\text{Re}(s)$  can be 1 or 0 as shown in (a) or (f). However, it has been known that there are no nontrivial zeros for  $s = 1$  or  $s = 0$  because of Property iv (Critical Strip) in Section 4-1-6.

### Case 4: $1 < \sigma < 3/2$

<b>Table V</b>		
<b>If <math>\sigma &lt; 3/2</math> then</b>		<b>Functions and Factors of Equation (4.17)</b>

$0 < -1 + \sigma < \frac{1}{2}$	$\Rightarrow$	(a) $\zeta(-1 + \sigma + i\tau)$ is anticipated to have nontrivial zeros for $\text{Re}(s) \in (0, 1)$ .
$-\frac{3}{2} < -\sigma < -1$	$\Rightarrow$	(b) $\zeta(-\sigma - i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0, 1)$ .
$1 < \sigma < \frac{3}{2}$	$\Rightarrow$	(c) $(\sigma + i\tau)$ has no zeros.
$-\frac{1}{2} < 1 - \sigma < 0$	$\Rightarrow$	(d) $(1 - \sigma - i\tau)$ has no zeros.
$2 < 1 + \sigma < \frac{5}{2}$	$\Rightarrow$	(e) $\zeta(1 + \sigma + i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0, 1)$ .
$\frac{1}{2} < 2 - \sigma < 1$	$\Rightarrow$	(f) $\zeta(2 - \sigma - i\tau)$ is expected to have nontrivial zeros for $\text{Re}(s) \in (0, 1)$ .

Similar to Case 1 of Section 4-2-4, only (a) and (f) are anticipated to have nontrivial zeros due to the absence of zeros from one pair of zeta functions in the conjugate form  $-1 + \sigma - i\tau$  and  $2 - \sigma + i\tau$ , which do not conform to (4.12c) and Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6. Therefore, the nontrivial zeros cannot be established for both (a) and (f). Hence, it follows that there are no nontrivial zeros for  $1 < \sigma < \frac{3}{2}$ .

### Case 5: $0 \leq \sigma \leq 1$

<b>Table VI</b>		
<b>If <math>0 \leq \sigma \leq 1</math> then</b>	<b>Functions and Factors of Equation (4.17)</b>	
$-1 \leq -1 + \sigma \leq 0$	$\Rightarrow$	(a) $\zeta(-1 + \sigma + i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0, 1)$ .
$-1 \leq -\sigma \leq 0$	$\Rightarrow$	(b) $\zeta(-\sigma - i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0, 1)$ .
$0 \leq \sigma \leq 1$	$\Rightarrow$	(c) $(\sigma + i\tau)$ has no zeros.
$0 \leq 1 - \sigma \leq -1$	$\Rightarrow$	(d) $(1 - \sigma - i\tau)$ has no zeros.
$1 \leq 1 + \sigma \leq 2$	$\Rightarrow$	(e) $\zeta(1 + \sigma + i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0, 1)$ , and a simple pole at $\sigma = 0$ .
$1 \leq 2 - \sigma \leq 2$	$\Rightarrow$	(f) $\zeta(2 - \sigma - i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0, 1)$ , and a simple pole at $\sigma = 1$ .

There are no nontrivial zeros exist for  $0 \leq \sigma \leq 1$ .

**Case 6:  $-\frac{1}{2} < \sigma < 0$** 

<b>Table VII</b>	
<b>If <math>-\frac{1}{2} &lt; \sigma &lt; 0</math> then</b>	<b>Functions and Factors of Equation (4.17)</b>
$-\frac{3}{2} < -1+\sigma < -1 \Rightarrow$	(a) $\zeta(-1+\sigma+i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .
$0 < -\sigma < \frac{1}{2} \Rightarrow$	(b) $\zeta(-\sigma-i\tau)$ is anticipated to have nontrivial zeros for $\text{Re}(s) \in (0,1)$ .
$-\frac{1}{2} < \sigma < 0 \Rightarrow$	(c) $(\sigma+i\tau)$ has no zeros.
$1 < 1-\sigma < \frac{3}{2} \Rightarrow$	(d) $(1-\sigma-i\tau)$ has no zeros.
$\frac{1}{2} < 1+\sigma < 1 \Rightarrow$	(e) $\zeta(1+\sigma+i\tau)$ is anticipated to have nontrivial zeros for $\text{Re}(s) \in (0,1)$ .
$2 < 2-\sigma < \frac{5}{2} \Rightarrow$	(f) $\zeta(2-\sigma-i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .
<p>Similar to Case 1 of Section 4-2-4, we see that (b) and (e) cannot establish the nontrivial zeros due to the absence of zeros from one pair of zeta functions in the conjugate form <math>-\sigma+i\tau</math> and <math>1+\sigma-i\tau</math>, which do not conform to (4.12c) and Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6. Hence, there are no nontrivial zeros exist for <math>-\frac{1}{2} &lt; \sigma \leq 0</math>.</p>	

**Case 7:  $\sigma = -\frac{1}{2}$** 

Equation (4.18) shows that only  $\zeta(\frac{1}{2}+i\tau)$  and  $\zeta(\frac{1}{2}-i\tau)$  contribute nontrivial zeros on the critical line at  $\tau=\tau_k$  for  $k \in \mathbb{N}$  while other functions and factors do not have nontrivial zeros for all real  $\tau$ .

**Case 8:  $\sigma < -\frac{1}{2}$** 

<b>Table VIII</b>	
<b>If <math>\sigma &lt; -\frac{1}{2}</math> then</b>	<b>Functions and Factors of Equation (4.17)</b>
$-1+\sigma < -\frac{3}{2} \Rightarrow$	(a) $\zeta(-1+\sigma+i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .
$-\sigma > \frac{1}{2} \Rightarrow$	(b) $\zeta(-\sigma-i\tau)$ is anticipated to have nontrivial zeros for $\text{Re}(s) \in (0,1)$ .
$\sigma < -\frac{1}{2} \Rightarrow$	(c) $(\sigma+i\tau)$ has no zeros.
$1-\sigma > \frac{3}{2} \Rightarrow$	(d) $(1-\sigma-i\tau)$ has no zeros.

$1+\sigma < 1/2$	$\Rightarrow$	(e) $\zeta(1+\sigma+i\tau)$ is anticipated to have nontrivial zeros for $\text{Re}(s) \in (0,1)$ .
$2-\sigma > 5/2$	$\Rightarrow$	(f) $\zeta(2-\sigma-i\tau)$ has no nontrivial zeros for $\text{Re}(s) \notin (0,1)$ .

Similar to Case 1 of Section 4-2-4, we see that (b) and (e) cannot establish the nontrivial zeros due to the absence of zeros from one pair of zeta functions in the conjugate form  $-\sigma+i\tau$  and  $1+\sigma-i\tau$ , which do not conform to (4.12c) and Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6. Hence, there are no nontrivial zeros exist for  $\sigma < -1/2$ .

Based on the results obtained from Case 1 to Case 8, the zeta function has nontrivial zeros only when either  $\sigma = 3/2$  as described in equation (4.17) or  $\sigma = -1/2$  as described in equation (4.18). The shift (by replacing  $s$  by  $1-s$ ) affects only the domain, causing the critical points to shift accordingly. Our analysis from these two equations (4.17) and (4.18) shows no nontrivial zeros are found off the critical line.

### Section 4-4. Proof of Riemann Hypothesis - No Nontrivial Zeros Off the Critical Line

This section plays a key role in proving the Riemann Hypothesis. We show that for  $0 < \sigma < 1$  and  $\sigma \neq 1/2$ , **all possible nontrivial zeros** to the zeta equation  $\zeta(s) = 0$  have imaginary part in  $6.28318534... < \tau \leq \tau_s \approx 6.28983598...$  in upper half-plane, and therefore by conjugate symmetry there is a corresponding possible nontrivial zeros with imaginary part in  $-6.28983598... \approx \tau_s \leq \tau < -6.28318534...$  in lower half-plane.

We start with equation (4.6) that contains product factors such as  $2^s$ ,  $\pi^{s-1}$  and  $\sin(\pi s/2)$ , which do not yield any nontrivial zeros except  $\zeta(s)$  and  $\zeta(1-s)$ . Now we consider all zeta functions that produce zeros on one side as a ratio  $\zeta(s)/\zeta(1-s)$ , and examine its magnitude  $|\zeta(s)/\zeta(1-s)|$ . The concept is to utilize the magnitude of the ratio  $|\zeta(s)/\zeta(1-s)|$  to equal 1 when  $|\zeta(s)|$  and  $|\zeta(1-s)|$  approach  $|0+i0|$ . We establish a theorem that helps to identify  $s$  as a possible nontrivial zero if the absolute value of  $|\zeta(s)/\zeta(1-s)|$  intersects with  $|G(s)| = 1$ . We discover that either  $\zeta(s)$  or  $\zeta(1-s)$  with  $\text{Re}(s) \neq 1/2$  cannot have nontrivial zero, they do not satisfy zeta functional equation (4.6). In conjunction with this research, we uncover a constant number,  $6.289835...$ , which relates to the boundary limit of nontrivial zeros. It is

astonishing to us that any other nontrivial zeros with an imaginary part, where  $0 < \text{Re}(s) < 1$  and excluding  $\text{Re}(s) = 1/2$ , cannot exceed this constant.

### Briefly, what will we delivery in this section?

- **Examining the Function  $\left| \frac{\zeta(s)}{\zeta(1-s)} \right|$  and Its Relation**
- **Derivative of  $\left| \frac{\zeta(s)}{\zeta(1-s)} \right| = \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  Respect to  $\tau$**
- **Behavior of  $\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  For  $0 < \sigma < 1$**
- **Graphs of  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)| = 1$**
- **Existence of  $\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right|$  for  $0 < \text{Re}(s_0) < 1$**
- **Intersecting Points of  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)| = 1$**
- **Saturation Constant**
- **Conclusion: Riemann Hypothesis Is True**

### Assumptions

#### 1. First Nontrivial Zero of the Riemann Zeta Function

It is widely supported by extensive numerical computations that the first nontrivial zero lies at  $s_0 = 1/2 + i14.134725\dots$ . While this value is not yet formally proven, it is treated here as a foundational premise.

#### 2. Validity of Numerical Methods

In addressing transcendental equations where closed-form solutions are unavailable, numerical methods are assumed to provide reliable and valid approximations. This approach aligns with standard practices in both theoretical and applied mathematics.

#### 3. Role of Graphical Representations

Graphs derived from closed-form expressions, such as those expressed in elementary functions, are considered rigorous and can serve as formal proofs. In contrast, numerical graphs, which depend on approximations from numerical computations, are utilized primarily for exploratory analysis and hypothesis formation and do not constitute formal proofs.

#### 4. Numerical Method

It's important to note that the process of finding intersection points may involve solving transcendental equations, which typically lack closed-form solutions. In pure mathematics, such equations are often addressed using numerical methods. These methods, including the bisection method, Newton-Raphson method, and secant method, are widely accepted for approximating solutions to transcendental equations.

### 5. Notations Use in this Section

- Let  $s = \sigma + i\tau$  be a complex variable, where  $\sigma$  and  $\tau$  real.
- $s_0 = \sigma_0 + i\tau_0$  denote a nontrivial zero of the zeta function, where  $\sigma_0$  and  $\tau_0$  real.

#### 4-4-1. Examining the Function $\zeta(s)/\zeta(1-s)$ and Its Relation

Let  $s = \sigma + i\tau$ , where  $\sigma, \tau \in \mathbb{R}$ . Equation (4.6) gives the ratio

$$\frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \quad (4.19)$$

$$= \frac{2^{s-1} \pi^s}{\cos\left(\frac{\pi s}{2}\right) \Gamma(s)}. \quad (4.20)$$

(Use the reflection gamma [see **Appendix 4-A-3**].)

Taking the magnitude of both sides of equation (4.19), which gives

$$\left| \frac{\zeta(s)}{\zeta(1-s)} \right| = \left| 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \right|. \quad (4.21)$$

We break down equation (4.21) into two distinct functions, examining each one independently. Let's call the term

$$|F(\sigma + i\tau)| = \left| 2^{\sigma + i\tau} \pi^{\sigma + i\tau - 1} \sin\left(\frac{\pi(\sigma + i\tau)}{2}\right) \Gamma(1 - \sigma - i\tau) \right| \quad (4.22)$$

$$= \left| 2^\sigma \pi^{\sigma-1} \right| \left| \sin\left(\frac{\pi(\sigma + i\tau)}{2}\right) \right| \left| \Gamma(1 - \sigma - i\tau) \right| \quad (4.22a)$$

$$= \left| \frac{2^{\sigma+i\tau-1} \pi^{\sigma+i\tau}}{\cos\left(\frac{\pi(\sigma+i\tau)}{2}\right) \Gamma(\sigma+i\tau)} \right|. \quad (4.22b)$$

Then this term is also equal to the ratio

$$|F(\sigma+i\tau)| = \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|. \quad (4.23)$$

The benefit of this method is that the findings derived from one function can provide insights or be utilized for the other, and vice versa. Next, we take derivative of (4.23) to explore whether equation (4.23) displays local extrema, along with its increasing or decreasing trends, for values of  $\text{Re}(s)$  within the range  $(0, 1)$ .

### Notes:

- Equation (4.21) has a simple pole at  $s = 1$  [see **Appendix 4-A-4**]. This equation is also true when  $|\zeta(1-s)| = 0$ , a special case when  $s$  is on the critical line where  $\text{Re}(s) = 1/2$ .
- The function  $|\zeta(s)|$  is not analytic because the absolute value operation is not complex differentiable. However, it is a real-valued function on the  $\sigma\tau$ -plane, so plotting it as a surface or a heatmap or a line plot (fixed  $\sigma$  or  $\tau$ ) is valid for analysis and visualization purposes.
- The plot does not depict the complex function as a single 3D surface in the complex plane but instead shows the magnitude over the  $(\sigma, \tau)$  domain, which helps visualize features like zeros, poles (if extended), and growth behavior. For instance, the graphs of (4.22a), (4.22b) and (4.23) are expected to be identical.

#### 4-4-1-A. Derivative of $\left| \frac{\zeta(s)}{\zeta(1-s)} \right| = \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$ **Respect to $\tau$**

Examining the derivative of  $|\zeta(\sigma + i\tau)/\zeta(1-\sigma -i\tau)|$  respect to  $\tau$ . We have:

$$|2^{\sigma+i\tau}| = 2^\sigma,$$

$$|\pi^{\sigma+i\tau-1}| = \pi^{\sigma-1},$$

$$\left| \cos\left(\frac{\pi s}{2}\right) \right| = \left| \cos\left(\frac{\pi(\sigma+i\tau)}{2}\right) \right| = \sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)}.$$

By substituting the above expressions in (4.22b) gives

$$|F(\sigma+i\tau)| = \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| = \frac{2^{\sigma-1} \pi^\sigma}{\sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} |\Gamma(\sigma+i\tau)|} \quad (4.24)$$

Use the identity  $\frac{d}{d\tau} |\Gamma(\sigma+i\tau)| = -|\Gamma(\sigma+i\tau)| \cdot \Im(\Psi(\sigma+i\tau))$  [see **Appendix 4-A-7**], and take derivative of (4.24) respect to  $\tau$ , which give

$$\begin{aligned} \frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| &= \frac{d}{d\tau} \left[ \frac{2^{\sigma-1} \pi^\sigma}{\sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} |\Gamma(\sigma+i\tau)|} \right] \\ &= -2^{\sigma-1} \pi^\sigma \left[ \frac{2 \sinh\left(\frac{\pi\tau}{2}\right) \cosh\left(\frac{\pi\tau}{2}\right) \cdot \frac{\pi}{2} |\Gamma(\sigma+i\tau)| - \sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} |\Gamma(\sigma+i\tau)| \cdot \Im(\Psi(\sigma+i\tau))}{2 \sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} \left( \cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right) \right) |\Gamma(\sigma+i\tau)|^2} \right] \quad (4.25) \end{aligned}$$

After obtaining a common denominator and simplifying the gamma functions in both the numerator and denominator of (4.25), we deduce that

$$\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| = -2^{\sigma-1} \pi^\sigma \frac{\left[ \pi \sinh(\pi\tau) - 4 \left( \cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right) \right) \cdot \Im(\Psi(\sigma+i\tau)) \right]}{4 \left( \cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right) \right)^{3/2} |\Gamma(\sigma+i\tau)|}.$$

Factor out  $2^{\sigma-1}/4 = 2^{a-3}$  and move the negative sign inside the parentheses, result in

$$\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| = 2^{\sigma-3} \pi^\sigma \frac{\left[ 4 \left( \cos^2 \left( \frac{\pi\sigma}{2} \right) + \sinh^2 \left( \frac{\pi\tau}{2} \right) \right) \Im \left( \Psi(\sigma+i\tau) \right) - \pi \sinh(\pi\tau) \right]}{\left( \cos^2 \left( \frac{\pi\sigma}{2} \right) + \sinh^2 \left( \frac{\pi\tau}{2} \right) \right)^{3/2} |\Gamma(\sigma+i\tau)|} \quad (4.26)$$

#### 4-4-1-B. Behavior of $\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$ for $0 < \sigma < 1$

We study the behavior of (4.26) for  $0 < \sigma < 1$ . It consists of three cases.

