FINITE PRODUCT LIMITS FOR ELEMENTARY AND GAMMA FUNCTIONS

EDIGLES GUEDES

Abstract. This paper presents a series of novel finite product limit representations for several fundamental functions. For any complex number z and nonnegative integer k, we prove that

$$z^k = \lim_{n\to\infty} \prod_{j=0}^{k-1} \left[1 + \frac{n(z-1)}{n+j}\right].$$

An asymptotic equivalence $\prod_{k=n}^{n^2-1} \frac{z+k}{k} \sim n^z$ is established and used to obtain a finite product for the exponential function, first for real arguments and then extended to the whole complex plane

$$e^z = \lim_{n\to\infty} \prod_{k=0}^{n-1} \left(1 + \frac{nz}{n^2+k}\right).$$

Analogous limit expressions are derived for the gamma function,

$$\Gamma(z) = \frac{1}{z} \lim_{n\to\infty} \prod_{k=1}^{n^2-1} \left(\frac{z+k}{k}\right)^{\operatorname{sgn}(k-n)},$$

and for its reciprocal,

$$\frac{1}{z\,\Gamma(z)} = \lim_{n\to\infty} \left[\left(\prod_{k=1}^{n-1} \frac{k+z}{k}\right)\left(\prod_{k=n}^{n^2-1} \frac{k}{k+z}\right)\right],$$

both valid for $z \neq 0, -1, -2, \ldots$ and admitting entire extensions. The proofs combine elementary algebraic manipulations, integral estimates for finite sums, expansions of the logarithm, and classical Stirling asymptotics. These representations provide direct connections between finite products and transcendental and special functions; thereby serving as a basis for further asymptotic analysis.

1. Introduction

Infinite product representations, such as the Weierstrass product for the gamma function and Euler's product for the exponential function, are cornerstones of real analysis, complex analysis and the theory of special functions. In many contexts, however, it is desirable to approximate these functions by finite products depending on a discrete parameter n that tends to infinity, combining the flexibility of algebraic manipulation with the rigor of limit processes.

Date: May 14, 2026.

2000 *Mathematics Subject Classification.* Primary: 33B15; Secondary: 33B10, 30E10, 30E15, 26A09, 26A12.

Key words and phrases. finite product; limit representation; power function; exponential function; gamma function; reciprocal gamma function; asymptotic equivalence; sign function; harmonic numbers.

In this paper we introduce several new finite product limit representations for the power function, the exponential, the gamma function, and the reciprocal gamma function. The main results are as follows. Theorem 1 gives a direct limit formula for an arbitrary complex power with a nonnegative integer exponent k, namely,

$$z^k = \lim_{n\to\infty} \prod_{j=0}^{k-1} \left[1 + \frac{n(z-1)}{n+j}\right].$$

Theorem 2 establishes the asymptotic equivalence

$$\prod_{k=n}^{n^2-1} \frac{z+k}{k} \sim n^z \qquad (n\to\infty),$$

which plays a key role in subsequent proofs.

Building on this, Theorems 3 and 4 provide a finite product representation for the exponential function, first for real x and then for all complex z via uniform convergence on compact sets

$$e^z = \lim_{n\to\infty} \prod_{k=0}^{n-1} \left(1 + \frac{nz}{n^2+k}\right).$$

Theorems 5 and 6 extend the approach to the gamma function and its reciprocal. Using the sign function $\operatorname{sgn}(k-n)$, the gamma function is expressed as

$$\Gamma(z) = \frac{1}{z} \lim_{n\to\infty} \prod_{k=1}^{n^2-1} \left(\frac{z+k}{k}\right)^{\operatorname{sgn}(k-n)} \qquad (z \neq 0, -1, -2, \dots),$$

while the reciprocal gamma function admits the simplified product

$$\frac{1}{z\,\Gamma(z)} = \lim_{n\to\infty} \left[\left(\prod_{k=1}^{n-1} \frac{k+z}{k}\right)\left(\prod_{k=n}^{n^2-1} \frac{k}{k+z}\right)\right],$$

which extends to an entire function equal to $1/[z\,\Gamma(z)]$ on the whole complex plane.

