TWO GENERALIZED BINOMIAL IDENTITIES WITH A FREE PARAMETER

EDIGLES GUEDES

Abstract. Two remarkable combinatorial identities are established, expressing the power $(x \pm y)^n$ for arbitrary nonnegative integers n and complex numbers x, y, z as triple sums of products of binomial coefficients and monomials in x, y, z.

In each identity the auxiliary variable z appears explicitly in the summand but cancels completely when the summations are performed, so that the value of the sum is independent of z and reduces exactly to $(x \pm y)^n$.

The proofs are elementary and consist of successive applications of the classical binomial theorem together with reindexing and factorization. Several related exercises are provided that extend the method to integral analogs involving expressions of the form $\frac{(x\pm y)^{n+1}-x^{n+1}}{n+1}$, further illustrating the cancellation phenomenon.

There is neither Jew nor Greek, there is neither bond nor free, there is neither male nor female: for ye are all one in Christ Jesus.

Galatians 3:28

1. Introduction

The binomial theorem is one of the most fundamental results in algebra and combinatorial analysis, and its numerous generalizations and corollaries continue to inspire new identities. This short note presents two such identities, Theorems 1 and 2, that express $(x \pm y)^n$ as a triple sum over three indices, where the summand contains an additional free complex parameter z. The striking feature of these formulas is that, although z is woven into the individual terms of the sum, it disappears entirely after evaluating the nested summations, leaving the elementary binomial expansion independent of z.

The proofs are straightforward: by isolating the innermost sum and repeatedly applying the binomial theorem in the form $\sum_{i=0}^{k} \binom{k}{i} A^i = (1 + A)^k$, one reduces the triple sum first to a double sum, then to a single sum, and finally to the closed form $(x \pm y)^n$. The calculations require only careful handling of signs and ranges of summation, and no advanced machinery is invoked.

After establishing the main identities, four exercises are offered. Exercises 3 at 6 propose analogous triple-sum representations for integral expressions such as $\frac{(x+y)^{n+1}-x^{n+1}}{n+1}$ and $\frac{x^{n+1}-(x-y)^{n+1}}{n+1}$, again featuring the cancellation of z. These exercises illustrate that the underlying combinatorial mechanism is flexible and can be adapted

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to obtain integral variants where the exponents are shifted and denominators involving the summation indices appear.

The note is self-contained and accessible to anyone familiar with basic properties of binomial coefficients and finite series. It highlights an elegant cancellation phenomenon that may find use in the manipulation of multiple sums, in the derivation of convolution identities, or in teaching combinatorial proof techniques.

2. Theorems

Theorem 1. *For every nonnegative integer n and any complex numbers x, y, z, the following identity holds*

$$(x \pm y)^n = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{k} (-1)^{j+i} \binom{n}{k} \binom{n-k}{j} \binom{k}{i} x^{n-k-j} (\mp y)^i z^{k+j-i}, \tag{2.1}$$

where the upper signs or the lower signs are chosen simultaneously on both sides. In particular, the right-hand side does not depend on z, since the variable z cancels out when performing the summations.

Proof. Consider the right-hand side of the equality in (2.1) and denote it by S, defined as

$$S := \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{k} (-1)^{j+i} \binom{n}{k} \binom{n-k}{j} \binom{k}{i} x^{n-k-j} (\mp y)^i z^{k+j-i}. \tag{2.2}$$

Fix k and j. Examine the innermost sum in (2.2), naming it S_i and defining it as

$$S_i := \sum_{i=0}^{k} (-1)^i \binom{k}{i} (\mp y)^i z^{k+j-i}. \tag{2.3}$$

Factor out z^{k+j} in (2.3) and write the general term as a power

$$\begin{aligned} S_i &= z^{k+j} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left(\frac{\mp y}{z}\right)^i \\ &= z^{k+j} \sum_{i=0}^{k} \binom{k}{i} \left(-\frac{\mp y}{z}\right)^i \\ &= z^{k+j} \sum_{i=0}^{k} \binom{k}{i} \left(\pm \frac{y}{z}\right)^i. \end{aligned} \tag{2.4}$$

By the binomial theorem, $\sum_{i=0}^{k} \binom{k}{i} A^i = (1+A)^k$. With $A = \pm \frac{y}{z}$ in (2.4), we obtain

$$\begin{aligned} S_i &= z^{k+j} \left(1 \pm \frac{y}{z}\right)^k \\ &= z^{k+j} \left(\frac{z \pm y}{z}\right)^k \\ &= z^j (z \pm y)^k. \end{aligned} \tag{2.5}$$

Substitute the right-hand side of (2.5) into the expression for S in (2.2) and factor out the terms that do not depend on j

$$\begin{aligned} S &= \sum_{k=0}^{n} \binom{n}{k} (z \pm y)^k \left[\sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} x^{n-k-j} z^j \right] \\ &= \sum_{k=0}^{n} \binom{n}{k} (z \pm y)^k \left[\sum_{j=0}^{n-k} \binom{n-k}{j} x^{n-k-j} (-z)^j \right]. \end{aligned} \tag{2.6}$$

The bracketed sum is a direct application of binomial theorem [1]; thus the inner sum in (2.6) gives

$$\sum_{j=0}^{n-k} \binom{n-k}{j} x^{n-k-j} (-z)^j = (x-z)^{n-k}. \tag{2.7}$$

Substitute the right-hand side of (2.7) into the right-hand side of (2.6)

$$S = \sum_{k=0}^{n} \binom{n}{k} (x-z)^{n-k} (z \pm y)^k. \tag{2.8}$$

Recognize and apply binomial theorem in (2.8) to obtain

$$\begin{aligned} S &= \sum_{k=0}^{n} \binom{n}{k} (x-z)^{n-k} (z \pm y)^k \\ &= \left[(x-z) + (z \pm y) \right]^n \\ &= (x - z + z \pm y)^n \\ &= (x \pm y)^n. \end{aligned} \tag{2.9}$$