##### (a) Case 1: $\sigma = 1/2$

By substituting  $\sigma = 1/2$  in (4.31) gives

$$\begin{aligned} \frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|_{\sigma = \frac{1}{2}} &= 2^{-5/2} \pi^{1/2} \frac{\left[ 4 \left( \frac{1}{2} + \sinh^2 \left( \frac{\pi\tau}{2} \right) \right) \Im \left( \Psi \left( \frac{1}{2} + i\tau \right) \right) - \pi \sinh(\pi\tau) \right]}{\left( \frac{1}{2} + \sinh^2 \left( \frac{\pi\tau}{2} \right) \right)^{3/2} \left| \Gamma \left( \frac{1}{2} + i\tau \right) \right|} \\ &= 2^{-5/2} \pi^{1/2} \frac{\left[ 2 \cosh(\tau) \Im \left( \Psi \left( \frac{1}{2} + i\tau \right) \right) - \pi \sinh(\pi\tau) \right]}{\left( \frac{1}{2} + \sinh^2 \left( \frac{\pi\tau}{2} \right) \right)^{3/2} \left| \Gamma \left( \frac{1}{2} + i\tau \right) \right|} \end{aligned} \quad (4.27)$$

Substituting  $\Im \left( \Psi \left( \frac{1}{2} + i\tau \right) \right) = \frac{\pi}{2} \tanh(\pi\tau)$  in (4.27) yields

$$\begin{aligned} &= 2^{-5/2} \pi^{1/2} \frac{\left[ 2 \cosh(\pi\sigma) \frac{\pi}{2} \tanh(\pi\sigma) - \pi \sinh(\pi\sigma) \right]}{\left( \frac{1}{2} + \sinh^2 \left( \frac{\pi\sigma}{2} \right) \right)^{3/2} \left| \Gamma \left( \frac{1}{2} + i\sigma \right) \right|} \\ &= 2^{-5/2} \pi^{1/2} \frac{\left[ \pi \sinh(\pi\tau) - \pi \sinh(\pi\tau) \right]}{\left( \frac{1}{2} + \sinh^2 \left( \frac{\pi\tau}{2} \right) \right)^{3/2} \left| \Gamma \left( \frac{1}{2} + i\tau \right) \right|} \\ &= 0. \end{aligned}$$

The result leads to conclude that the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  has a local extreme point(s) at  $\sigma = 1/2$  for all relevant values of  $\tau$ .

**(b) Case 2:  $\sigma > 1/2$**

We use the identity,

$$\Im(\Psi(\sigma+i\tau)) = \sum_{k=0}^{\infty} \frac{\tau}{(\sigma+k)^2 + \tau^2},$$

to examine the vary of the digamma function as  $\sigma$  varies. When  $\sigma > 1/2$ , we have

$$\Im(\Psi(\sigma+i\tau)) = \sum_{k=0}^{\infty} \frac{\tau}{(\sigma+k)^2 + \tau^2} < \sum_{k=0}^{\infty} \frac{\tau}{(1/2+k)^2 + \tau^2} \text{ for all } \tau > 0.$$

Therefore, it follows that

$$\Im(\Psi(\sigma+i\tau)) < \Im\left(\Psi\left(\frac{1}{2}+i\tau\right)\right) \quad (4.28)$$

Since Case 1 establishes that

$$4\left(\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)\right) \Im(\Psi(1/2+i\tau)) - \pi \sinh(\pi\tau) = 0$$

for  $\sigma = 1/2$ , and this result is combined with inequality (4.28), it follows that the numerator of the right-hand side of (4.26) must be less than 0, namely

$$4\left(\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)\right) \Im(\Psi(\sigma+i\tau)) - \pi \sinh(\pi\tau) < 0, \text{ for all } \tau > 0,$$

which gives

$$\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|_{\sigma > \frac{1}{2}} < 0, \text{ for all } \tau > 0.$$

Thus, for  $\sigma > 1/2$ , the derivative  $\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| < 0$  for all  $\tau > 0$ , the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  is strictly decreasing on this interval.

### (c) Case 3: $0 < \sigma < 1/2$

Similarly, for  $0 < \sigma < 1/2$ , we have

$$\Im(\Psi(\sigma+i\tau)) = \sum_{k=0}^{\infty} \frac{\tau}{(\sigma+k)^2 + \tau^2} > \sum_{k=0}^{\infty} \frac{\tau}{(1/2+k)^2 + \tau^2}, \text{ for all } \tau > 0,$$

from which it follows that

$$\Im(\Psi(\sigma+i\tau)) > \Im\left(\Psi\left(\frac{1}{2}+i\tau\right)\right).$$

Therefore, the numerator of the right-hand side of (4.26) must be positive, namely

$$4\left(\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)\right) \Im(\Psi(\sigma+i\tau)) - \pi \sinh(\pi\tau) > 0, \quad \text{for } \tau > 0,$$

which gives

$$\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|_{0 < \sigma < \frac{1}{2}} > 0 \quad \text{for all } \tau > 0.$$

Thus, as  $\sigma$  increases from 0 to  $1/2$ , the derivative  $\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|_{0 < \sigma < \frac{1}{2}} > 0$  for all  $\tau > 0$ .

It implies that the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  is strictly increasing for all relevant values of  $\tau > 0$ .

## Summary

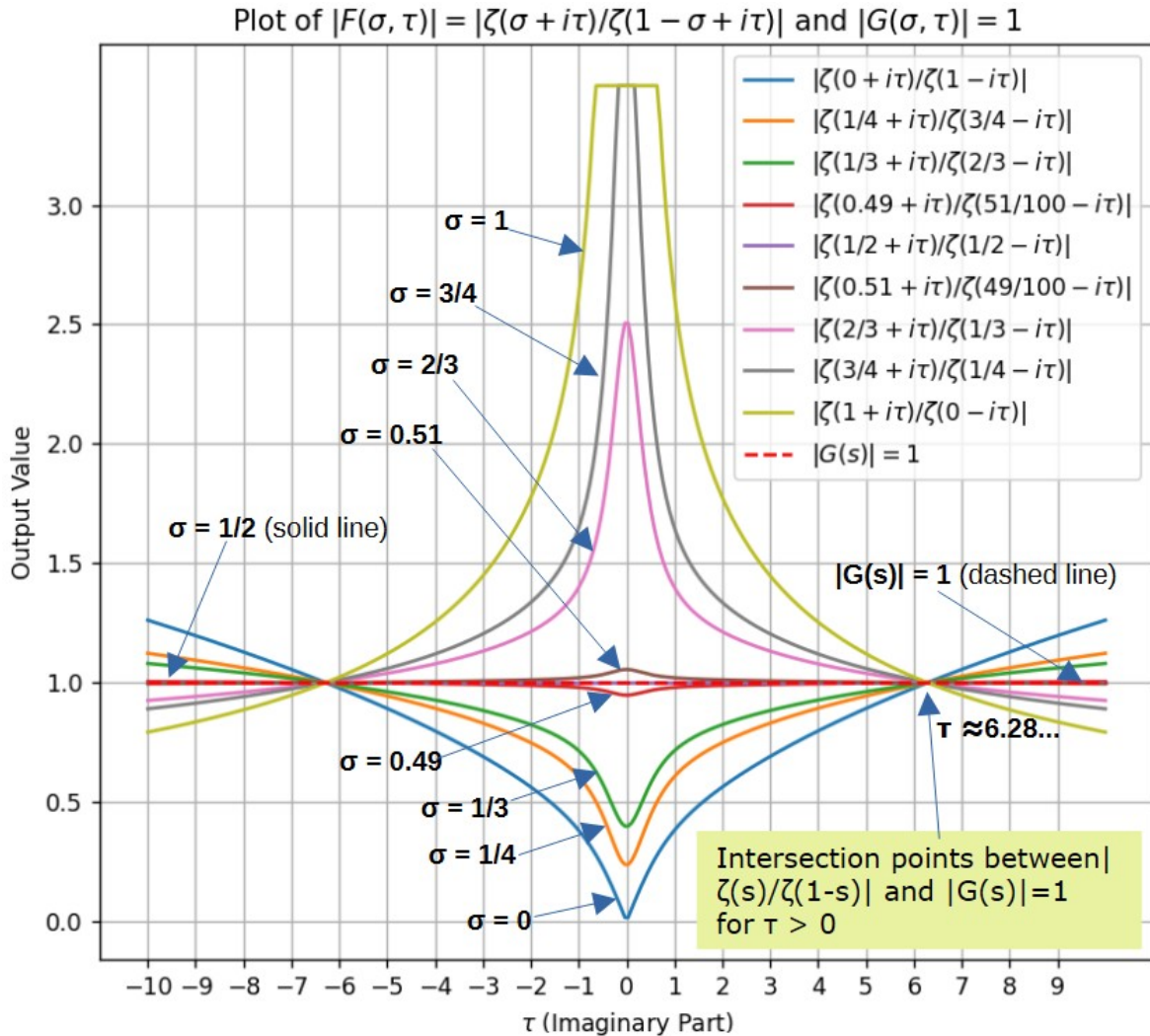
- For  $0 < \sigma < 1/2$ ,  $\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| > 0$ , the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  strictly increases, and  $\lim_{\tau \rightarrow \infty} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|_{0 < \sigma < 1/2} = \infty$ .
- For  $\sigma = 1/2$ ,  $\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| = 0$ , the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  has a local extreme point.
- For  $\sigma > 1/2$ ,  $\frac{d}{d\tau} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right| < 0$ , the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  strictly decreases, and  $\lim_{\tau \rightarrow \infty} \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|_{\sigma > 1/2} = 0$ .

Furthermore, we observe that the sign of the derivative changes from positive to negative as  $\sigma$  passes through  $1/2$ , then the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  has a local maximum  $\sigma = 1/2$ . Overall, the function  $\left| \frac{\zeta(s)}{\zeta(1-s)} \right| = \left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  **is strictly monotonic for  $0 < \sigma < 1$ ,  $\sigma \neq 1/2$ ,  $\tau > 0$ .**

Next, we plot the graphs of  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)| = 1$  for specific values of  $\sigma$  within the interval  $(0, 1)$ . Since  $|\zeta(s)/\zeta(1-s)|$  increases on  $(0, 1/2)$  and decreases on  $(1/2, 1)$ , **the curve  $|\zeta(s)/\zeta(1-s)|$  intersects  $|G(s)|$  only once within  $(0, 1)$ , except at  $\sigma = 1/2$ .** As  $\sigma$  varies from 0 to  $1/2$  or from  $1/2$  to 1, these curves illustrate how  $|\zeta(s)/\zeta(1-s)|$  approaches a 'saturation point' as  $\sigma$  approaches  $1/2$  (critical line). We identify key points on the  $|\zeta(s)/\zeta(1-s)|$  graph and predict a **potential limiting value** it may reach as  $\sigma$  approaches this boundary.

#### 4-4-1-C. Graphs of $|\zeta(s)/\zeta(1-s)|$ and $|G(s)| = 1$

The graphs of  $|\zeta(s)/\zeta(1-s)|$  for  $\sigma = 0, 1/4, 1/3, 0.49, 1/2, 0.51, 2/3, 3/4$ , and 1 alongside the graph of  $|G(s)| = 1$  are shown in Figure 1.



**Figure 1:** The graph of  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  for  $\sigma = 0, 1/4, 1/3, 0.49, 1/2, 0.51, 2/3, 3/4,$  and  $1$  alongside of  $|G(s)|=1$ .

**Source Code Note:** To view the graphs locally, visit the online version [see **Appendix 4-B-1**]. Readers can download the source code and run it locally using Python and scientific libraries Plotly, SciPy, NumPy and Matplotlib.

### Properties of the Graph of $|\zeta(s)/\zeta(1-s)|$

Through the graph as  $\sigma$  varies from 0 to 1, we recognize the following key points:

1. The function  $|\zeta(s)/\zeta(1-s)|$  shown in Figure 1 is an even function of  $\tau$ , and its magnitude in the upper half plane is the reciprocal of its magnitude in the lower half plane. It's suffice to consider the right-half plane for  $\tau > 0$  because of this existing inversion symmetry.

2. The function  $|\zeta(s)/\zeta(1-s)|$  intersects with the function  $|G(s)| = 1$  at the points  $\tau$  where  $6 < \tau < 7$ .
3. When  $\sigma$  approaches  $1/2$  (for instance,  $\sigma = 0.49$ ), the graph of  $|\zeta(s)/\zeta(1-s)|$  gradually transforms into a linear form. The graph becomes a line at the point where  $\sigma = 1/2$ , resulting in  $|\zeta(1/2-i\tau)/\zeta(1/2+i\tau)| = 1$  for  $\tau > 0$ . At this stage, we observe the variable  $\tau$  and its right-hand expression of (4.24) disappear, making it impossible to derive the value of  $\tau$ . It appears that  $\tau$  must approach an unknown maximum value, but the challenge lies in figure out to determine  $\tau$  under this scenario.
4. As  $\tau \rightarrow \infty$ , the ratio  $|\zeta(s)/\zeta(1-s)|$  tends to zero when  $\sigma < 1/2$ , and tends to infinity when  $\sigma > 1/2$ .

Below, we provide the proof for statements #1, #3 and #4. The proof of the remaining one will be addressed in Sections 4-4-3 and 4-4-4.

### Proof

1. Replacing  $\sigma$  and  $\tau$  with  $-\sigma$  and  $-\tau$  in (4.22a) gives

$$|F(-\sigma - i\tau)| = \left| 2^{-\sigma} \pi^{-\sigma-1} \left| \sin\left(\frac{\pi(-\sigma - i\tau)}{2}\right) \right| \Gamma(1 + \sigma + i\tau) \right|$$

Use  $|\sin(\pi(-\sigma - i\tau)/2)| = |\sin(-\pi(\sigma + i\tau)/2)| = |\sin(\pi(\sigma + i\tau)/2)|$  and  $|\Gamma(\sigma - i\tau)| = |\Gamma(\sigma + i\tau)|$ , and by substitution, we get

$$= \left| 2^{-\sigma} \pi^{-\sigma-1} \left| \sin\left(\frac{\pi(\sigma + i\tau)}{2}\right) \right| \Gamma(1 + \sigma - i\tau) \right|$$

This result follows that

$$\Im(|F(\sigma + i\tau)|) = \Im(|F(-\sigma - i\tau)|).$$

We conclude that  $|\zeta(s)/\zeta(1-s)|$  is an even function of  $\tau$ .

3. The right expression of (4.24) is disappeared when  $\sigma = 1/2$ . Indeed, we have

$$\frac{2^{\sigma-1} \pi^{\sigma}}{\sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} |\Gamma(\sigma + i\tau)|} = \frac{\sqrt{\pi}}{\sqrt{2} \sqrt{\cos^2\left(\frac{\pi}{4}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} \left| \Gamma\left(\frac{1}{2} + i\tau\right) \right|}$$

$$\begin{aligned}
\text{Applying } \left| \Gamma\left(\frac{1}{2} + i\tau\right) \right| &= \sqrt{\frac{\pi}{\cosh \pi \tau}} \text{ and } \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \text{ to the above expression gives} \\
&= \frac{\sqrt{\pi}}{\sqrt{2} \sqrt{\frac{1}{2} + \sinh^2\left(\frac{\pi \tau}{2}\right)} \sqrt{\frac{\pi}{\cosh \pi \tau}}} \\
&= \frac{1}{\sqrt{1 + 2 \sinh^2\left(\frac{\pi \tau}{2}\right)} \sqrt{\frac{1}{\cosh \pi \tau}}} \\
&= \frac{1}{\sqrt{\cosh(\pi \tau)} \sqrt{\frac{1}{\cosh \pi \tau}}} \\
&= 1,
\end{aligned}$$

which indicates the variable  $\tau$  and its expression from the right-hand side of (4.24) disappear.