The proofs are elementary yet rich in technique. They combine algebraic factorization, integral bounds for monotone functions to estimate sums, the Taylor expansion of the logarithm, harmonic numbers, and the classical Stirling asymptotics for the gamma function. The appendix contains two auxiliary claims: a detailed derivation of the inequality needed for the exponential product, and a self-contained proof of the asymptotic ratio $\Gamma(x+a)/\Gamma(x) \sim x^a$.

The paper is organized as follows. Section 2 contains the main theorems and their complete proofs. Section 3 offers two exercises that generalize the exponential product to rational exponents. The appendix presents Claims 9 and 10, which justify the key inequalities and the gamma-ratio asymptotic used in the body of the paper.

2. Finite Product Representations for Powers, Exponential, Gamma Function, Reciprocal Gamma Function and Asymptotic Behavior

Theorem 1. *Let $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$. Then*

$$z^k = \lim_{n\to\infty} \prod_{j=0}^{k-1} \left[1 + \frac{n(z-1)}{n+j}\right]. \tag{2.1}$$

Proof. We rewrite each factor in (2.1) as

$$1 + \frac{n(z-1)}{n+j} = \frac{n+j+n(z-1)}{n+j} = \frac{nz+j}{n+j}. \tag{2.2}$$

Substitute the right-hand side of (2.2) into the finite product in (2.1) and define P_n as follows

$$P_n := \prod_{j=0}^{k-1}\left[1 + \frac{n(z-1)}{n+j}\right] = \prod_{j=0}^{k-1} \frac{nz+j}{n+j} = \frac{\prod_{j=0}^{k-1}(nz+j)}{\prod_{j=0}^{k-1}(n+j)}. \tag{2.3}$$

In (2.3), we factor out n from each term, both in the numerator and in the denominator, obtaining

$$\prod_{j=0}^{k-1}(nz+j) = n^k \prod_{j=0}^{k-1}\left(z + \frac{j}{n}\right) \tag{2.4}$$

and

$$\prod_{j=0}^{k-1}(n+j) = n^k \prod_{j=0}^{k-1}\left(1 + \frac{j}{n}\right). \tag{2.5}$$

Substitute the right-hand sides of (2.4) and (2.5) into the right-hand side of (2.3)

$$P_n = \frac{n^k \prod_{j=0}^{k-1}\left(z + \frac{j}{n}\right)}{n^k \prod_{j=0}^{k-1}\left(1 + \frac{j}{n}\right)} = \frac{\prod_{j=0}^{k-1}\left(z + \frac{j}{n}\right)}{\prod_{j=0}^{k-1}\left(1 + \frac{j}{n}\right)}. \tag{2.6}$$

Since k is fixed, as $n \to \infty$ and taking into account the continuity of the finite product, in (2.6), we have

$$\begin{aligned} \lim_{n\to\infty} P_n &= \frac{\lim_{n\to\infty} \prod_{j=0}^{k-1} \left(z + \frac{j}{n}\right)}{\lim_{n\to\infty} \prod_{j=0}^{k-1} \left(1 + \frac{j}{n}\right)} \\ &= \frac{\prod_{j=0}^{k-1} \lim_{n\to\infty} \left(z + \frac{j}{n}\right)}{\prod_{j=0}^{k-1} \lim_{n\to\infty} \left(1 + \frac{j}{n}\right)} \\ &= \frac{\prod_{j=0}^{k-1} z}{\prod_{j=0}^{k-1} 1} \\ &= z^k. \end{aligned} \tag{2.7}$$

Substitute the right-hand side of (2.3) into the left-hand side of (2.7) and find

$$\lim_{n\to\infty} \prod_{j=0}^{k-1} \left[1 + \frac{n(z-1)}{n+j}\right] = z^k. \tag{2.8}$$