Note that the right-hand side of (2.9) is exactly the same mathematical expression as the left-hand side of (2.1). Thus we conclude that the triple sum reduces, independently of the value of z, to $(x \pm y)^n$, proving the validity of the identity for every $n \in \mathbb{N}_0$ and all $x, y, z \in \mathbb{C}$. This completes the proof. □

Theorem 2. *For every nonnegative integer n and any complex numbers x, y, z, the following identity holds*

$$(x \pm y)^n = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{k} (-1)^k \binom{n}{k} \binom{n-k}{j} \binom{k}{i} x^{n-k-j} (\mp y)^{k-i} z^{j+i}, \tag{2.10}$$

where the upper signs or the lower signs are chosen simultaneously on both sides. In particular, the right-hand side does not depend on z, since the variable z cancels out when performing the summations.

Proof. Denote the right-hand side of (2.10) by S

$$\begin{aligned} S &:= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{k} (-1)^k \binom{n}{k} \binom{n-k}{j} \binom{k}{i} x^{n-k-j} (\mp y)^{k-i} z^{j+i} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \underbrace{\sum_{j=0}^{n-k} \binom{n-k}{j} x^{n-k-j} z^j}_{S_j} \underbrace{\sum_{i=0}^{k} \binom{k}{i} (\mp y)^{k-i} z^i}_{S_i}. \end{aligned} \tag{2.11}$$

Fix k and j. Examine the innermost sum in (2.11), naming it S_i and defining it as

$$S_i := \sum_{i=0}^{k} \binom{k}{i} (\mp y)^{k-i} z^{i}. \tag{2.12}$$

Apply binomial theorem to the right-hand side of (2.12) to find

$$S_i = (\mp y + z)^k. \tag{2.13}$$

Fix k. Define a new inner sum in (2.11) by S_j as follows

$$S_j := \sum_{j=0}^{n-k} \binom{n-k}{j} x^{n-k-j} z^{j}. \tag{2.14}$$

Apply binomial theorem to the right-hand side of (2.14) to obtain

$$S_j = (x+z)^{n-k}. \tag{2.15}$$

Substitute the right-hand side of (2.13) and the right-hand side of (2.15) into the right-hand side of (2.11)

$$\begin{aligned} S &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+z)^{n-k} (\mp y + z)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (x+z)^{n-k} (-1)^k (\mp y + z)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (x+z)^{n-k} (\pm y - z)^k. \end{aligned} \tag{2.16}$$

Apply Newton's binomial theorem to the right-hand side of (2.16)

$$\begin{aligned} S &= \sum_{k=0}^{n} \binom{n}{k} (x+z)^{n-k} (\pm y - z)^k \\ &= [(x+z) + (\pm y - z)]^n \\ &= (x \pm y)^n. \end{aligned} \tag{2.17}$$

Note that the right-hand side of (2.17) is exactly the same mathematical expression as the left-hand side of (2.10). Thus we conclude that the triple sum reduces, independently of the value of z, to $(x \pm y)^n$, proving the validity of the identity for every $n \in \mathbb{N}_0$ and all $x, y, z \in \mathbb{C}$. This completes the proof. □

3. Exercises

Exercise 3. Prove that

$$\frac{(x+y)^{n+1} - x^{n+1}}{n+1} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{k} (-1)^j \binom{n}{k} \binom{n-k}{j} \binom{k}{i} x^{n-k-j} \frac{y^{1+i}}{1+i} z^{k+j-i},$$

for $n = 0, 1, 2, \ldots$ and $x, y, z \in \mathbb{C}$ (or $\mathbb{R}$) any.

Exercise 4. Prove that

$$\frac{(x+y)^{n+1}-x^{n+1}}{n+1} = \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{i=0}^{k}(-1)^{i}\binom{n}{k}\binom{n-k}{j}\binom{k}{i}x^{n-k-j}\frac{y^{1+k-i}}{1+k-i}z^{j+i},$$

for $n = 0, 1, 2, \ldots$ and $x, y, z \in \mathbb{C}$ (or $\mathbb{R}$) any.

Exercise 5. Prove that

$$\frac{x^{n+1}-(x-y)^{n+1}}{n+1} = \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{i=0}^{k}(-1)^{j}(-1)^{i}\binom{n}{k}\binom{n-k}{j}\binom{k}{i}x^{n-k-j}\frac{y^{1+i}}{1+i}z^{k+j-i},$$

for $n = 0, 1, 2, \ldots$ and $x, y, z \in \mathbb{C}$ (or $\mathbb{R}$) any.

Exercise 6. Prove that

$$\frac{x^{n+1}-(x-y)^{n+1}}{n+1} = \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{i=0}^{k}(-1)^{k}\binom{n}{k}\binom{n-k}{j}\binom{k}{i}x^{n-k-j}\frac{y^{1+k-i}}{1+k-i}z^{j+i},$$

for $n = 0, 1, 2, \ldots$ and $x, y, z \in \mathbb{C}$ (or $\mathbb{R}$) any.

References

[1] Wikipedia contributors. "Binomial theorem." *Wikipedia, The Free Encyclopedia.* Wikipedia, The Free Encyclopedia, 12 May. 2026. Web. 15 May. 2026.