4. Since  $|\zeta(s)/\zeta(1-s)| > 0$  for all  $\tau$ , and  $|\zeta(s)/\zeta(1-s)|$  is decreasing for  $0 < \sigma < 1/2$  due to its negative derivative, it follows that  $\lim_{\tau \rightarrow \infty} |\zeta(s)/\zeta(1-s)| = 0$ , as the function asymptotically approaches the  $\tau$ -axis. And  $|\zeta(s)/\zeta(1-s)|$  is increasing for  $\sigma > 1/2$  due to its positive derivative, which implies  $\lim_{\tau \rightarrow \infty} |\zeta(s)/\zeta(1-s)| = \infty$ .

**Note.** Observing the function  $|\zeta(s)/\zeta(1-s)|$  in Figure 1, and noting that its derivative is strictly monotonic for  $0 < \sigma < 1$  except at  $\sigma = 1/2$ , it seems that  $|\zeta(s)|$  alone cannot have zeros within this interval, as there are no zeros or singularities behavior or present.

### Closed-Forms of $|\zeta(s)/\zeta(1-s)|$ for Some $\sigma$

We use the identity

$$|\Gamma(n+1+i\tau)| = \sqrt{\frac{\pi \tau}{\sinh(\pi \tau)}} \prod_{k=1}^n \sqrt{\tau^2 + k^2} \text{ for positive integer } n \geq 1$$

to find some closed-form of  $|\zeta(s)/\zeta(1-s)|$  for specific  $\sigma$ . At  $\sigma = 0$ , we have

$$|\Gamma(1+i\tau)| = \sqrt{\frac{\pi\tau}{\sinh \pi\tau}} \quad (4.29)$$

Applying  $\Gamma(x+1) = x\Gamma(x)$  to the left expression of (4.29) gives

$$|\Gamma(i\tau)| = \sqrt{\frac{\pi}{\tau \sinh \pi\tau}} \quad (4.30)$$

**At  $\sigma = 0$ :**

Substituting  $\Gamma$  from (4.30) in (4.25) gives

$$\begin{aligned} \left| \frac{\zeta(i\tau)}{\zeta(1-i\tau)} \right| &= \frac{1}{2\sqrt{1+\sinh^2\left(\frac{\pi\tau}{2}\right)} \cdot \sqrt{\frac{\pi}{\tau \sinh \pi\tau}}} \\ &= \frac{1}{2\sqrt{\cosh^2\left(\frac{\pi\tau}{2}\right)} \cdot \sqrt{\frac{\pi}{2\tau \sinh \frac{\pi\tau}{2} \cosh \frac{\pi\tau}{2}}}}, \end{aligned}$$

which can be simplified as

$$\left| \frac{\zeta(i\tau)}{\zeta(1-i\tau)} \right| = \sqrt{\frac{\tau}{2\pi} \tanh\left(\frac{\pi\tau}{2}\right)}. \quad (4.31)$$

**At  $\sigma = 1$ :**

Substituting  $\Gamma$  from (4.29) in (4.25) gives

$$\begin{aligned} \left| \frac{\zeta(1+i\tau)}{\zeta(-i\tau)} \right| &= \frac{\pi}{\sqrt{\cos^2\left(\frac{\pi}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} \cdot |\Gamma(1+i\tau)|} \\ &= \frac{\pi}{\sqrt{\sinh^2\left(\frac{\pi\tau}{2}\right)} \cdot \sqrt{\frac{\pi\tau}{\sinh \pi\tau}}} \end{aligned}$$

(Use the identity  $\sinh(\pi\tau) = 2\sinh(\pi\tau/2)\cosh(\pi\tau/2)$ , rewrite the denominator)

$$= \frac{\pi}{\sqrt{\sinh^2\left(\frac{\pi\tau}{2}\right) \cdot \sqrt{\frac{\pi\tau}{2\sinh\frac{\pi\tau}{2}\cosh\frac{\pi\tau}{2}}}},$$

which arrives at the simplified form

$$\left|\frac{\zeta(1+i\tau)}{\zeta(-i\tau)}\right| = \sqrt{\frac{2\pi}{\tau} \coth\frac{\pi\tau}{2}}. \quad (4.31 a)$$

### At $\sigma = 2$ :

We have

$$|\Gamma(2+i\tau)| = \sqrt{\frac{\pi\tau}{\sinh\pi\tau}} \sqrt{\tau^2+1}.$$

Substituting the above expression in (4.25) gives

$$\left|\frac{\zeta(2+i\tau)}{\zeta(-1-i\tau)}\right| = \frac{2\pi^2}{\cosh\left(\frac{\pi\tau}{2}\right) \cdot \sqrt{\frac{\pi\tau}{\sinh\pi\tau}} \sqrt{(\tau^2+1)}}.$$

By applying identity  $\sinh(\pi\tau) = 2\sinh(\pi\tau/2)\cosh(\pi\tau/2)$  and then simplifying, we obtain

$$\left|\frac{\zeta(2-i\tau)}{\zeta(-1+i\tau)}\right| = (2\pi)^{3/2} \sqrt{\frac{\tanh\left(\frac{\pi\tau}{2}\right)}{\tau(\tau^2+1)}}. \quad (4.32)$$

### At $\sigma = 3$ :

Proceeding in the same manner as above yields

$$\left|\frac{\zeta(3-i\tau)}{\zeta(-2+i\tau)}\right| = (2\pi)^{5/2} \sqrt{\frac{\coth\left(\frac{\pi\tau}{2}\right)}{\tau(\tau^2+1)(\tau^2+4)}}. \quad (4.33)$$

If we continue to let  $\sigma = m+1$ , the resulting expressions depend on whether  $m$  is odd or even:

$$\left| \frac{\zeta(m+1-i\tau)}{\zeta(-m+i\tau)} \right| = (2\pi)^{m+1/2} \sqrt{\frac{\tanh\left(\frac{\pi\tau}{2}\right)}{\tau \prod_{k=1}^m (\tau^2+k^2)}} \quad \text{when } m \text{ is odd,} \quad (4.34)$$

and

$$\left| \frac{\zeta(m+1-i\tau)}{\zeta(-m+i\tau)} \right| = (2\pi)^{m+1/2} \sqrt{\frac{\coth\left(\frac{\pi\tau}{2}\right)}{\tau \prod_{k=1}^m (\tau^2+k^2)}} \quad \text{when } m \text{ is even.} \quad (4.35)$$

Below, we present the graphs showing the magnitudes of  $\zeta(s)$ ,  $\zeta(s)/\zeta(1-s)$  and the critical line in 2D and 3D as follows: Figure 2 presents the 3D graph of  $|\zeta(s)/\zeta(1-s)|$ , Figure 3 displays the 3D graph of  $|\zeta(s)|$ , with the critical line (in red) highlighted, and Figure 4 shows the 2D graph of  $|\zeta(s)|$ . [See source code, **Appendix 4-B-1**].

**3D Graph of  $|\zeta(s)/\zeta(1-s)|$  (Magnitude Truncated at 5)**

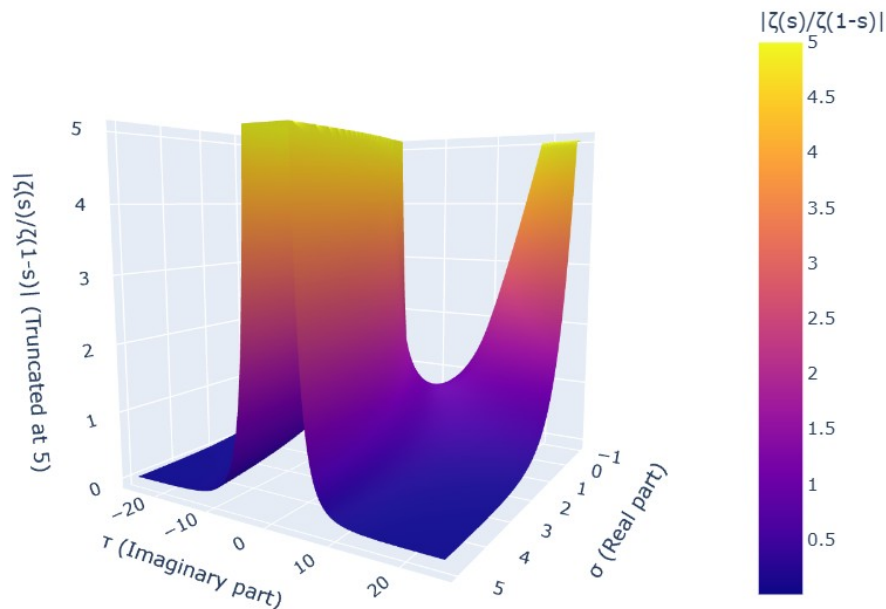
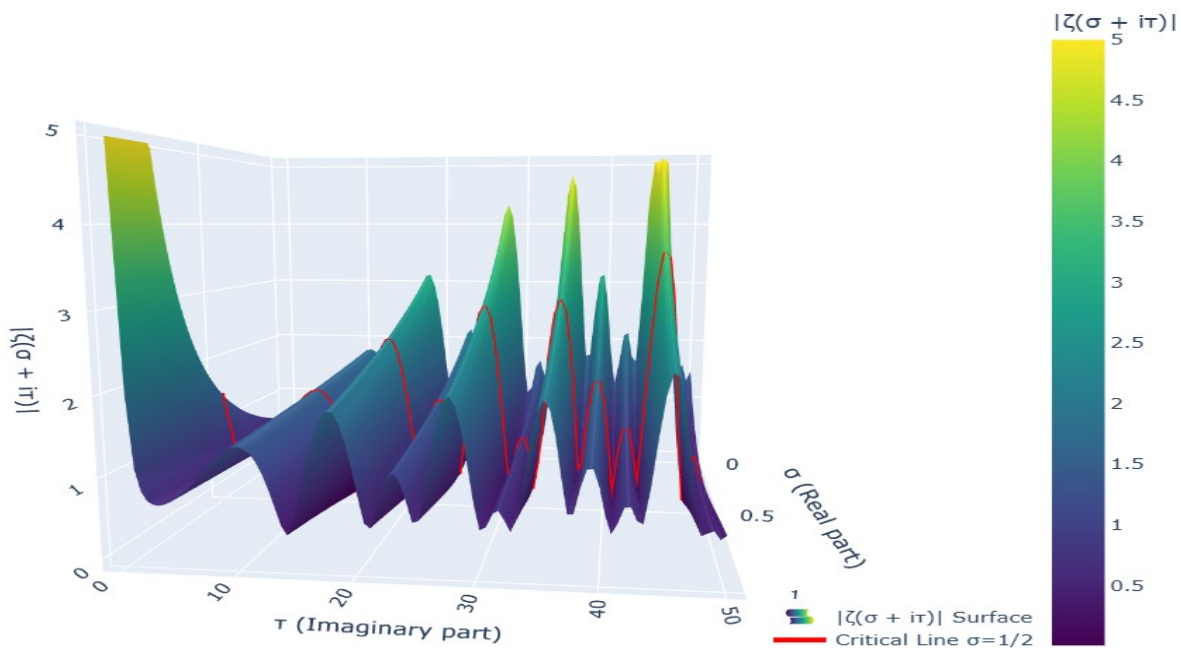
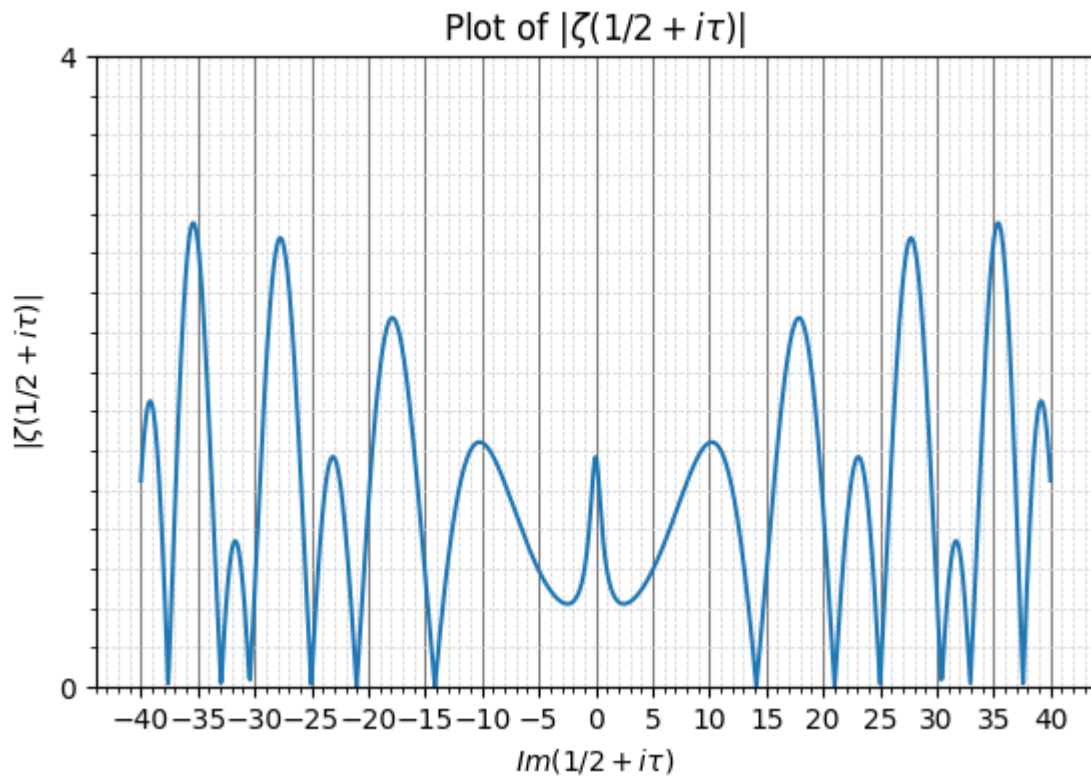


Figure 2 – illustrates the 3D Graph of  $|\zeta(s)/\zeta(1-s)|$ .

Graphs of  $|\zeta(\sigma + i\tau)|$  and the Critical Line

**Figure 3** – illustrates the 3D graph of  $|\zeta(s)|$ , with the critical line (red) indicating points where all nontrivial zeros are lied on.



**Figure 4** – illustrates the 2D graph of  $|\zeta(s)|$ , displaying the locations of the

nontrivial zeros along the line  $\text{Re}(s) = 1/2$ .

Next, we examine the values of  $\tau$  to determine how far it extends, by analyzing the intersections between  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)|$  through solving the relevant equations for  $\tau$ . Finally, we summarize the overall behavior of the graphs and discuss the implications for the Riemann Hypothesis.

#### 4-4-2. Existence of $\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = 1$ for $0 < \text{Re}(s_0) < 1$

As  $s_0$  is assumed nontrivial zeros, we aim to prove that  $\lim_{s \rightarrow s_0} |\zeta(s)/\zeta(1-s)| = 1$  for  $s$  approaching  $s_0$  within the critical strip.

##### 4-4-2-A. Evaluating $\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = 1$ at $\text{Re}(s) = 1/2$ (on the Critical Line)

On the critical line, using the fact that the Riemann zeta function satisfies the conjugate symmetry  $\overline{\zeta(s)} = \zeta(\bar{s})$ , we obtain

$$\zeta\left(\frac{1}{2} + i\tau\right) = \overline{\zeta\left(\frac{1}{2} + i\tau\right)} = \zeta\left(\frac{1}{2} - i\tau\right).$$

It follows immediately that

$$\left| \frac{\zeta\left(\frac{1}{2} + i\tau\right)}{\zeta\left(\frac{1}{2} - i\tau\right)} \right| = 1 \quad \text{for all } \tau > 0. \quad (4.36)$$

Result (4.36) shows that the left-hand side of (4.21) equals 1. This implies that the right-hand side of (4.21) also equals 1 because equation (4.21) is a meromorphic continuation. To validate this independently, we then prove separately that the right-hand side of (4.11) equals 1, without relying on result (4.36). Indeed, we have

$$\left| 2^{1/2+i\tau} \pi^{i\tau-1/2} \sin\left(\frac{\pi(1/2+i\tau)}{2}\right) \Gamma\left(\frac{1}{2}-i\tau\right) \right| = 2^{1/2} \pi^{-1/2} \left| \sin\left(\frac{\pi(1/2+i\tau)}{2}\right) \right| \left| \Gamma\left(\frac{1}{2}-i\tau\right) \right|$$

$$\begin{aligned}
&= 2^{1/2} \pi^{-1/2} \sqrt{\frac{1}{2} + \sinh^2\left(\frac{\pi \tau}{2}\right)} \left| \Gamma\left(\frac{1}{2} - i \tau\right) \right| \\
&= \pi^{-1/2} \sqrt{\cosh\left(\frac{\pi \tau}{2}\right)} \left(\frac{\pi}{\cosh \pi \tau}\right)^{1/2} \\
&= 1 \quad \text{for all } \tau > 0.
\end{aligned} \tag{4.37}$$

We highlight this result<sup>[4\*]</sup> because equation (4.21) represents a meromorphic continuation, ensuring that the two separate functions (4.22) and (4.23) also produce the same value for all  $\tau > 0$ . We will apply this property in the upcoming sections.