Note that the mathematical expression in (2.8) is identical to (2.1). A small observation: the case $k = 0$ corresponds to the empty product, whose value is 1 by convention on the right-hand side of the equality in (2.1). Furthermore, on the left-hand side of the equality in (2.1) we use the identity $z^0 = 1$. This completes the proof. □

Theorem 2. *Let $z \in \mathbb{C}$ be a fixed complex number. Then*

$$\lim_{n\to\infty} n^{-z} \prod_{k=n}^{n^2-1} \frac{z+k}{k} = 1. \tag{2.9}$$

We can also write

$$\prod_{k=n}^{n^2-1} \frac{z+k}{k} \sim n^z, \tag{2.10}$$

as $n \to \infty$.

Proof. Define P_n from (2.9) and (2.10) as follows

$$\begin{aligned} P_n &:= \prod_{k=n}^{n^2-1} \frac{z+k}{k} \\ &= \prod_{k=n}^{n^2-1} \left(1 + \frac{z}{k}\right). \end{aligned} \tag{2.11}$$

Observe that for sufficiently large n, we have $|z/k| \leq 1/2$ for all $k \geq n$. Apply the logarithm to both sides of (2.11) and write

$$\ln P_n = \ln\left[\prod_{k=n}^{n^2-1}\left(1+\frac{z}{k}\right)\right] = \sum_{k=n}^{n^2-1}\ln\left(1+\frac{z}{k}\right). \tag{2.12}$$

Use the expansion $\ln(1+w) = w + O(|w|^2)$, valid uniformly for $|w| \leq 1/2$, on the right-hand side of (2.12) and obtain

$$\ln P_n = \sum_{k=n}^{n^2-1}\left(\frac{z}{k} + O\left(\frac{1}{k^2}\right)\right) = z\left(\sum_{k=n}^{n^2-1}\frac{1}{k}\right) + O\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right). \tag{2.13}$$

The second sum in (2.13) is given by

$$O\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right) = O\left(\frac{1}{n}\right), \tag{2.14}$$

because

$$\sum_{k=n}^{\infty}\frac{1}{k^2} \leq \int_{n-1}^{\infty}\frac{\mathrm{d}x}{x^2} = \frac{1}{n-1}.$$

For the first sum in (2.13), recall the asymptotic expansion for the harmonic numbers [2]

$$H_N = \sum_{k=1}^{N}\frac{1}{k} \sim \ln N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + \frac{1}{120N^4} - \ldots \tag{2.15}$$

Use the big-O notation on the right-hand side of (2.15) and find

$$H_N = \sum_{k=1}^{N}\frac{1}{k} = \ln N + \gamma + O\left(\frac{1}{N}\right). \tag{2.16}$$

Manipulate the first finite sum in (2.13) by expanding it into the difference of two harmonic numbers; then apply (2.16) to these harmonic numbers, obtaining the following expression

$$
\begin{aligned}
(2.17)\qquad \sum_{k=n}^{n^2-1}\frac{1}{k} &= H_{n^2-1} - H_{n-1} \\
&= \left[\ln(n^2-1) + \gamma + O\left(\frac{1}{n^2}\right)\right] - \left[\ln(n-1) + \gamma + O\left(\frac{1}{n}\right)\right] \\
&= \left[\ln(n^2-1) + O\left(\frac{1}{n^2}\right)\right] - \left[\ln(n-1) + O\left(\frac{1}{n}\right)\right].
\end{aligned}
$$

On the other hand, we know that

$$
\begin{aligned}
(2.18)\qquad \ln(n^2-1) &= \ln\left[n^2\left(1-\frac{1}{n^2}\right)\right] \\
&= 2\ln n + \ln\left(1-\frac{1}{n^2}\right) \\
&= 2\ln n + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

and

$$
\begin{aligned}
(2.19)\qquad \ln(n-1) &= \ln\left[n\left(1-\frac{1}{n}\right)\right] \\
&= \ln n + \ln\left(1-\frac{1}{n}\right) \\
&= \ln n + O\left(\frac{1}{n}\right).
\end{aligned}
$$

