PARTIAL DIFFERENTIAL EQUATIONS ARISING FROM THE CHAIN RULE FOR DIRECT DEPENDENCIES

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Abstract. This paper presents two partial differential equations obtained from the chain rule for functions with direct dependencies among the variables. For a function $V = V(x, y, z)$ with continuous and non-zero partial and total derivatives, we demonstrate that the equation

$$\frac{dV}{dy}\frac{dV}{dz}\frac{\partial V}{\partial x} + \frac{dV}{dx}\frac{dV}{dz}\frac{\partial V}{\partial y} + \frac{dV}{dx}\frac{dV}{dy}\frac{\partial V}{\partial z} = \frac{dV}{dx}\frac{dV}{dy}\frac{dV}{dz}$$

holds.

Next, we prove that the two-variable analogue for $V = V(x, y)$

$$\frac{dV}{dy}\frac{\partial V}{\partial x} + \frac{dV}{dx}\frac{\partial V}{\partial y} = \frac{dV}{dx}\frac{dV}{dy},$$

also holds.

The proofs utilize the chain rule and relationships between differential ratios, culminating in the verification of the fundamental identity.

But of that day and hour knoweth no man, no, not the angels of heaven, but my Father only.

Matthew 24:36 (KJV)

1. Introduction

In this paper, we will explore the chain rule for differentiation with direct dependencies and arrive at the two partial differential equations. The first one is

$$\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\partial V}{\partial x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\partial V}{\partial y} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z},$$

for V dependent on the variables x, y and z, that is, $V = V(x, y, z)$. The second one is

$$\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\partial V}{\partial x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\partial V}{\partial y} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}$$

for V dependent on the variables x and y, namely, $V = V(x, y)$.

2. Theorems

Theorem 1. *If $V = f(x, y, z)$ is a continuous function of three variables x, y, z, with continuous partial derivatives $\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z$ and the total derivatives $\mathrm{d}V/\mathrm{d}x$, $\mathrm{d}V/\mathrm{d}y$, $\mathrm{d}V/\mathrm{d}z$ are known, continuous and nonzero, then the following partial differential equation*

$$\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\partial V}{\partial x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\partial V}{\partial y} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\partial V}{\partial z} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z}. \tag{2.1}$$

holds.

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Proof. Denote the partial derivatives by

$$A = \frac{\partial V}{\partial x}, \quad B = \frac{\partial V}{\partial y}, \quad C = \frac{\partial V}{\partial z}.$$

Let $p = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $q = \frac{\mathrm{d}z}{\mathrm{d}x}$. Then, using the chain rule [1] e substituting the variables above,

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

becomes

$$\frac{\mathrm{d}V}{\mathrm{d}x} = A + Bp + Cq. \tag{2.2}$$

For the total derivative with respect to y, we utilize $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{p}$ and $\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} = q \cdot \frac{1}{p} = \frac{q}{p}$; thus, employing the chain rule and substituting the variables above,

$$\frac{\mathrm{d}V}{\mathrm{d}y} = \frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}y}$$

becomes

$$\frac{\mathrm{d}V}{\mathrm{d}y} = B + A\frac{1}{p} + C\frac{q}{p} = \frac{A + Bp + Cq}{p} = \frac{1}{p}\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{2.3}$$

For the total derivative with respect to z, we apply $\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{1}{q}$ and $\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}z} = p \cdot \frac{1}{q} = \frac{p}{q}$; and, again, using the chain rule and substituting the variables above,

$$\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{\partial V}{\partial z} + \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}z}$$

becomes

$$\frac{\mathrm{d}V}{\mathrm{d}z} = C + A\frac{1}{q} + B\frac{p}{q} = \frac{A + Bp + Cq}{q} = \frac{1}{q}\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{2.4}$$

From (2.3) and (2.4) we obtain the simple relations

$$\frac{\mathrm{d}V}{\mathrm{d}y} = \frac{\mathrm{d}V}{\mathrm{d}x} \cdot \frac{1}{p}, \qquad \frac{\mathrm{d}V}{dz} = \frac{\mathrm{d}V}{\mathrm{d}x} \cdot \frac{1}{q}.$$