If  $s_0 = \sigma_0 + i\tau_0$  is a nontrivial zero of the zeta function on the critical line, then  $\zeta(1/2 + i\tau_0) = 0 + i0$  and  $\zeta(1/2 - i\tau_0) = 0 + i0$ . Substituting into (4.36), we derive the following important property:

$$\left| \frac{\zeta\left(\frac{1}{2} + i\tau_0\right)}{\zeta\left(\frac{1}{2} - i\tau_0\right)} \right| = \left| \frac{0 + i \cdot 0}{0 - i \cdot 0} \right| \equiv 1. \tag{4.38}$$

*(We will rewrite 0/0 form in terms of the limit form.)*

*This indeterminate form,  $|0+i0|/|0-i0|$ , indeed exists and equals to 1 along the critical line. It uniquely characterizes the behavior of the zeta function on  $\text{Re}(s) = 1/2$ .*

### Important Clarification:

- *The expression 0/0 itself, as a standalone algebraic quantity, is not defined in standard arithmetic.*
- *Its significance lies in the context of limits and analytic continuation, where it signals the need for further analysis rather than an actual number.*
- *The concept of assigning a value to  $|0+i0|/|0-i0|$  at a certain point of view is invalid in the Riemann zeta function unless within a limit process that yields a specific value.*

Rewriting (4.38) in terms of the limit form as a formal way gives

$$\frac{\zeta\left(\frac{1}{2}+i\tau_0\right)}{\zeta\left(\frac{1}{2}-i\tau_0\right)} = \lim_{\lambda \rightarrow 0} \left| \frac{\lambda+i\lambda}{\lambda-i\lambda} \right| = 1. \quad (4.39)$$

**4-4-2-B. Evaluating**  $\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = 1$  **for**  $0 < \text{Re}(s_0) < 1$  **and**  $\text{Re}(s_0) \neq \frac{1}{2}$  **(Off the Critical Line)**

Since  $s_0$  is nontrivial zero off the critical line, we present three methods to prove

that  $\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = 1.$

### 1<sup>st</sup> Method

Using the Laurent series expansion<sup>[27]</sup> for the Riemann zeta function about  $s = 0$  gives

$$\zeta(1+s) = \frac{1}{s} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k s^k}{k!}, \quad (4.40)$$

where  $\gamma$  is the Euler constant and  $\gamma_1, \gamma_2, \gamma_3, \dots$  are the Stieltjes constants. The constraint  $\text{Re}(s) > 0$  is removed because  $\zeta(s)$  is meromorphic with a simple pole at  $s = 1$ .

We aim to derive formula series for  $\zeta(1-s)$  and  $\zeta(s)$ . Replacing  $s$  with  $-s$  into (4.40) gives

$$\begin{aligned} \zeta(1-s) &= -\frac{1}{s} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k (-s)^k}{k!} \\ &= -\frac{1}{s} + \sum_{k=0}^{\infty} \frac{\gamma_k s^k}{k!} \end{aligned} \quad (4.40 a)$$

Replacing  $s$  with  $s-1$  in (4.40) gives

$$\begin{aligned}
\zeta(s) &= \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k (s-1)^k}{k!} \\
&= \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k (-1)^k (1-s)^k}{k!} && \text{[note: } (s-1) = -(1-s)\text{]} \\
&= -\frac{1}{1-s} + \sum_{k=0}^{\infty} \frac{\gamma_k (1-s)^k}{k!}
\end{aligned} \tag{4.40 b}$$

**i> Observing that if  $0 < \text{Re}(s) < 1$ ,** then  $-1 < \text{Re}(-s) < 0$  or  $0 < \text{Re}(1-s) < 1$ . It follows that every term in expressions (40a) and (40b) has the same sign and coefficient within the critical strip. Now, let  $s$  near  $s_0$  so that  $\zeta(s_0) = 0$ , then  $\zeta(1-s_0) = 0$  (functional equation), and take the absolute of the ratio:

$$\begin{aligned}
\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| &= \lim_{s \rightarrow s_0} \left| \frac{-\frac{1}{1-s} + \sum_{k=0}^{\infty} \frac{\gamma_k (1-s)^k}{k!}}{-\frac{1}{s} + \sum_{k=0}^{\infty} \frac{\gamma_k s^k}{k!}} \right| \\
&= \lim_{s \rightarrow s_0} \left| \frac{-\frac{1}{1-s} + \gamma + \frac{\gamma_1(1-s)}{1!} + \frac{\gamma_2(1-s)^2}{2!} + \frac{\gamma_3(1-s)^3}{3!} + \dots}{-\frac{1}{s} + \gamma + \frac{\gamma_1 s}{1!} + \frac{\gamma_2 s^2}{2!} + \frac{\gamma_3 s^3}{3!} + \dots} \right| = 1.
\end{aligned} \tag{4.41}$$

The limit exists and equals 1 for the following reasons:

- Both  $\zeta(s_0) = 0$  and  $\zeta(1-s_0) = 0$  share the same nontrivial zero, each with the same multiplicity.
- The coefficients in the numerator associated with  $(1-s_0)^k$  and in the denominator associated with  $s_0^k$  are identical.

**ii> If  $\text{Re}(s) > 1$ ,** then  $\text{Re}(1-s) < 0$ . It follows that every term in the numerator and denominator of (4.41) has same coefficient but does not have same sign for the terms at the odd power in the numerator. We conclude

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| \neq 1. \tag{4.41 a}$$

**iii> If  $\text{Re}(s) < 0$ ,** then  $\text{Re}(1-s) > 1$ . This case is analogous to (ii) but it does not have the same sign for terms at the odd power in the denominator. We conclude

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| \neq 1.$$

The result of (4.41) also implies<sup>[4\*]</sup>  $|F(s_0)| = \left| 2^{s_0} \pi^{s_0-1} \sin\left(\frac{\pi s_0}{2}\right) \Gamma(1-s_0) \right| = 1$  because of (4.21).

### 2<sup>nd</sup> Method (and 3<sup>rd</sup> Method, reference only)

Assume  $s_0$  is a simple nontrivial zero of the zeta function. It follows that  $\zeta(s_0) = 0$  and  $\zeta'(s_0) \neq 0$ . Then, as  $s$  approaches  $s_0$ , we can write Taylor expansions for  $\zeta(s)$  and  $\zeta(1-s)$  [see **Appendix 4-A-5**]:

- $\zeta(s) \approx c_1(s-s_0)$  and by symmetry, we also have
- $\zeta(1-s) \approx c_2(1-s-(1-s_0))$ ,

where  $c_1$  and  $c_2$  are constants that can be determined by the derivative of the functions at  $s_0$ . **Note that the Taylor series converges absolutely only when  $\text{Re}(s) > 1$ .** Since the indeterminate form exists in (4.25), therefore, the ratio:

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| \approx \left| \frac{c_1(s-s_0)}{c_2(1-s-(1-s_0))} \right| = \left| \frac{c_1}{c_2} \right|. \quad (\text{reference only; valid for } \text{Re}(s) > 1.)$$

Since both constants  $c_1$  and  $c_2$  are finite and nonzero, their ratio is finite. Due to the existing symmetry of the zero in the zeta function, we expect  $|c_1| = |c_2|$ . Indeed, we know  $|\zeta(s)/\zeta(1-s)|$  is even function of  $\tau$  (see the proof of statement #1 of **Properties of the Graph of  $|\zeta(s)/\zeta(1-s)|$** ), hence,  $\text{Im}(c_1(s-s_0)) = \text{Im}(c_2(1-s-(1-s_0)))$ . This holds if only if  $c_1 = c_2$ . Thus,

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = 1.$$

The preceding analysis (1<sup>st</sup> Method) leads to the following theorem.

#### 4-4-2-C. Indeterminate Ratio Theorem

Let  $s_0$  be nontrivial zero of the Riemann zeta function. **If  $s_0$  exists off the critical line**, where  $0 < \text{Re}(s) < 1$  and  $\text{Re}(s) \neq 1/2$ , then

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = 1. \quad (4.42)$$

#### Proof

Please see the proof as shown in **1<sup>st</sup> Method** of **Section 4-4-2-B**.

The existence of  $\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = 1$ , where  $0 < \text{Re}(s_0) < 1$ ,  $\text{Re}(s_0) = 0$ , holds an important role to identify nontrivial zeros where solutions may exist, thereby reducing the probability of discovering solutions throughout the entire complex plane when  $\text{Re}(s_0)$  lies within the critical strip. Based on this fact, we arrive another theorem for determination and localization of the nontrivial zero.

#### 4-4-2-D. Candidate Nontrivial Zero Theorem

Let  $s = \sigma + i\tau$  for real  $\sigma$  and  $\tau$ . A pair  $(\sigma, \tau)$  with  $0 < \sigma < 1$  can potentially correspond to a nontrivial zero of  $\zeta(s)$  if it meets the following conditions:

- (a)  $s_0$  satisfies the zeta functional relational equation (4.6)
- (b)  $\left| \frac{\zeta(\sigma + i\tau)}{\zeta(1 - \sigma - i\tau)} \right| = 1$  for  $0 < \sigma < 1$
- (c)  $s_0$  is considered a nontrivial zero if  $\zeta(s_0) = 0$ .

#### Proof

This theorem searches for the locations of nontrivial zeros of  $\zeta(s)$  in the complex plane for  $0 < \sigma < 1$ . It distinguishes between possible and actual nontrivial zeros of  $\zeta(s)$ :

1. Conditions (a) and (b) are necessary for  $(\sigma, \tau)$  to be a possible zero of  $\zeta(s)$  in the critical strip, according to Indeterminate Ratio Theorem.
2. Condition (c) is a verification step. If  $\zeta(s_0) = 0$  at  $s_0 = \sigma + i\tau$  or  $s = \sigma - i\tau$ , then  $s_0$  is a nontrivial zero of  $\zeta(s)$ . Notice that finding candidate solutions is only part of the job. As always, the final step is verification against the zeta equation.

We note that, for each  $\sigma$  in the interval  $(0, 1)$ , there may be multiple intersects between  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)| = 1$ . For example,  $s = s_0 = 1/2$ , there are infinite nontrivial zeros on the critical line. Additionally, when  $s$  is off the critical line, there may also be infinitely many intersections between  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)| = 1$  within the critical strip, where  $0 < \sigma < 1/2$  or  $1/2 < \sigma < 1$ . **Fortunately, the derivative of  $|\zeta(s)/\zeta(1-s)|$  exists and we have proved this ratio is strictly increasing for  $0 < \sigma < 1/2$  and decreasing for  $1/2 < \sigma < 1$ . This ensures that there is only one intersection between the functions  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)| = 1$  for every  $\sigma \neq 1/2$  within the interval  $(0, 1)$ . This leads us to solve the transcendental function for the unique  $\tau$ , a candidate solution of  $\zeta(s)$ .**

Next, we examine the values of  $\tau$  to determine how far it extends, by analyzing the intersection points between  $|\zeta(s)/\zeta(1-s)|$  and  $|G(s)| = 1$  through solving the transcendental equations for  $\tau$ . Finally, we summarize the overall behavior of the graphs and discuss the implications for the Riemann Hypothesis.

#### 4-4-3. Intersecting Points of $|\zeta(s)/\zeta(1-s)|$ and $|G(s)| = 1$

According to the Part(a) of Section 4-4-2-D, "Possible Nontrivial Zero Theorem", the possible nontrivial zeros satisfy the equation:

$$\left| \frac{\zeta(\sigma + i\tau)}{\zeta(1 - \sigma - i\tau)} \right| = 1 \quad (4.43)$$

and therefore any solution of (4.43) must also satisfy

$$\frac{2^{\sigma-1} \pi^\sigma}{\sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} \cdot |\Gamma(\sigma + i\tau)| = 1, \quad (4.44)$$

where the expression containing  $\Gamma$  is the expression appearing on the right-hand side of (4.24).

Both equations (4.43) and (4.44) are the transcendental equations and  $\tau$  can be only solved by numerically. As  $\sigma$  varies from 0 to 1, equation (4.43) (or equivalently, equation (4.44)) yields infinite number of intersection points  $\tau$ . Some of these points may represent nontrivial zeros if they satisfy the zeta equation  $\zeta(s) = 0$ .

Next, we determine  $\tau$  for particular values of  $\sigma$  that lead to a closed-form of function  $|\zeta(s)/\zeta(1-s)|$ , and then compute  $\tau$  for other values of  $\sigma$  as illustrated in Figure 1.

### 1. Particular values of $\sigma$

#### (a) $\sigma = 0$

At  $\sigma = 0$ , we have:

$$\left| \frac{\zeta(s)}{\zeta(1-s)} \right| = \left| \frac{\zeta(i\tau)}{\zeta(1-i\tau)} \right| = \frac{\tau}{2\pi} \tanh\left(\frac{\pi\tau}{2}\right) \quad (\text{See formula (4.31)})$$

Therefore, the intersection point between  $\left| \frac{\zeta(i\tau)}{\zeta(1-i\tau)} \right|$  and  $|G(s)| = 1$  is given by

$$\tau \tanh\left(\frac{\pi\tau}{2}\right) - 2\pi = 0. \quad (4.45)$$

The *transcendental* equation can be solved by numerically, yielding

$$\tau \approx +6.28318534 \dots \quad (4.46)$$

#### (b) $\sigma = 1$

Substituting the result from (4.31a) in equation (4.43) gives

$$\sqrt{\frac{2\pi}{\tau} \coth\left(\frac{2\pi}{\tau}\right)} = 1 \Rightarrow 2\pi \coth\left(\frac{2\pi}{\tau}\right) - \tau = 0. \quad (4.47)$$

The equation has a numerical solution at approximately

$$\tau \approx +6.28318534 \dots \quad (4.47 a)$$

*Readers may be surprised by the same results of (a) and (b) since their equations are identical. Indeed, we express  $\coth()$  in terms of  $\tanh()$  in equation (4.45) and rearrange the terms, resulting to equation (4.47). This indicates that it is sufficient to examine  $\sigma$  in either  $(0, 1/2)$  or  $(1/2, 1)$  since one of these is a reflection of the other, a known property called the reflection symmetry of the zeta function across the*

critical line.

## 2. Other values of $\sigma$

Numerical solutions of  $\tau$  for equation  $|\zeta(s)/\zeta(1-s)| - 1 = 0$  are presented in **Table IX** for other values of  $\sigma$ .

<b>Table IX – Intersect Points</b>	
$\sigma$	$\tau$
1/5	$\approx 6.2874412551730\dots$
1/4	$\approx 6.2881729195707\dots$
1/3	$\approx 6.2890968133855\dots$
0.49	$\approx 6.2898333277089\dots$
2/3	$\approx 6.2890968133855\dots$
3/4	$\approx 6.2881729195707\dots$
4/5	$\approx 6.2874412551730\dots$

[See **Appendix 4-B-2**]

We observe the same values of  $\tau$  for  $\sigma = 1/5$  and  $4/5$ ,  $\sigma = 1/4$  and  $3/4$ , and  $\sigma = 1/3$  and  $2/3$ , respectively. This consistency aligns with the well-known reflection symmetry of the zeta function across the critical line. When  $\sigma = 0.49$ , which is close to the real line  $1/2$ , the value of  $\tau$  to be approaching a contextual constant, which we call it a "saturation constant."