Substitute the right-hand sides of (2.18) and (2.19) into the right-hand side of (2.17)

$$
\begin{aligned}
(2.20)\qquad \sum_{k=n}^{n^2-1}\frac{1}{k} &= \left[\ln(n^2-1) + O\left(\frac{1}{n^2}\right)\right] - \left[\ln(n-1) + O\left(\frac{1}{n}\right)\right] \\
&= \left[2\ln n + O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^2}\right)\right] - \left[\ln n + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n}\right)\right] \\
&= \left[2\ln n + O\left(\frac{1}{n^2}\right)\right] - \left[\ln n + O\left(\frac{1}{n}\right)\right] \\
&= \ln n + O\left(\frac{1}{n}\right).
\end{aligned}
$$

Replace the right-hand sides of (2.14) and (2.20) into the right-hand side of (2.13)

$$\begin{aligned}\ln P_n &= z\left(\sum_{k=n}^{n^2-1}\frac{1}{k}\right)+O\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\\ &= z\left[\ln n+O\left(\frac{1}{n}\right)\right]+O\left(\frac{1}{n}\right)\\ &= z\ln n+zO\left(\frac{1}{n}\right)+O\left(\frac{1}{n}\right)\\ &= z\ln n+O\left(\frac{1}{n}\right)+O\left(\frac{1}{n}\right)\\ &= z\ln n+O\left(\frac{1}{n}\right).\end{aligned} \tag{2.21}$$

Rearrange (2.21) and obtain

$$\ln P_n - z\ln n = O\left(\frac{1}{n}\right). \tag{2.22}$$

Take the limit as $n\to\infty$ on both sides of (2.22)

$$\lim_{n\to\infty}(\ln P_n - z\ln n) = \lim_{n\to\infty}\left[O\left(\frac{1}{n}\right)\right]. \tag{2.23}$$

On the other hand, we have

$$\lim_{n\to\infty}\left[O\left(\frac{1}{n}\right)\right] = 0. \tag{2.24}$$

Substitute the right-hand side of (2.24) into the right-hand side of (2.23)

$$\lim_{n\to\infty}(\ln P_n - z\ln n) = 0. \tag{2.25}$$

Exponentiate both sides of (2.25) and conclude that

$$\begin{aligned}\lim_{n\to\infty} n^{-z}P_n &= e^0\\ &= 1.\end{aligned} \tag{2.26}$$

Substitute the right-hand side of (2.11) into the left-hand side of (2.26)

$$\lim_{n\to\infty} n^{-z}\prod_{k=n}^{n^2-1}\frac{z+k}{k} = 1, \tag{2.27}$$

and this proves the theorem, because the mathematical expression in (2.27) is identical to (2.9).

Two brief remarks:

a) For complex z, $n^{-z} = e^{-z\ln n}$, and exponentiation is continuous; in this way, this justifies our procedure and prevents other complications arising from complex analysis.

b) The mathematical expression (2.27) is an equivalent form of identity (2.10). □

Theorem 3. *For every real number x, the following identity holds*

$$e^x = \lim_{n\to\infty} \prod_{k=0}^{n-1} \left(1 + \frac{nx}{n^2+k}\right). \tag{2.28}$$

Proof. Step 1. Fix $x \in \mathbb{R}$. For each integer $n \geq 1$, define the finite product on the right-hand side of (2.28) by

$$P_n(x) := \prod_{k=0}^{n-1} \left(1 + \frac{nx}{n^2+k}\right). \tag{2.29}$$

For sufficiently large n, all factors in (2.29) are positive. Observe that if $x \geq 0$, the factors in (2.29) are obviously positive; if $x < 0$, note that the smallest factor occurs at $k = n-1$ and

$$1 + \frac{nx}{n^2+n-1} > 0 \quad \text{for } n > |x|.$$

Therefore, $\ln P_n(x)$ is well defined for large n.