Now compute the left-hand side of (2.1)

$$\begin{aligned} \text{LHS} &= \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z}A + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{dV}{\mathrm{d}z}B + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}C \\ &= \left(\frac{\mathrm{d}V}{\mathrm{d}x}\frac{1}{p}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}x}\frac{1}{q}\right)A + \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}x}\frac{1}{q}\right)B + \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}x}\frac{1}{p}\right)C \\ &= \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2\left(\frac{A}{pq} + \frac{B}{q} + \frac{C}{p}\right). \end{aligned} \tag{2.5}$$

The right-hand side of (2.1) is

$$\text{RHS} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{\mathrm{d}V}{\mathrm{d}x}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\frac{1}{p}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}x}\frac{1}{q}\right) = \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^3\frac{1}{pq}. \tag{2.6}$$

Thus the equality LHS = RHS, from (2.5) and (2.6), is equivalent to

$$\begin{aligned} \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2\left(\frac{A}{pq} + \frac{B}{q} + \frac{C}{p}\right) &= \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^3\frac{1}{pq} \\ \iff \frac{A}{pq} + \frac{B}{q} + \frac{C}{p} &= \frac{\mathrm{d}V}{\mathrm{d}x}\frac{1}{pq}. \end{aligned} \tag{2.7}$$

Multiply (2.7) by pq yields

$$A + Bp + Cq = \frac{\mathrm{d}V}{\mathrm{d}x}.$$

This is exactly the chain rule expression of (2.2). And the proof is complete. □

Theorem 2. *If $V = f(x, y)$ is a continuous function of two variables x and y, with continuous partial derivatives $\partial V/\partial x$ and $\partial V/\partial y$ and the total derivatives $\mathrm{d}V/\mathrm{d}x$, $\mathrm{d}V/\mathrm{d}y$ are known, continuous and nonzero, then the following partial differential equation*

$$\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\partial V}{\partial x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\partial V}{\partial y} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y} \tag{2.8}$$

holds.

Proof. By the chain rule for total derivatives [1], we find

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}x} = \dfrac{\partial V}{\partial x} + \dfrac{\partial V}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}x} \\ \dfrac{\mathrm{d}V}{\mathrm{d}y} = \dfrac{\partial V}{\partial y} + \dfrac{\partial V}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}y}. \end{cases} \tag{2.9}$$

Let $p = \frac{\mathrm{d}y}{\mathrm{d}x}$. Because the total derivatives are nonzero and the curves are regular, we have $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{p}$. Denote $A = \frac{\partial V}{\partial x}$, $B = \frac{\partial V}{\partial y}$. Then (2.9) becomes

$$\begin{cases} \frac{\mathrm{d}V}{\mathrm{d}x} = A + Bp \\ \frac{\mathrm{d}V}{\mathrm{d}y} = B + \frac{A}{p}. \end{cases} \tag{2.10}$$

Now compute the left-hand side of (2.8)

$$\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\partial V}{\partial x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\partial V}{\partial y} = \left(B + \frac{A}{p}\right)A + (A + Bp)B = AB + \frac{A^2}{p} + AB + B^2 p = 2AB + \frac{A^2}{p} + B^2 p. \tag{2.11}$$

The right-hand side of (2.8) is

$$\frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y} = (A + Bp)\left(B + \frac{A}{p}\right) = AB + \frac{A^2}{p} + B^2 p + AB = 2AB + \frac{A^2}{p} + B^2 p. \tag{2.12}$$

The expressions (2.11) and (2.12) are identical, hence

$$\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\partial V}{\partial x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\partial V}{\partial y} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y},$$

which is exactly (2.8). This completes the proof. □

3. Exercise

Exercise 3. Prove that

$$\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\mathrm{d}V}{\mathrm{d}t}\frac{\partial V}{\partial x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\mathrm{d}V}{\mathrm{d}t}\frac{\partial V}{\partial y} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}t}\frac{\partial V}{\partial z} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\partial V}{\partial t} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}V}{\mathrm{d}z}\frac{\mathrm{d}V}{\mathrm{d}t},$$

for similar conditions of Theorem 2.1.

References

[1] Wikipedia contributors, *Total derivative*, Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/w/index.php?title=Total_derivative&oldid=1348401533` (accessed April 24, 2026).