A CONJECTURE ABOUT PRIMES WITH PRIMITIVE ROOT 2

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Abstract. In this paper, we conjecture that a Collatz-like ("if odd/if even") function can be used to test whether a prime number has primitive root 2.

Introduction

In 1937 Lothar Collatz proposed that, for any arbitrary positive number, n, iterative operations can be made such that, when even, n is divided by two, and when odd, it is multiplied by three and added to one, and that when this process is sufficiently repeated, the sequence will always reach 1. This iterative function is normally stated as follows:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 3x+1 & \text{if } x \text{ is odd} \end{cases}$$

So far all known sequences always ends in the loop $1 \to 4 \to 2 \to 1$. A counterexample sequence would either continue *ad infinitum* without converging to 1, or it would end in another loop (not ending in 1). According to Paul Erdős "Mathematics may not be ready for such problems." Jeffrey Lagarias stated in 2010 that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present day mathematics."

A similar function states that for any positive integer, x, if even is divided by two, and if odd is added to d, and that when this process is sufficiently repeated, the sequence will always reach 1. The general iterative function can be stated as follows:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x+d & \text{if } x \text{ is odd.} \end{cases}$$

Here, we conjecture that iff d is a prime with primitive root 2, then a 1-1 loop (of length $\frac{3(d-1)}{2}$) is created which includes all the odd numbers less that d. In number theory, a full reptend prime in base b is an odd prime number p such that the Fermat quotient

$$q_p(b) = \frac{b^{p-1}-1}{p},$$

(where p does not divide b) gives a cyclic number. In base 2, the first few full reptend primes are:

3, 5, 11, 13, 19, 29, 37, 53, 59, 61, 67, 83, 101, 107, 131, 139, 149, 163, 173, 179, 181, 197, 211, 227, 269, 293, 317, 347, 349, 373, 379, 389, 419, 421, 443, 461, 467, 491, 509, 523, 541, 547, 557, 563, 587, 613, 619, 653, 659, 661, 677, 701, 709,

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Date: 2026.
2010 *Mathematics Subject Classification.* Primary 11D41.
Key words and phrases. Number theory, Collatz Conjecture.

757, 773, 787, 797, 821, 827, 829, 853, 859, 877, 883, 907, 941, 947, ... (sequence A001122 in the OEIS).

These are primes, p, such that $\frac{1}{p}$, when written in base 2, has period $p-1$, which is the greatest period possible for any integer. Binary full reptend prime sequences (also called maximum-length decimal sequences) have found cryptographic and error-correction coding applications.[1]

THE COLLATZ-LIKE FUNCTION APPLIED.
We define the function as follows:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ d+x & \text{if } x \text{ is odd.} \end{cases}$$

We observe that, for any value of d, when $x_0 = 1$ the sequence always returns to 1. To give a few examples:
Let d=5.
1 6 3 16 8 4 2 1. (2 odd steps: 1 3 1).

Let d=11.
1 12 6 3 14 7 18 9 20 10 5 16 8 4 2 1. (5 odd steps: 1 3 7 9 5 1).

Let d=13.
1 14 7 20 10 5 18 9 22 11 24 12 6 3 16 8 4 2 1. (6 odd steps: 1 7 5 9 11 3 1).

Let d=19.
1 20 10 5 24 12 6 3 22 11 30 15 34 17 36 18 9 28 14 7 26 13 32 16 8 4 2 1. (9 odd steps: 1 5 3 11 15 17 9 7 13 1).

This process is equivalent to one known way of converting $\frac{1}{p}$ to reptend binary. For example, for $p = 11$, $\frac{1}{11} = 0.\overline{0001011101}$ in base 2. The following steps produce the binary reptend (where odd numerators are allocated 1, and even numerators are allocated 0):
$1/11 * 2 = 2/11$ (0)
$2/11 * 2 = 4/11$ (0)
$4/11 * 2 = 8/11$ (0)
$8/11 * 2 = 16/11 = 1 + 5/11$ (1)
$5/11 * 2 = 10/11$ (0)
$10/11 * 2 = 20/11 = 1 + 9/11$ (1)
$9/11 * 2 = 18/11 = 1 + 7/11$ (1)
$7/11 * 2 = 14/11 = 1 + 3/11$ (1)
$3/11 * 2 = 6/11$ (0)
$6/11 * 2 = 12/11$ (0)
$1/11 * 2$ returns to $1/11$ (1)
Note how all the numerators in base 10 produce the same sequence in reverse order: 1 12 6 3 14 7 18 9 20 10 5 16 8 4 2 1.

[1]Kak, Subhash, "Encryption and error-correction using d-sequences". IEEE Trans. On Computers, vol. C-34, pp. 803–809, 1985.

Conjectures.
Conjecture 1: when this process is sufficiently repeated for any odd d, the sequence will always return to 1.
Conjecture 2: we conjecture that d is a prime with a primitive root of 2 (cf. A001122) iff the following conditions apply:
a) the total number of steps to reach 1 is $\frac{3*(d-1)}{2}$;
b) the number of odd steps in a loop is always equal to $\frac{(d-1)}{2}$;
c) the peak odd value reached is d–2; and
d) every odd number $< d$ and every even number up $< 2d$ is included, then

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