Apply the natural logarithm to both sides of (2.29)

$$\ln P_n(x) = \sum_{k=0}^{n-1} \ln\left(1 + \frac{nx}{n^2+k}\right). \tag{2.30}$$

Use the Taylor expansion of the natural logarithm, for $|u| \leq 1/2$,

$$\ln(1+u) = u - \frac{u^2}{2} + O(u^3), \tag{2.31}$$

where the O-constant is uniform.

Now choose N such that $\frac{|x|}{N} \leq \frac{1}{2}$. Thus, for every $n \geq N$ and $0 \leq k \leq n-1$,

$$\left|\frac{nx}{n^2+k}\right| \leq \frac{n|x|}{n^2} = \frac{|x|}{n} \leq \frac{1}{2}, \tag{2.32}$$

and we may write, using the right-hand side of (2.31) and the inequality in (2.32),

$$\ln\left(1 + \frac{nx}{n^2+k}\right) = \frac{nx}{n^2+k} - \frac{1}{2}\left(\frac{nx}{n^2+k}\right)^2 + r_{n,k}, \tag{2.33}$$

where $|r_{n,k}| \leq C\left(\frac{n|x|}{n^2+k}\right)^3$ for some absolute constant $C > 0$.

Sum with respect to k from 0 at $n-1$ in (2.33)

$$\sum_{k=0}^{n-1} \ln\left(1 + \frac{nx}{n^2+k}\right) = x\sum_{k=0}^{n-1} \frac{n}{n^2+k} - \frac{x^2}{2}\sum_{k=0}^{n-1}\left(\frac{n}{n^2+k}\right)^2 + \sum_{k=0}^{n-1} r_{n,k}. \tag{2.34}$$

Substitute the left-hand side of (2.30) into the left-hand side of (2.34) and represent each sum on the right-hand side by S_n, T_n, and R_n, respectively, as indicated below

$$\ln P_n(x) = x\underbrace{\sum_{k=0}^{n-1} \frac{n}{n^2+k}}_{S_n} - \frac{x^2}{2}\underbrace{\sum_{k=0}^{n-1}\left(\frac{n}{n^2+k}\right)^2}_{T_n} + \underbrace{\sum_{k=0}^{n-1} r_{n,k}}_{R_n}. \tag{2.35}$$

Step 2. On the other hand, since the function $t \mapsto \frac{1}{n^2+t}$ is decreasing, we have the following integral inequalities for the sum S_n in (2.35)

$$\int_0^n \frac{dt}{n^2+t} \le \sum_{k=0}^{n-1} \frac{1}{n^2+k} \le \frac{1}{n^2} + \int_0^{n-1} \frac{dt}{n^2+t}. \tag{2.36}$$

Compute the integral

$$\begin{aligned} \int_0^a \frac{dt}{n^2+t} &= \ln\left(n^2+t\right)\Big|_{t=0}^{t=a} \\ &= \ln\left(n^2+a\right) - \ln\left(n^2\right) \\ &= \ln\left(1+\frac{a}{n^2}\right). \end{aligned} \tag{2.37}$$

Set $a = n$ in (2.37) and deduce the first integral in the inequality (2.36)

$$\begin{aligned} \int_0^n \frac{dt}{n^2+t} &= \ln\left(1+\frac{n}{n^2}\right) \\ &= \ln\left(1+\frac{1}{n}\right). \end{aligned} \tag{2.38}$$

Set $a = n-1$ in (2.37) and derive the second integral in the inequality (2.36)

$$\begin{aligned} \int_0^{n-1} \frac{dt}{n^2+t} &= \ln\left(1+\frac{n-1}{n^2}\right) \\ &= \ln\left(1+\frac{1}{n}-\frac{1}{n^2}\right). \end{aligned} \tag{2.39}$$