### 4-4-4. Saturation Constant

We intend to utilize the term "saturation constant" in Chemistry to refer to a saturation number that signifies the transformation of the function  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  curve into a straight line. This situation may arise as  $\sigma$  approaches  $1/2$ . In Section 4-4-1-B, we demonstrated that  $\left| \frac{\zeta(\sigma+i\tau)}{\zeta(1-\sigma-i\tau)} \right|$  has a local maximum at  $\sigma = 1/2$ , as evidenced by the change in sign of its derivative from positive to negative as  $\sigma$  passes through  $1/2$ . In this case,

$$\left| \frac{\zeta(1/2+i\tau)}{\zeta(1/2-i\tau)} \right| = 1,$$

a constant, indicating that the equation is independent of  $\tau$ . This result is also found the same on the right-hand side of (4.24). Therefore, the equation provides no information to determine, as  $\tau$  and its associated expression have been eliminated.

The statement #3 from section "Properties of the Graph of  $|\zeta(s)/\zeta(1-s)|$ " predicts a saturation point as  $\sigma$  approaches  $1/2$ . We know the existence of this point because the graphs presented in Figure 1 indicate that there exists a value of  $\tau$ , which must be a specific number that serves as the last point for the curve of the function  $\left| \frac{\zeta(1/2+i\tau)}{\zeta(1/2-i\tau)} \right|$  to transform into a straight line. This particular number does not appear to be represented by any mathematical expression. How can we determine this number?

Assuming there exists a very small epsilon  $\varepsilon$  such that a corresponding real value  $\tau$  exists to satisfy equation (4.43)

$$\lim_{\varepsilon \rightarrow 0} \left| \frac{\zeta\left(\frac{1}{2}-\varepsilon+i\tau\right)}{\zeta\left(\frac{1}{2}+\varepsilon-i\tau\right)} \right| = 1.$$

By choosing, for instance,  $\varepsilon = 0.00001$ , then  $\sigma = 1-\varepsilon = 1/2-0.00001 = 0.49999$  and  $1/2 + \varepsilon = 0.50001$ . This ensures that  $\tau$  exists in the following equation and can be solved using a high-precision numerical method:

$$\lim_{\varepsilon \rightarrow 0} \left\{ \left| \zeta\left(\frac{1}{2}-\varepsilon+i\tau\right) \right| - \left| \zeta\left(\frac{1}{2}+\varepsilon-i\tau\right) \right| \right\} = 0. \quad (4.48)$$

Likewise, the same result can be obtained from alternative equation (4.44), namely

$$\frac{2^{\sigma-1} \pi^{\sigma}}{\sqrt{\cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} |\Gamma(\sigma+i\tau)|} = 1$$

Therefore, squaring both sides and moving all terms to one side, we get

$$\left[ \cos^2\left(\frac{\pi\sigma}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right) \right] |\Gamma(\sigma+i\tau)|^2 - 2^{2\sigma-2} \pi^{2\sigma} = 0. \quad (4.49)$$

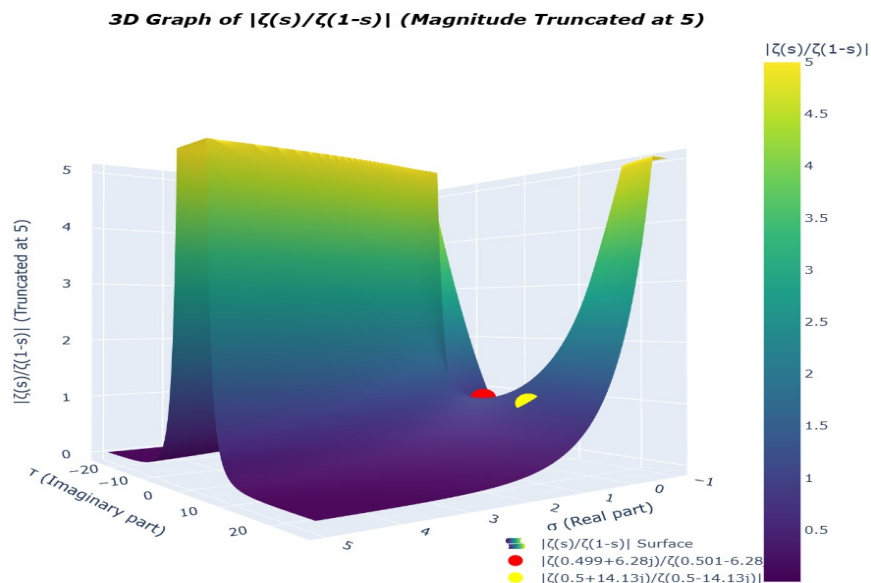
Let  $\sigma = 1/2 - \varepsilon$ . Similarly, by choosing a small  $\varepsilon$  such that a corresponding real value  $\tau$  exists, we ensure that  $\tau$  appears in the following equation and can be solved:

$$\left[ \cos^2\left(\frac{\pi(1/2-\varepsilon)}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right) \right] |\Gamma(1/2-\varepsilon+i\tau)|^2 - 2^{-1-2\varepsilon} \pi^{1-2\varepsilon} = 0. \quad (4.50)$$

Both equations (4.48) and (4.50) are of the transcendental equations that usually do not have closed-form solutions. By using an existing high-precision numerical method implemented Python libraries [see **Appendix 4-B-3**], we find that

$$\tau = \tau_s \approx 6.289835988836902779665090100821853... \quad (\text{Saturation Constant}) \quad (4.51)$$

The results from (4.47a) and (4.51) indicate that for  $0 < \sigma < 1$  and  $\sigma \neq 1/2$ , **the set of possible nontrivial zeros of the Riemann zeta function lies between  $\tau \approx 6.28318534...$  and  $\tau = \tau_s \approx 6.289835988...$ , a narrow band of 0.00665**. According to "4-4-2-D. Candidate Nontrivial Zero Theorem", no pairs within this set satisfy the criteria in Part (c), because the final possible zero at  $\tau = \tau_s \approx 6.289835988...$  lies below the first actual nontrivial zero at  $\tau_0 = 14.134725141...$ . This means that there is no nontrivial zero off the critical line to be discovered beyond this point. Since  $|\zeta(s)/\zeta(1-s)|$  is an even function of  $\tau$ , the other half of the domain yields a similar result. The 3D graph shown in Figure 5 illustrates  $|\zeta(s)/\zeta(1-s)|$ , *highlighting the saturation point (red circle) and the first nontrivial zero (yellow circle) on this surface, respectively*.



**Figure 5** – illustrates the saturation point (in red circle) and the first nontrivial zero (in yellow circle) on the surface of  $|\zeta(s)/\zeta(1-s)|$  [see **Appendix 4-B-1**].

Thus, we conclude that all possible nontrivial zeros off the critical line lie within the bands  $6.28318534... \leq \tau \leq \tau_s \approx 6.28983598...$  or  $-6.28983598... \approx \tau_s \leq \tau < -6.28318534...$ , and none of them provide as solutions to the Riemann zeta function because their imaginary parts are below the first non-trivial zeros. This also implies that there is no nontrivial zero off the critical line to be discovered beyond the points  $\tau = \tau_s \approx 6.289835988...$  or  $\tau = \tau_s \approx -6.289835988...$  as illustrated in **Figure 6**. As noted in the **Key Note [3\*]** of Section 4-3, we now return to this matter to present a clear and rigorous proof that neither  $\zeta(\sigma+i\tau)$  nor  $\zeta(1-\sigma-i\tau)$  can simultaneously be zero, nor can either  $\zeta$  have nontrivial zeros when  $\text{Re}(s)$  lies within the intervals  $(0, 1/2)$  or  $(1/2, 1)$ . The Key Note is valid, as there is no such nontrivial zero off the critical line existed for which  $\zeta(s_0) = 0$ , implying that  $\zeta(1-s_0) = 0$  ( $\sigma_0 \neq 1/2$ )!

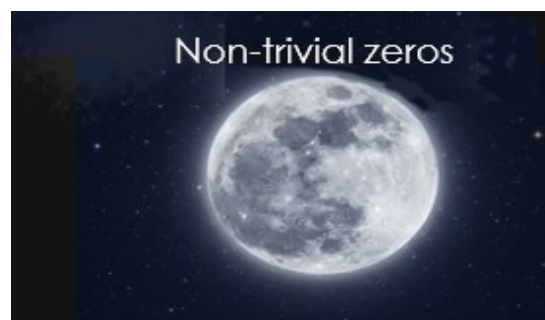


the zeta function lie on the critical line.

**We summarize the facts that**

- the trivial zeros are located on the left half-plane at  $s = -2n$  for positive integers  $n$ . (Known fact)
- infinitely many zeros lie on the critical line  $\text{Re}(s) = 1/2$ . (Known fact)
- the first nontrivial zero of  $\zeta(s) = 0$  is  $s = 1/2 + i\tau$ , where  $\tau = 14.134725141\dots$  (Known fact)
- for  $0 < \sigma < 1$  and  $\sigma \neq 1/2$ , all possible nontrivial zeros lie in the narrow bands  $6.28318534\dots < \tau \leq \tau_s \approx 6.28983598\dots$  or  $\tau_s \approx -6.28983598\dots \leq \tau < -6.28318534\dots$ . None of them provide as solutions to  $\zeta(s) = 0$  because their imaginary parts are below the first nontrivial zero.
- the nontrivial zeros cannot lie in the critical strip  $0 < \text{Re}(s) < 1$  but are located solely at  $\text{Re}(s) = 1/2$ .

**The Riemann conjecture has been confirmed as correct.**



*To seek the moon beneath the lake is vain,  
No truth beneath its mirrored gleam.*

## Appendix 4-A: Miscellaneous Proofs

### [4-A-1]

#### Zeta Function Has Nontrivial Zeros for $\sigma = \text{Re}(s) = 1/2$

To find solutions of equation (4.6), we establish a system of two simultaneous equations by setting both its sides to zero:

$$\begin{cases} \zeta(s) = 0 & (4.6a) \\ 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) = 0 & (4.6b) \end{cases}$$

Equation (4.6a) is anticipated to have the nontrivial zeros for  $0 < \text{Re}(s) < 1$ , according to Table I. We observe that no nontrivial zeros for  $0 < \text{Re}(s) < 1$  are found from  $2^s$ ,  $\pi^{s-1}$ ,  $\sin(\pi s/2)$ , and  $\Gamma(1-s)$  in equation (4.6b) except for  $\zeta(1-s)$ . On the other hand, we know that  $\zeta(s)$  has infinite nontrivial zeros on  $\text{Re}(s) = 1/2$  by Hardy's proof<sup>[21]</sup>. Hence, let  $s = \sigma + i\tau_k$ , where  $\sigma$  and  $\tau_k$  are real, and  $k$  is integer. By substituting  $s$  into (4.6a) and (4.6b), we obtain

$$\begin{cases} \zeta(\sigma + i\tau_k) = 0 & (4.6c) \\ 2^{\sigma + i\tau_k} \pi^{\sigma + i\tau_k - 1} \sin\left(\frac{\pi(\sigma + i\tau_k)}{2}\right) \Gamma(1 - \sigma + i\tau_k) \zeta(1 - \sigma + i\tau_k) = 0 & (4.6d) \end{cases}$$

A solution to equations (4.6c) and (4.6d) exists if that solution satisfies both equations and shares the same conjugate form, such that  $\sigma$  from (4.6c) equals to  $1 - \sigma$  from (4.6d), namely

$$\sigma = 1 - \sigma \Rightarrow \sigma = \frac{1}{2}. \quad (4.6e)$$

By substituting  $\sigma = 1/2$  into (4.6c) and (4.6d), we obtain

$$\left\{ \zeta\left(\frac{1}{2} + i\tau_k\right) = 0 \right. \quad (4.6f)$$

$$\left. 2^{\frac{1}{2} + i\tau_k} \pi^{\frac{1}{2} + i\tau_k} \sin\left(\frac{\pi}{2}\left(\frac{1}{2} + i\tau_k\right)\right) \Gamma\left(\frac{1}{2} - i\tau_k\right) \zeta\left(\frac{1}{2} - i\tau_k\right) = 0 \right. \quad (4.6g)$$

We see that the conjugate forms,  $\zeta\left(\frac{1}{2} + i\tau_k\right)$  from (4.6f) and  $\zeta\left(\frac{1}{2} - i\tau_k\right)$  from (4.6g), do exist for  $\text{Re}(s) = \sigma = 1/2$  and for certain real  $\tau_k$ . By applying Property (ii) (Zeta conjugate and symmetry) in Section 4-1-6, it follows that  $s=1/2+i\tau_k$  and  $s=1/2-i\tau_k$  are the nontrivial zeros of the zeta functions in (4.6f) and (4.6g).

#### [4-A-2]

##### Zeta conjugate and symmetry property

The zeta functional equation (4.6) satisfies the complex conjugate identity,  $\overline{\zeta(s)} = \zeta(\bar{s}) \Leftrightarrow \zeta(s) = \overline{\zeta(\bar{s})}$  for all  $s \neq 1$ . It follows that

a. If  $\zeta(\bar{s}) = a + ib$ , where  $a$  and  $b$  are real numbers, then

$$\zeta(s) = \overline{\zeta(\bar{s})} = \overline{a + ib} = a - ib.$$

b. If  $s$  is a nontrivial zero then so is  $\bar{s}$ . It is equivalent to if  $\zeta(s) = 0$  then

$$\zeta(\bar{s}) = \overline{\zeta(s)} = \overline{0} = 0.$$

It means the set of all nontrivial zeros is symmetric about the real axis, indicating that these zeros appear in conjugate pairs. In fact, the complex conjugate identity,  $\zeta(s) = \overline{\zeta(\bar{s})}$ , is a fundamental property of the Riemann zeta function. This is often referred to as the Schwarz Reflection Principle <sup>[25][26]</sup> or the Conjugate Property. We prove this fundamental property as follows:

The Riemann zeta function is defined for  $\text{Re}(s) > 1$ ,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Let  $s = a + ib$ , where  $a$  and  $b$  are real numbers. Then  $\bar{s} = a - ib$ . Therefore,

$$\zeta(\bar{s}) = \sum_{k=1}^{\infty} \frac{1}{k^{\bar{s}}} = \sum_{k=1}^{\infty} \frac{1}{k^{a-ib}}.$$

Now, let's take the complex conjugate of  $\zeta(\bar{s})$ :

$$\begin{aligned} \overline{\zeta(\bar{s})} &= \overline{\sum_{k=1}^{\infty} \frac{1}{k^{a-ib}}} \\ &= \overline{\sum_{k=1}^{\infty} \frac{1}{e^{(a-ib)\ln k}}} \\ &= \overline{\sum_{k=1}^{\infty} \frac{1}{e^{a \ln k} e^{-ib \ln k}}} \\ &= \overline{\sum_{k=1}^{\infty} \frac{1}{k^a (\cos(-b \ln k) + i \sin(-b \ln k))}} \\ &= \overline{\sum_{k=1}^{\infty} \frac{1}{k^a (\cos(b \ln k) - i \sin(b \ln k))}} \\ &= \overline{\sum_{k=1}^{\infty} \frac{1}{k^a (\cos(b \ln k) + i \sin(b \ln k))}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k^a e^{ib \ln k}} \\ &= \sum_{k=1}^{\infty} \frac{1}{e^{a+ib \ln k}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k^{a+ib}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s). \end{aligned}$$

Thus, we demonstrate the property  $\zeta(s) = \overline{\zeta(\bar{s})}$  for  $\text{Re}(s) > 1$ . Recall that the Schwarz Reflection Principle <sup>[25][26]</sup> states that if a function  $f$  is analytic on a domain symmetric about the real axis and takes real values on a segment of the real axis, then it satisfies

$$f(\bar{s}) = \overline{f(s)}.$$

Since the analytic continuation of  $\zeta(s)$  is real for real  $s > 1$  and analytic on  $\mathbb{C} \setminus \{1\}$ , applying the principle gives

$$\zeta(\bar{s}) = \overline{\zeta(s)} \Rightarrow \overline{\zeta(\bar{s})} = \zeta(s),$$

which extends the identity to all  $s \in \mathbb{C} \setminus \{1\}$ .

### Summary:

1. For  $\text{Re}(s) > 1$ , the identity  $\zeta(s) = \overline{\zeta(\bar{s})}$  follows directly from the series definition.
2. The analytic continuation of  $\zeta(s)$  and the Schwarz Reflection Principle <sup>[25][26]</sup> allow us to extend this property to the entire domain of  $\zeta(s)$  in the complex plane for all  $s \neq 1$ .

### [4-A-3]

We prove (4.20) by using the reflection gamma  $\Gamma(1-s) = \frac{\pi}{\Gamma(s) \sin(\pi s)}$ . From (4.19), we have:

$$\begin{aligned} \frac{\zeta(s)}{\zeta(1-s)} &= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \\ &= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \frac{\pi}{\Gamma(s) \sin(\pi s)} \end{aligned}$$

$$= \frac{2^{s-1} \pi^s}{\cos\left(\frac{\pi s}{2}\right) \Gamma(s)}$$

**[4-A-4]**

The validation of the pole depends on the position of the gamma function within the expression. For instance, the ratio involving the zeta function and the gamma function in the following expression is valid at the pole  $s = 1$ :

$$\begin{aligned} \lim_{s \rightarrow 1} \frac{\zeta(s)}{\Gamma(1-s)} &= \lim_{s \rightarrow 1} \left\{ 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) \right\} \\ &= 2 \cdot 1 \cdot 1 \cdot \left(-\frac{1}{2}\right) \quad (\zeta(0) = -1/2) \\ &= -1. \end{aligned}$$

Numerical computation confirms that  $\frac{\zeta(s)}{\Gamma(1-s)} = \frac{\zeta(0.999)}{\Gamma(1-0.999)} = 0.99999\dots$ , supporting the consistency of the formulation.

At this result, we can appreciate the *elegance* of the zeta functional equation, which can accommodate all values of  $s$  due to its rich analytic characteristics in the complex plane.

**[4-A-5]**

The zeta functional equation exhibits conjugate symmetry, which stems from the identity  $\zeta(s) = \overline{\zeta(\bar{s})}$ . So, the statement that if  $s = \sigma + i\tau$  is a zero of  $\zeta(s)$ , then  $\bar{s} = \sigma - i\tau$  is also a zero. This is robust. In contrast, the statement that if  $\zeta(s) = 0$  then  $\zeta(1-s) = 0$  relies solely the zeta functional equation (4.6). Any zero of  $\zeta(s)$  belongs to a quartet automatically, namely  $s, \bar{s}, 1-s, 1-\bar{s}$ , based on the symmetries of the zeta function.

If the zero is not simple, then near  $s = s_0$ ,  $\zeta(s)$  has a nontrivial zero of order  $m$  and  $\zeta(1-s)$  has a nontrivial zero of order  $n$ , where  $m$  and  $n$  are integer. We can write

$\zeta(s) \approx c_1(s-s_0)^m$  and  $\zeta(1-s) \approx c_2(1-s-(1-s_0))^n$  near  $s = s_0$ , where  $c_1$  and  $c_2$  are constants.

Therefore, the ratio

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| \approx \left| \frac{c_1(s-s_0)^m}{c_2(1-s-(1-s_0))^n} \right|$$

By the symmetry and the continuous of  $|\zeta(s)/\zeta(1-s)|$ , the orders of zero must be the same multiplicity, namely  $m = n$ :

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| \approx \left| \frac{c_1(s-s_0)^n}{c_2(1-s-(1-s_0))^n} \right| = \left| \frac{c_1}{c_2} \right|$$

At this point, we are back to the case where the zero is considered as a simple, and we find its magnitude is equal to 1.

**3<sup>rd</sup> Method.** We provide another simple prove as follows: Since  $s_0$  and  $1-s_0$  are the nontrivial zeros the zeta function within  $0 < \text{Re}(s) < 1$ , we have  $\zeta(s_0) = a + ia$  and  $\zeta(1-s_0) = b+ib$ , where  $a$  and  $b$  approach 0. For  $\mu$  be real number and assuming that  $b = \mu a$ , then, as  $s \rightarrow s_0$ , the ratio

$$\lim_{s \rightarrow s_0} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = \lim_{a \rightarrow 0, b \rightarrow 0} \left| \frac{a+ia}{b+ib} \right| = \lim_{a \rightarrow 0} \left| \frac{a+ia}{\mu(a+ia)} \right| = \left| \frac{1}{\mu} \right|.$$

We also know  $|\zeta(s)/\zeta(1-s)|$  is even function of  $\tau$  (see the proof of statement #1 of **Properties of the Graph of  $|\zeta(s)/\zeta(1-s)|$** ), hence,  $\text{Im}(a+ia) = \text{Im}(\mu(a+ia))$ . This holds if only if  $\mu=1$ .

**[4-A-6]**

The expression,

$$\left| \frac{0+i0}{0-i0} \right| \equiv \lim_{\lambda \rightarrow 0} \left| \frac{\lambda+i\lambda}{\lambda-i\lambda} \right| = 1 \quad \text{or not} \quad \frac{0}{0} = 1,$$

has the potential mislead readers, as it is typically considered undefined in standard mathematical contexts. However, in certain specialized equations, such as especially within the analysis of the zeta functional equation, this indeterminate form can arise naturally and often yields results that align with well-known mathematical constants. We emphasize here that such expressions are not to be disregarded entirely. Instead, their existence should be acknowledged explicitly, as they can carry meaningful mathematical significance in certain analytical frameworks.

**[4-A-7]**

We have

$$\ln |\Gamma(s)| = \Re(\ln \Gamma(s))$$

Differentiate both sides with respect to  $\tau$

$$\begin{aligned} \frac{\frac{d}{d\tau} |\Gamma(s)|}{|\Gamma(s)|} &= \Re \left( \frac{\Gamma'(s)}{\Gamma(s)} i \right) \\ &= \Re(i \Psi(s)) \\ &= -\Im(\Psi(s)) \\ \Rightarrow \frac{d}{d\tau} |\Gamma(s)| &= -|\Gamma(s)| \Im(\Psi(s)). \end{aligned}$$

## Appendix 4-B: Graphs & Python Source Code

### [4-B-1]

**1. The Python source code (chapter3.zip)** used to generate Figures 1–5 is available for download at [https://seriesmathstudy.com/sms/part2\\_download](https://seriesmathstudy.com/sms/part2_download), allowing you to run and explore each graph locally. We use the following versions of Python and math scientific libraries: Python 3.13.7, Plotly 6.3.1, mpmath 1.3.0, numpy 2.3.3, scipy 1.16.2 and tqdm 4.67.1.

### 2. Figure 1 (file name: chapter4\_graph\_figure1.py)

#### Source Code:

```
import numpy as np
from scipy.special import zeta
import matplotlib.pyplot as plt
from fractions import Fraction
# Plot the ratio  $|\zeta(s)/\zeta(1-s)|$  in 2D
def plot_ratio_zetas():
    max_height = 3.5
    start = -10
    end = 10
    num_points = 300
    step = 1
    # Define and plot  $|G(s)| = 1$ 
    def plot_line(ax):
        t = np.linspace(start, end, num_points)
        g = np.full_like(t, 1)
        ax.plot(t, g, label='$|G(s)| = 1$', color='red', linestyle='--')
        ax.legend()
    # Define and plot  $|\zeta(s)/\zeta(1-s)|$ 
    def plot_abs_ratio_zetas(str_real_part, ax):
        # Define variables for displaying the labels
        sigma = "σ"
```

## Appendix 4-B: Graphs & Python Source Code

```
tau = "τ"
# Convert to number
real_part = eval(str_real_part)
# Generate a range of imaginary values (t)
t = np.linspace(start, end, num_points)

# Define complex number
s = real_part + 1j * t
# Compute the absolute value of the zeta function
# abs_zeta = [|ζ(val)/ζ(1-val)| for val in s]
# Plot τ for τ < max_height
abs_zeta = [
    max_height
    if abs(zeta(val)/zeta(1-val)) > max_height
    else abs(zeta(val)/zeta(1-val)) for val in s
]
# Plot |ζ(val)/ζ(1-val)|
ax.plot(t, abs_zeta, label=fr'$|\zeta(\{\text{str\_real\_part}\} + i\{\text{tau}\})/\zeta(\{\text{convert\_fraction(str\_real\_part)} - i\{\text{tau}\})|$',
ax.set_title(fr'Plot of $|F(\{\sigma\},\{\text{tau}\})| = |\zeta(\{\sigma\} + i\{\text{tau}\})/\zeta(1-\{\sigma\} + i\{\text{tau}\})|$ and $|G(\{\sigma\},\{\text{tau}\})| = 1 $')
ax.set_xlabel(fr'$\{\text{tau}\}$ (Imaginary Part)')
ax.set_ylabel('Output Value')
# Set the tick locations for the τ-axis to be integers from 0 to 10
ax.set_xticks(np.arange(start, end, step))
# Set the tick locations for the y-axis to be integers
ax.set_yticks(np.arange(0, max_height, 0.5))
ax.grid(True)
ax.legend()
# Create the figure and axes objects
fig, ax = plt.subplots(figsize=(8, 7))
# Plot |ζ(s)/ζ(1-s)|
plot_abs_ratio_zetas("0", ax)      # σ = 0
plot_abs_ratio_zetas("1/4", ax)    # σ = 1/4
plot_abs_ratio_zetas("1/3", ax)    # σ = 1/3
```

```

plot_abs_ratio_zetas("0.49", ax)    #  $\sigma = 0.49$ 
plot_abs_ratio_zetas("1/2", ax)    #  $\sigma = 1/2$ 
plot_abs_ratio_zetas("0.51", ax)    #  $\sigma = 0.51$ 
plot_abs_ratio_zetas("2/3", ax)    #  $\sigma = 2/3$ 
plot_abs_ratio_zetas("3/4", ax)    #  $\sigma = 3/4$ 
plot_abs_ratio_zetas("1", ax)      #  $\sigma = 1$ 
# Plot  $|G(s)| = 1$ 
plot_line(ax)
plt.show()

def convert_fraction(str_):
    # Convert the string to a Fraction object
    str_fraction = Fraction(str_)
    # Perform the calculation, resulting in a new Fraction object
    result_frac = 1 - str_fraction
    # Convert the result back to a string
    return str(result_frac)
plot_ratio_zetas()

```

### 3. Figure 2 (file name: chapter4\_graph\_figure2.py)

#### Source Code:

```

import numpy as np
from scipy.special import gamma, zeta
import plotly.graph_objects as go
from tqdm import tqdm
# Plot the ratio  $|\zeta(s)/\zeta(1-s)|$  in 3D
def plot_abs_ratio_zetas():
    # Define  $\zeta(s)/\zeta(1-s)$  complex function. Input: s - complex number
    def ratio_complex_function(s):
        try:
            return zeta(s)/zeta(1-s)
        except ValueError:
            return np.nan # If s = 1, return NaN.

```

```

# Create a grid of complex numbers
real_min, real_max = -1, 5
imag_min, imag_max = -25, 25
grid_points_σ, grid_points_τ = 500, 500
real_grid = np.linspace(real_min, real_max, grid_points_σ)
imag_grid = np.linspace(imag_min, imag_max, grid_points_τ)
s_grid = real_grid[np.newaxis, :] + 1j * imag_grid[:, np.newaxis]

# Compute the ζ(s)/ζ(1-s) function values over the grid
print("Computing the values of |ζ(s)/ζ(1-s)|...")
zeta_values = np.zeros_like(s_grid, dtype=np.complex128)
for i in tqdm(range(grid_points_τ)):
    for j in range(grid_points_σ):
        zeta_values[i, j] = ratio_complex_function(s_grid[i, j])

# Take magnitude of ζ(s)/ζ(1-s)
magnitude = np.abs(zeta_values)

# Truncate magnitude values greater than 5
truncated_magnitude = np.where(magnitude > 5, 5, magnitude)

# Create the Plotly 3D surface figure, critical point and first nontrivial zero
fig = go.Figure(data=[go.Surface(
    x = real_grid,
    y = imag_grid,
    z = truncated_magnitude,
    colorscale = 'Plasma',
    name='|ζ(s)/ζ(1-s)| Surface',
    colorbar = dict(title='|ζ(s)/ζ(1-s)|'),
    showlegend=False
)])

# Update layout

```

```

fig.update_layout(
    title=dict(
        text='<b><i>3D Graph of  $|\zeta(s)/\zeta(1-s)|$  (Magnitude Truncated at 5)</i></b>',
        font=dict(size=16, family='Verdana', color='black'),
        x=0.5, # 0: left, 0.5: center, 1: right
        y=0.9
    ),
    scene=dict(
        xaxis_title = 'σ (Real part)',
        yaxis_title = 'τ (Imaginary part)',
        zaxis_title = ' $|\zeta(s)/\zeta(1-s)|$  (Truncated at 5)',
    ),
    legend=dict(
        x=0.6,
        y=0 # Vertical position (0 = bottom, 1 = top)
    ),
    width = 700,
    height = 600,
)

# Display the ratio  $|\zeta(s)/\zeta(1-s)|$ , critical point and first nontrivial zero.
fig.show()

# Or save the figure as a local HTML file, and display it in the browser.
# fig.write_html("ratio_zetas_plot.html", auto_open=True)
# print("Plot saved to plot_abs_ratio_zetas.html")

```

#### 4. Figure 3 (file name: chapter4\_graph\_figure3.py)

##### Source Code:

```

import numpy as np
import plotly.graph_objects as go
from mpmath import zeta, mp

def plot_abs_zeta():

```

## Appendix 4-B: Graphs & Python Source Code

```
# Set precision for mpmath
mp.dps = 15
# Define grid (avoid sigma = 1 to prevent zeta(1) pole)
sigma_vals = np.linspace(0, 0.9999, 100)
tau_vals = np.linspace(0, 50, 100)
# Initialize empty array for zeta magnitudes
zeta_vals = np.zeros((len(tau_vals), len(sigma_vals)))
# Compute |zeta(sigma + i*tau)| with truncation and error handling
for i, tau in enumerate(tau_vals):
    for j, sigma in enumerate(sigma_vals):
        try:
            s = complex(sigma, tau)
            value = abs(zeta(s))
            zeta_vals[i, j] = min(value, 5) # Truncate height to max 5
        except ValueError:
            zeta_vals[i, j] = np.nan # Handle pole at s = 1
# Create meshgrid for plotting
sigma_grid, tau_grid = np.meshgrid(sigma_vals, tau_vals)
# Create 3D surface plot
fig = go.Figure(data=[go.Surface(
    x=sigma_grid,
    y=tau_grid,
    z=zeta_vals,
    colorscale='Viridis', # Magma, Plasma, Cividis, Viridis, Inferno
    name='|ζ(σ + iτ)| Surface',
    showlegend=True,
    colorbar=dict(
        title='|ζ(σ + iτ)|',
        x=0.8 # Moves the colorbar farther to the left
    )
)])

# Add critical line at σ = 0.5
critical_sigma = 0.5
```

## Appendix 4-B: Graphs & Python Source Code

```
critical_τ = τ_vals
critical_z = []
# Find the column index closest to  $\sigma = 0.5$ 
j_critical = np.argmin(np.abs(sigma_vals - critical_sigma))
for i in range(len(τ_vals)):
    z = zeta_vals[i, j_critical]
    critical_z.append(z if not np.isnan(z) else 0)
fig.add_trace(go.Scatter3d(
    x=[critical_sigma] * len(critical_τ),
    y=critical_τ,
    z=critical_z,
    mode='lines',
    line=dict(color='red', width=4),
    name='Critical Line  $\sigma=1/2$ ',
    showlegend=True
))
# Update layout
fig.update_layout(
    title=dict(
        text='<b><i>Graphs of  $|\zeta(\sigma + i\tau)|$  and the Critical Line</i></b>',
        font=dict(size=16, family='Verdana', color='black'),
        x=0.5, # 0: left, 0.5: center, 1: right
        y=0.9
    ),
    scene=dict(
        xaxis_title='σ (Real part)',
        yaxis_title='τ (Imaginary part)',
        zaxis_title=' $|\zeta(\sigma + i\tau)|$ '
    ),
    legend=dict(
        x=0.66,
        y=0 # Vertical position (0 = bottom, 1 = top)
    )
)
```

```
fig.show()
plot_abs_zeta()
```

## 5. Figure 4 (file name: chapter4\_graph\_figure4.py)

### Source Code:

```
import numpy as np
from scipy.special import zeta
import matplotlib.pyplot as plt
from matplotlib.ticker import MultipleLocator
def plot_zeta_2d():
    # Set the real part of the complex number
    real_part = 1/2
    τ = np.linspace(-40, 40, 1000)
    s = real_part + 1j * τ
    abs_zeta = [abs(zeta(val)) for val in s]
    # Create a figure and axes
    fig, ax = plt.subplots(figsize=(8, 6))
    # Plot abs_zeta
    ax.plot(τ, abs_zeta)
    # Set the major and minor tick locations
    ax.xaxis.set_major_locator(MultipleLocator(5))
    ax.xaxis.set_minor_locator(MultipleLocator(1))
    ax.yaxis.set_major_locator(MultipleLocator(4))
    ax.yaxis.set_minor_locator(MultipleLocator(0.25))
    # Set gridlines, styling major and minor
    ax.grid(which='major', color='gray', linestyle='-')
    ax.grid(which='minor', color='lightgray', linestyle='--', linewidth=0.5)
    ax.set_ylim(0, 4) # Set the τ-axis limits to 0 and 4
    # Add a title and labels
    ax.set_title(fr'Plot of  $|\zeta(1/2 + i\tau)|$ ')
    ax.set_xlabel(fr' $\text{Im}(1/2 + i\tau)$ ')
    ax.set_ylabel(fr' $|\zeta(1/2 + i\tau)|$ ')
```

```
plt.grid(True)
plt.show()
plot_zeta_2d()
```

## 6. Figure 5 (file name: chapter4\_graph\_figure5.py)

### Source Code:

```
import numpy as np
from scipy.special import gamma, zeta
import plotly.graph_objects as go
from tqdm import tqdm

# Plot the ratio  $|\zeta(s)/\zeta(1-s)|$ , the saturation point and the first nontrivial zero in 3D
def plot_abs_ratio_zetas_and_2_points():
    # Define  $\zeta(s)/\zeta(1-s)$  complex function. Input: s - complex number
    def ratio_complex_function(s):
        try:
            return zeta(s)/zeta(1-s)
        except ValueError:
            return np.nan # If s = 1, return NaN.

    # Create a grid of complex numbers
    real_min, real_max = -1, 5
    imag_min, imag_max = -25, 25
    grid_points_σ, grid_points_τ = 500, 500
    real_grid = np.linspace(real_min, real_max, grid_points_σ)
    imag_grid = np.linspace(imag_min, imag_max, grid_points_τ)
    s_grid = real_grid[np.newaxis, :] + 1j * imag_grid[:, np.newaxis]
    # Compute the  $\zeta(s)/\zeta(1-s)$  function values over the grid
    print("Computing the values of  $|\zeta(s)/\zeta(1-s)|$ ...")
    zeta_values = np.zeros_like(s_grid, dtype=np.complex128)
    for i in tqdm(range(grid_points_τ)):
        for j in range(grid_points_σ):
            zeta_values[i, j] = ratio_complex_function(s_grid[i, j])
    # Take magnitude of  $\zeta(s)/\zeta(1-s)$ 
    magnitude = np.abs(zeta_values)
```

## Appendix 4-B: Graphs & Python Source Code

```
# Truncate magnitude values greater than 5
truncated_magnitude = np.where(magnitude > 5, 5, magnitude)
# Highlight the critical point (saturation constant) at  $\sigma = 0.499$  and  $\tau = 6.28$ 
_σ = [0.499]
_τ = [6.28]
critical_point = 0.499+6.28j
_z = [abs(zeta(critical_point)/zeta(1-critical_point))]
# Create a separate trace for the single, large point
highlight_critical_point = go.Scatter3d(
    x = _σ,
    y = _τ,
    z = _z,
    mode = 'markers',
    marker=dict(
        size = 12,
        color = 'red', # Use a contrasting color
        symbol = 'circle'
    ),
    name='|ζ(0.499+6.28j)/ζ(0.501-6.28j)|' # Label for the legend
)
# Show the first nontrivial zero at  $\sigma = 0.5$  and  $\tau = 14.1347$ 
σ_1st_zero = [0.5]
τ_1st_zero = [14.1347]
s_1st_zero = 0.5 + 14.1347j
z_1st_zero = [abs(zeta(s_1st_zero)/zeta(1-s_1st_zero))]
# Create a blue circle presenting for the first nontrivial zero
show_1st_zero = go.Scatter3d(
    x = σ_1st_zero,
    y = τ_1st_zero,
    z = z_1st_zero,
    mode='markers',
    marker=dict(
        size = 12,
        color = 'yellow',
```

## Appendix 4-B: Graphs & Python Source Code

```
        symbol = 'circle'
    ),
    name = '|ζ(0.5+14.13j)/ζ(0.5-14.13j)|' # Label for the legend
)
# Create the Plotly 3D surface figure, critical point and first nontrivial zero
fig = go.Figure(data=[go.Surface(
    x = real_grid,
    y = imag_grid,
    z = truncated_magnitude,
    colorscale = 'Viridis',
    name='|ζ(s)/ζ(1-s)| Surface',
    colorbar = dict(title='|ζ(s)/ζ(1-s)|'),
    showlegend=True
), highlight_critical_point, show_1st_zero])
# Update layout
fig.update_layout(
    title=dict(
        text='<b><i>3D Graph of |ζ(s)/ζ(1-s)| (Magnitude Truncated at 5)</i></b>',
        font=dict(size=16, family='Verdana', color='black'),
        x=0.5, # 0: left, 0.5: center, 1: right
        y=0.9
    ),
    scene=dict(
        xaxis_title = 'σ (Real part)',
        yaxis_title = 'τ (Imaginary part)',
        zaxis_title = '|ζ(s)/ζ(1-s)| (Truncated at 5)',
    ),
    legend=dict(
        x=0.66,
        y=0 # Vertical position (0 = bottom, 1 = top)
    ),
    width = 800,
    height = 800,
)
```

```
# Display the ratio  $|\zeta(s)/\zeta(1-s)|$ , critical point and first nontrivial zero.
fig.show()
# Or save the figure as a local HTML file, and display it in the browser.
# fig.write_html("ratio_zetas_plot.html", auto_open=True)
# print("Plot saved to plot_abs_ratio_zetas.html")
```

## [4-B-2]

We present both the source code for solving equation and its corresponding output to demonstrate the numerical solution and provide clear evidence of the results. See **chapter4.zip**. Link: [https://seriesmathstudy.com/sms/part2\\_download](https://seriesmathstudy.com/sms/part2_download).

### Source Code (file name: *chapter4\_solve\_equation.py*)

```
import mpmath
mpmath.mp.dps = 20 # Set the desired precision
def find_root( $\sigma$ , initial_guess=1.0):
    # Find root of the equation:  $|\zeta(\sigma + i\tau) / \zeta(1 - (\sigma + i\tau))| - 1 = 0$ 
    def equation( $\tau$ ):
        s = mpmath.mpc( $\sigma$ , - $\tau$ )
        ratio = abs(mpmath.zeta( $\sigma + \tau * 1j$ ))/abs(mpmath.zeta(1 - s))
        return ratio - 1
    try:
        # Utilize findroot with default hybrid. Other methods: newton, secant, muller, deb
        root = mpmath.findroot(equation, initial_guess, verbose=False)
        check_value = equation(root)
        print(f"For  $\sigma=\{\sigma\}$ ,  $\tau=\{\text{root}\}$ . Check: equation( $\{\text{root}\}$ )= $\{\text{check\_value}\}$ ")
    except (mpmath.NoConvergence, ValueError) as e:
        print(f"Failed to find root for  $\sigma=\{\sigma\}$ :  $\{e\}$ ")
# Solutions of  $|\zeta(\sigma + i\tau) / \zeta(1 - (\sigma + i\tau))| - 1 = 0$  for  $\sigma = 0, 1/5, 1/4, \text{etc.}$ 
find_root(0)
find_root(1/5)
find_root(1/4)
find_root(1/3)
find_root(0.49)
find_root(0.51)
```

```
find_root(2/3)
find_root(3/4)
find_root(4/5)
find_root(1)
```

## Output

```
For  $\sigma=0$ ,  $\tau=6.2831853407982434122$ . Check:  $\text{equation}(6.2831853407982434122)=0.0$ 
For  $\sigma=0.2$ ,  $\tau=6.2874412551730266978$ . Check:  $\text{equation}(6.2874412551730266978)=1.6940658945086e-21$ 
For  $\sigma=0.25$ ,  $\tau=6.2881729195707257949$ . Check:  $\text{equation}(6.2881729195707257949)=0.0$ 
For  $\sigma=0.3333333333333333$ ,  $\tau=6.2890968133855667714$ . Check:  $\text{equation}(6.2890968133855667714)=0.0$ 
For  $\sigma=0.49$ ,  $\tau=6.2898333277089574802$ . Check:  $\text{equation}(6.2898333277089574802)=0.0$ 
For  $\sigma=0.51$ ,  $\tau=6.2898333277089574802$ . Check:  $\text{equation}(6.2898333277089574802)=0.0$ 
For  $\sigma=0.6666666666666666$ ,  $\tau=6.2890968133855667719$ . Check:  $\text{equation}(6.2890968133855667719)=0.0$ 
For  $\sigma=0.75$ ,  $\tau=6.2881729195707257949$ . Check:  $\text{equation}(6.2881729195707257949)=0.0$ 
For  $\sigma=0.8$ ,  $\tau=6.2874412551730266969$ . Check:  $\text{equation}(6.2874412551730266969)=0.0$ 
For  $\sigma=1$ ,  $\tau=6.2831853407982434122$ . Check:  $\text{equation}(6.2831853407982434122)=0.0$ 
```

## [4-B-3]

*We present both the source code and its corresponding output to demonstrate the numerical solution and provide clear evidence of the results.*

### Source Code (file name: chapter4\_saturation\_num.py)

```
from mpmath import mp, mpc, mpf, zeta, gamma, findroot, pi, cos, sinh
```

```
# Set the desired precision (150 decimal places)
```

```
mp.dps = 150
```

```
# Define a symbolic math equation 1:  $|\zeta(s)/\zeta(1-s)| - 1 = 0$ 
```

```
def equation_1( $\sigma$ ,  $\tau$ ):
```

```
    # Convert the string to a high-precision mpf object
```

```
     $\sigma\_ = \text{mpf}(\sigma)$ 
```

```
    # Perform the subtraction
```

```
    one_minus_ $\sigma = \text{mpf}(1) - \sigma\_$ 
```

```
    # Convert the high-precision result back to a string
```

```
    one_minus_ $\sigma\_str = \text{str}(one\_minus\_sigma)$ 
```

```
    return  $\text{abs}(\text{zeta}(\text{mpc}(\sigma\_ , \tau)))/\text{abs}(\text{zeta}(\text{mpc}(one\_minus\_sigma\_str , -\tau))) - 1.0$ 
```



## Appendix 4-B: Graphs & Python Source Code

```
# Solve for  $\tau$  based on the ratio:  $|\zeta(s)/\zeta(1-s)| - 1 = 0$   
print_result_based_on_zetas()
```

```
# Solve for  $\tau$  based on the equation involved the gamma function:  
#  $(\cos(\pi\sigma/2)^2 + \sinh(\pi\tau/2)^2) * |\gamma(\sigma + i\tau)|^2 - 2^{(2\sigma-2)} * \pi^{2\sigma} = 0$   
print_result_based_on_gamma()
```

**Output** – highlights the sensitivity of the numerical method to input precision and demonstrates that higher input granularity can lead to more accurate computational results. We conducted a numerical experiment to observe how input precision affects the accuracy of computed results. The desired precision is set to `mp.dps = 150` decimal places, the routine increases the number of decimal digits in the input to evaluate the accuracy of the output. For example, inputs such as  $\sigma = 0.49, 0.499, 0.4999$ , and so on are used. As the input precision increases, the output converges toward a stable value with greater accuracy. Below are the outputs generated by the program, starting  $\sigma = 0.49999$  with 5 decimals:

Solving for  $\tau$  from the equation:  $|\zeta(s)/\zeta(1-s)| - 1 = 0$

```
 $\tau = 6.28983598883424165137173737407605741520851469359815511866732475589775254465077296978$   
 $\tau = 6.28983598883690277966482398799251808917835020619868972677349901975624904273757111421$   
 $\tau = 6.28983598883690277966509010082185339665831028406449619228296591160273238246346271869$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776551241619583026290399796$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776553902747876379365199397$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776553902747876379365199423$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776553902747876379365199423$ 
```

Solving for  $\tau$  from the alternative equation:

```
 $(\cos(\pi\sigma/2)^2 + \sinh(\pi\tau/2)^2) * |\gamma(\sigma + i\tau)|^2 - 2^{(2\sigma-2)} * \pi^{2\sigma} = 0$   
 $\tau = 6.28983598883424165137173737407605741520851469359815511866732475589775254465077296978$   
 $\tau = 6.28983598883690277966482398799251808917835020619868972677349901975624904273757111421$   
 $\tau = 6.28983598883690277966509010082185339665831028406449619228296591160273238246346271869$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776551241619583026290399796$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776553902747876379365199397$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776553902747876379365199423$   
 $\tau = 6.28983598883690277966509010082185339665831294519278954535776553902747876379365199423$ 
```

### Notes

- The alternative equation suffers from slow convergence and reduced floating-point precision due to the large values of  $\Gamma$  and  $\sinh$ . Small changes in  $\sigma$  can trigger large output variations, making root-finding numerically unstable.
- Equation (4.50) introduces squared gamma term. Use with care, such as large  $\tau$  values may cause numerical overflow. Prefer  $\log(|\Gamma(\sigma + i\tau)|)$  or scaled forms when implementing.

## Appendix 4-C: Miscellaneous Results

This Appendix 4-C presents the proofs of different results and new beautiful formulas associated with the zeta function.

### [4-C-1]

Handbook of Mathematical Functions<sup>[28]</sup> and Wikipedia<sup>[29]</sup>

$$|\Gamma(n+1+i\tau)| = \sqrt{\frac{\pi\tau}{\sinh(\pi\tau)}} \prod_{k=1}^n \sqrt{\tau^2+k^2} \text{ for positive integer } n \geq 1$$

$$\left| \Gamma\left(\frac{1}{2}+i\tau\right) \right| = \sqrt{\frac{\pi}{\cosh(\pi\tau)}}$$

$$|\Gamma(1+i\tau)| = \sqrt{\frac{\pi\tau}{\sinh(\pi\tau)}}$$

### [4-C-2]

Using results **[4-C-1]** and applying the method of induction, we derive

$$\left| \frac{\zeta(n-i\tau)}{\zeta(1-n+i\tau)} \right| = 2^n \pi^{n-1} \sqrt{\sin^2\left(\frac{\pi n}{2}\right) + \sinh^2\left(\frac{\pi\tau}{2}\right)} |\Gamma(1-n+i\tau)|, \quad (4.52)$$

which is also the general formula for (4.34) and (4.35), with integer n.

Since expression (4.52) no longer contains the product symbol, we can treat n as a real number. Let n = x and t = y, where x and y are real numbers. Then, equation (4.52) can be rewritten as follows:

$$\left| \frac{\zeta(x-iy)}{\zeta(1-x+iy)} \right| = 2^x \pi^{x-1} \sqrt{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)} |\Gamma(1-x+iy)|. \quad (4.53)$$

By applying the Gamma reflection formula, equation (4.53) can be

rewritten in the form

$$\begin{aligned} \left| \frac{\zeta(x-iy)}{\zeta(1-x+iy)} \right| &= 2^x \pi^{x-1} \sqrt{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)} \left| \frac{\pi}{\sin[\pi(x-iy)]\Gamma(x-iy)} \right| \\ &= \frac{2^x \pi^x \sqrt{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)}}{|\sin[\pi(x-iy)]|\Gamma(x-iy)}. \end{aligned}$$

We also have

$$\begin{aligned} |\sin[\pi(x-iy)]| &= |\sin(\pi x)\cosh(\pi y) - i\sinh(\pi y)\cos(\pi x)| \\ &= \sqrt{\sin^2(\pi x)\cosh^2(\pi y) + \sinh^2(\pi y)\cos^2(\pi x)} \\ &= \sqrt{\sin^2(\pi x) + \sinh^2(\pi y)}. \end{aligned}$$

Putting it all together gives

$$\begin{aligned} \left| \frac{\zeta(x-iy)}{\zeta(1-x+iy)} \right| &= \frac{(2\pi)^x \sqrt{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)}}{\sqrt{\sin^2(\pi x) + \sinh^2(\pi y)}|\Gamma(x-iy)|} \\ &= \frac{(2\pi)^x}{|\Gamma(x-iy)|} \sqrt{\frac{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)}{\sin^2(\pi x) + \sinh^2(\pi y)}}. \end{aligned} \tag{4.54}$$

The expression inside the square root on the right-hand side of (4.54) can be deduced to a simpler form

$$\frac{\sin^2(\pi x) + \sinh^2(\pi y)}{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)} = 4\sinh^2\left(\frac{\pi y}{2}\right) + 4\cos^2\left(\frac{\pi x}{2}\right). \tag{4.55}$$

### Proof

Let  $s = x+iy$ , we have

$$\begin{aligned} \frac{|\sin(\pi s)|}{\left|\sin\left(\frac{\pi s}{2}\right)\right|} &= 2 \left| \cos\left(\frac{\pi s}{2}\right) \right| \\ &= 2 \sqrt{\cos^2 \frac{\pi x}{2} \cosh^2 \frac{\pi y}{2} + \sin^2 \frac{\pi x}{2} \sinh^2 \frac{\pi y}{2}} \\ &= 2 \sqrt{1 + \sinh^2 \frac{\pi y}{2} - \sin^2 \frac{\pi x}{2}}. \end{aligned}$$

The result leads us to the expression

$$\sqrt{\frac{\sin^2(\pi x) + \sinh^2(\pi y)}{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)}} = 2 \sqrt{1 + \sinh^2 \frac{\pi y}{2} - \sin^2 \frac{\pi x}{2}}.$$

Squaring both sides of the equation gives

$$\frac{\sin^2(\pi x) + \sinh^2(\pi y)}{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)} = 4 \left( 1 + \sinh^2 \frac{\pi y}{2} - \sin^2 \frac{\pi x}{2} \right),$$

which leads to result (4.55).

Therefore, expression (4.54) can be rewritten in the simplified form

$$\left| \frac{\zeta(x-iy)}{\zeta(1-x+iy)} \right| = \frac{2^{x-1} \pi^x}{\sqrt{\cos^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)} |\Gamma(x-iy)|}. \quad (4.56)$$

**Notes:**

- The conjugate form obtained by replacing  $y$  with  $-y$  into (4.56) is:

$$\left| \frac{\zeta(x+iy)}{\zeta(1-x-iy)} \right| = \frac{2^{x-1} \pi^x}{\sqrt{\cos^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)} |\Gamma(x+iy)|} \quad (4.57)$$

- The Gamma function has the asymptotic expression,

$$|\Gamma(x+iy)| \sim \sqrt{2\pi} |y|^{x-1/2} e^{-|y|\pi/2}.$$

- Substituting this result into (4.53) gives the asymptotic expression for

$$\left| \frac{\zeta(x-iy)}{\zeta(1-x+iy)} \right| \sim 2^{x+1/2} \pi^{x-1/2} |y|^{1/2-x} e^{-|y|\pi/2} \sqrt{\sin^2\left(\frac{\pi x}{2}\right) + \sinh^2\left(\frac{\pi y}{2}\right)}. \quad (4.58)$$

### [4-C-3]

We now present several elegant formulas that, to the best of our knowledge, have not appeared previously in the literature. We begin by recalling that

$$\zeta(s-1)\zeta(-s) = \frac{s(1-s)}{4\pi^2} \zeta(1+s)\zeta(2-s). \quad (4.59)$$

Moving all zeta functions to one side gives

$$\frac{\zeta(s-1)\zeta(-s)}{\zeta(s+1)\zeta(2-s)} = \frac{s(1-s)}{4\pi^2}.$$

Substituting  $s = \frac{1}{2} + i\tau$  in the above expression yields

$$\frac{\zeta\left(-\frac{1}{2}+i\tau\right)\zeta\left(-\frac{1}{2}-i\tau\right)}{\zeta\left(\frac{3}{2}-i\tau\right)\zeta\left(\frac{3}{2}+i\tau\right)} = \frac{\left(\frac{1}{2}+i\tau\right)\left(\frac{1}{2}-i\tau\right)}{4\pi^2},$$

which can be simplified as

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$$\frac{\zeta^2\left(-\frac{1}{2}+i\tau\right)}{\zeta^2\left(\frac{3}{2}+i\tau\right)} = \frac{\frac{1}{4}+\tau^2}{4\pi^2} = \frac{1+4\tau^2}{16\pi^2}.$$

Taking the square root of both sides yields a beautiful formula,

$$\left| \frac{\zeta\left(-\frac{1}{2}+i\tau\right)}{\zeta\left(\frac{3}{2}+i\tau\right)} \right| = \frac{\sqrt{1+4\tau^2}}{4\pi}. \quad (4.60)$$

Substituting  $s = 3/2 + i\tau$  into (4.59) gives

$$\frac{\zeta\left(\frac{1}{2}+i\tau\right)}{\zeta\left(\frac{1}{2}-i\tau\right)} = \frac{16\pi^2\zeta\left(-\frac{3}{2}+i\tau\right)}{(4\tau^2+i8\tau-3)\zeta\left(\frac{5}{2}-i\tau\right)} \quad (4.61)$$

Taking the modulus of both sides of (4.61) gives another beautiful formula, namely

$$\left| \frac{\zeta\left(-\frac{3}{2}+i\tau\right)}{\zeta\left(\frac{5}{2}-i\tau\right)} \right| = \frac{1}{16\pi^2} \sqrt{16\tau^4+40\tau^2+9}. \quad (4.61a)$$

By substituting  $s = 2, 4, 6, \dots$  into (4.59), we obtain the indeterminate forms:

$$\zeta(-2)\zeta(1) = \frac{\zeta(3)}{4\pi^2}, \quad \zeta(-4)\zeta(1) = -\frac{3\zeta(5)}{4\pi^4}, \quad \zeta(-6)\zeta(1) = \frac{45\zeta(7)}{8\pi^6}, \dots$$

Then the general formula for integer  $n$  is given by:

$$\zeta(1)\zeta(-2n) = \frac{(-1)^{n+1}\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}}. \quad (4.62)$$

Interestingly, we arrive at results that exhibit the indeterminate form 0/0, such as:

$$\frac{\zeta(-4)}{\zeta(-2)} = -\frac{3}{\pi^2} \frac{\zeta(5)}{\zeta(3)}, \quad \frac{\zeta(-6)}{\zeta(-4)} = -\frac{15}{2\pi^2} \frac{\zeta(7)}{\zeta(5)}, \quad \frac{\zeta(-8)}{\zeta(-6)} = -\frac{14}{\pi^2} \frac{\zeta(9)}{\zeta(7)}, \quad \text{and}$$

$$\frac{\zeta(-4)}{\zeta(-8)} = \frac{\pi^4 \zeta(5)}{105 \zeta(9)}, \dots \quad (4.63)$$

Furthermore, we notice that the right-hand side of (4.62) is the popular result of  $-\zeta'(n)$ <sup>[29]</sup>, which leads us to conclude that

$$\zeta'(-2n) = -\zeta(1)\zeta(-2n), \quad \text{for } n \geq 1. \quad (4.64)$$

**Note.** This represents the author's personal viewpoint:

- The equation of the Riemann zeta function (4.6), (4.13) or (4.14) holds a unique significance in mathematics. It brings together both minimal and maximal values, which alternate in their roles of support and balance. The placement of the zeta function, namely whether in the numerator or denominator of the functional equation, gives rise to features such as singularities and poles. These, in turn, act as essential components that make the functional equation analytic and complete, ensuring its path is continuous and smooth. One could argue that this equation encapsulates all mathematical constants within its framework.
- The results involving the indeterminate form 0/0 in (4.38) and (4.42) differ significantly from those in (4.63), due to the fact that the real parts with  $\text{Im}(s)=0$  found in (4.63) lie outside the critical strip. Therefore, the indeterminate form 0/0 arising from the ratio  $\lim_{s \rightarrow s_0} |\zeta(s)/\zeta(1-s)|$  is not equal to 1, as demonstrated in (4.41a).
- Through formulas (4.62), (4.63) and (4.64), it appears that the

#### *Appendix 4-C: Miscellaneous Results*

indeterminate forms  $0/0$  and  $0 \times \infty$  seem to unfold into multiple layers of structure. Within these layers, expressions such as  $\zeta(-4)/\zeta(-2)$ ,  $\zeta(-8)/\zeta(-6)$ , and others, tend to converge to distinct values. We also observe that constants like Catalan's constant,  $\zeta(3)$ ,  $\zeta(5)$ , and others often emerge from the indeterminate form  $0/0$ . This suggests that within these layers, or perhaps within the form  $0/0$  itself, there may exist a constant that has yet to be discovered. This constant may underlie known values such as  $\zeta(3)$ ,  $\zeta(5)$ , and potentially even the nontrivial zeros of the Riemann zeta function, in the sense that they could be expressed in terms of it. This unknown constant, similar to the critical number  $\tau_s$  introduced in Section 4-4, may not appear in any currently known mathematical expression.

- One might say that zero ( $\zeta(-2)$ ,  $\zeta(-4)$ , ...) and infinity ( $\zeta(1)$ ) are the two mysterious figures inhabiting the miniature universe of the zeta functional equation. Their connection seems intentionally designed with remarkable order and amazement, a genuine work of art in mathematical ingenuity.
- All possible nontrivial zeros on the imaginary part include the real number 6.28 which is indeed a very close numerical approximation to the circumference of a unit circle.

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## **Authors and Commentary**

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***We would like to take a moment to create a respectful space to convey our gratitude and appreciation to the following benefactors:***

- Those who have dedicated their efforts to the development of mathematical libraries (Numpy, SciPy, SymPy, mpmath, math, Matplotlib) for Python, Graph 4.4.2 tool for plotting, Visual Studio Code/Jupyter Notebook tool for editing/running code, LibreOffice tool for writing documents.
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- Those who have created AI tools that aid in the search for references, the exchange of ideas, and the promotion of learning.

This book (Part II) could not have been completed without the assistance of these invaluable resources. The contents of this book do not belong to us; they belong to the mathematicians who preceded us.

*"Render unto Caesar the things that are Caesar's,  
and unto God the things that are God's."*

**Matthew 22:21**

As a result, the book is offered at no cost, unless you desire to obtain a physical copy, in which event you will be responsible for the printing and paper expenses through Amazon.

Below are the author's reflections recorded at various times in his notebook. We present them here as a reconstruction of his original thought process. Some viewpoints may appear more than once, as certain ideas naturally resurfaced during the course of his development.

- Just as a skyscraper requires a deeper and broader foundation, a basic mathematical function must be developed with greater breadth and depth to extend its reach. In Chapter 3, we introduce the fundamental structures underlying two-variable trigonometric and hyperbolic functions. These constructions may provide the groundwork for more advanced multivariable extensions. A nature direction for further development is to expand this foundation into a comprehensive calculus framework built upon these two-variable elementary functions, much as traditional calculus was developed from single-variable functions. We are confident that anyone familiar with standard calculus will be able to contribute to this effort. This undertaking will likely span many years, during which numerous definitions, theorems, and formulas will be examined, refined, and optimized. Through this process, we aim to achieve a more accurate understanding of the natural phenomena that shape the three dimensional world around us. Moreover, single-variable functions obscure structural features that only two-variable functions can reveal. For example, these extended bivariable functions expose natural curvature along the  $y$ -axis, behavior that remains hidden in the one dimensional case. This broader perspective opens the door to new insights and potentially significant discoveries.
- Mathematicians have established the fundamental single variable trigonometric functions on the two dimensional plane, which form the basis of modern mathematics. We continue their works by introducing trigonometric functions of two variables, extending the single variable case into three dimensional space. We analyze the basic properties of

these two variable functions in detail. It's worth noting that single variable trigonometric functions are actually a subset of the two variable ones. We view these two variable functions as both essential and foundational, since they will help us study other similar functions in three dimensional space. This development may pave the way for a deeper exploration of trigonometric functions in higher dimensions, potentially establishing them as inherently elementary functions for two variables in future mathematical frameworks. Once a function approaches to a unique of the object, more restrictions are appeared and applicable. It has no differ with more population in the city, the more rules and policies are required and applicable to them. Therefore, these enhanced functions may serve as tools to broaden our thinking and deepen our insights into the symbolic curvilinear objects around us. Our study proposes a framework for initially analyzing these enhanced functions and exploring their applications in trigonometry-geometry and beyond, opening up a wide new mathematical landscape, ready to be explored and applied.

- We introduce the extended trigonometric function with two variables  $x, y \equiv (x \ln(y))$ , which serves as a generalization of Euler's identity, hyperbolic cosine or sine function to two variables. Furthermore, we can keep applying the approach outlined in Chapter 3 to create a specific set of other standard functions based on  $x, y$  defined by an alternative function (not  $x \ln(y)$ ), provided each of them produces the appropriate atomic function. This new collection of functions would be constructed upon our extended functions to enable the generation manageable transcendental functions that we can regulate. Think of this layer as yet another enhancement of our extended layer. In our view, there is much of interest that we discover at this point.
- In 1957, Kolmogorov and Arnold<sup>[1]</sup> showed that single variable functions are sufficient to represent any continuous multivariable function. In principle, this means that truly fundamental two variable functions are not strictly necessary for describing the mathematical structure of the universe. In practice, however, expressing complex shapes or phenomena using only one-variable functions is extremely difficult. This

motivates the idea of introducing an **intermediate “layer”** built from a new family of two variable functions, namely functions constructed on top of the existing single variable ones. As we see in this Chapter 3, the very simple two variable functions may uncover geometric or structural patterns that remain hidden when we restrict ourselves to one variable building blocks. Here is the intuition behind this idea. Consider the one variable function  $f(u)=\tan(u)$ . Now embed this function into two-variable setting by defining  $F(x,y)=\tan(xy)$ . Although  $F$  is defined on the two dimensional  $xy$ -plane, it effectively depends on only one combination of the inputs, because if we introduce the substitution  $u=xy$ , then  $F(x,y)=\tan(xy)\equiv \tan(u)=f(u)$  (abstraction). What changes, however, is the geometry of the graph. The graph of  $f(u)$  is a curve in the  $uf(u)$ -plane, while the graph of  $F(x,y)$  is a surface in three dimensional  $xyz$ -space. *In this higher dimensional setting, new geometric patterns and structures appear, such as the way values repeat along curves  $xy$ , which do not show up in the usual graph of  $f(u)$ . In this sense, examining the same function through additional variables can reveal further geometric properties of its behavior. Establishing **intermediate layer(s)** is therefore necessary. Once this new layer(s) has been thoroughly studied, exploring deeper levels of the problem becomes much less complicated, and our work on other objects in space becomes more systematic.*

- By introducing a third variable,  $z$ , as outlined at the end of Chapter 3, we examine how its variation, from negative infinity to positive infinity or vice versa, affects the other two variables. This allows us to explore how the relationships between trigonometric and hyperbolic functions extend naturally into three variables, including their inverse forms. In this three-variable setting, the extended hyperbolic functions continue to produce real valued outputs, whereas the corresponding extended trigonometric functions yield exclusively complex values. Theoretically, this method of extending basic functions can be generalized to construct functions of  $n$  variables. Unlike parameters,  $n$ -variable functions exhibit a rigorous interdependence among variables within a single multivariable equation.

- In Chapter 4, demonstrating the Riemann Hypothesis (RH) through pure mathematics poses significant challenges. We determine that there is no nontrivial zero off the critical line to be discovered beyond the point  $\tau = \tau_s \approx 6.289835988 \dots$ . This point is not only a local maximum (extremum) of  $|\zeta(s)/\zeta(1-s)|$ , but also marks the boundary where the curve  $|\zeta(s)/\zeta(1-s)|$  transitions into a straight line; beyond this point,  $|\zeta(s)/\zeta(1-s)|$  behaves linearly rather than as a curve. We still do not fully understand whether the “saturation constant” involved are related to the values of the known nontrivial zeros or to other known mathematical constants. Based on our current understanding, we believe the proof as shown in Section 4-4 is correct. If our proof of the Riemann Hypothesis is found to be flawed because the limit  $|\zeta(s)/\zeta(1-s)| = 1$  as  $s$  approaches  $s_0$  within the critical strip, we accept that possibility with a mathematically sound justification and appreciate the scrutiny such a problem deserves. Notice that the Riemann zeta functional equation is a meromorphic identity. Then both sides of the equation represent the same underlying function across the complex plane. Consequently, if we plot the left side and the right side independently, the resulting graphs are identical.
- Readers may find the new relationships among zeta identities, presented in *Appendix 4-C*, of particular interest. Especially, the linear first-order ODE (4.64) gives its solutions (or “exponential decay function”) containing odd zeta values which connect zero (negative even zeta values) and infinity (pole) mathematically.
- Again, this book may contain errors or oversights resulting from misunderstanding or typographical mistakes. We are consistently enthusiastic about learning throughout our lives, especially from constructive criticism, as it will assist us in making essential improvements.