Substitute (2.38) and (2.39) into the inequality in (2.36) and find

$$\ln\left(1+\frac{1}{n}\right) \le \sum_{k=0}^{n-1} \frac{1}{n^2+k} \le \frac{1}{n^2} + \ln\left(1+\frac{1}{n}-\frac{1}{n^2}\right). \tag{2.40}$$

Multiply both sides of (2.40) by n

$$\begin{aligned} n\ln\left(1+\frac{1}{n}\right) &\le \sum_{k=0}^{n-1} \frac{n}{n^2+k} \le \frac{1}{n} + n\ln\left(1+\frac{1}{n}-\frac{1}{n^2}\right) \\ \therefore\ n\ln\left(1+\frac{1}{n}\right) &\le S_n \le \frac{1}{n} + n\ln\left(1+\frac{1}{n}-\frac{1}{n^2}\right) \qquad \text{[by virtue of (2.35)]}. \end{aligned} \tag{2.41}$$

Now, as $n \to \infty$ in the inequality (2.41), we have

$$\lim_{n\to\infty}\left[n\ln\left(1+\frac{1}{n}\right)\right] \le \lim_{n\to\infty} S_n \le \lim_{n\to\infty}\left[\frac{1}{n} + n\ln\left(1+\frac{1}{n}-\frac{1}{n^2}\right)\right]. \tag{2.42}$$

On the other hand, calculate

$$\begin{aligned}
\lim_{n\to\infty}\left[n\,\ln\left(1+\frac{1}{n}\right)\right] &= \lim_{n\to\infty}\left[\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right] \\
&= \lim_{n\to\infty}\left\{\frac{\frac{\mathrm{d}}{\mathrm{d}n}\left[\ln\left(1+\frac{1}{n}\right)\right]}{\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{n}\right)}\right\} \quad \text{[L'Hôpital's rule]} \\
&= \lim_{n\to\infty}\left[\frac{\frac{1}{1+\frac{1}{n}}\cdot\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}\right] \\
&= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) \\
&= \frac{1}{1+0} \\
&= 1.
\end{aligned} \tag{2.43}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\left[\frac{1}{n}+n\ln\left(1+\frac{1}{n}-\frac{1}{n^2}\right)\right] &= \lim_{x\to 0^+}\left[x+\frac{\ln\left(1+x-x^2\right)}{x}\right] \qquad x=\tfrac{1}{n},\; n\to\infty \Rightarrow x\to 0^+ \\
&= \lim_{x\to 0^+}\left[\frac{x^2+\ln\left(1+x-x^2\right)}{x}\right] \\
&= \lim_{x\to 0^+}\left\{\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2+\ln\left(1+x-x^2\right)\right]}{\frac{\mathrm{d}}{\mathrm{d}x}(x)}\right\} \quad \text{[L'Hôpital's rule]} \\
&= \lim_{x\to 0^+}\left(\frac{2x+\frac{1-2x}{1+x-x^2}}{1}\right) \\
&= \lim_{x\to 0^+}\left(2x+\frac{1-2x}{1+x-x^2}\right) \\
&= 2\cdot 0+\frac{1-2\cdot 0}{1+0-0^2} \\
&= 1.
\end{aligned} \tag{2.44}$$

Substitute the right-hand sides of (2.43) and (2.44) into the left-hand side and the right-hand side of the inequality in (2.42)

$$1 \;\le\; \lim_{n\to\infty} S_n \;\le\; 1. \tag{2.45}$$

Then, by the squeeze theorem [3] and the inequality (2.45), we may write that

$$\lim_{n\to\infty} S_n = 1. \tag{2.46}$$

Step 3. Considering the limit case as $n \to \infty$ for T_n, see (2.35), we may write

$$0 \le \lim_{n\to\infty} \left[T_n = \sum_{k=0}^{n-1} \frac{n^2}{(n^2+k)^2} \right] \le \lim_{n\to\infty} \left(\sum_{k=0}^{n-1} \frac{n^2}{n^4} \right) \tag{2.47}$$
$$0 \le \lim_{n\to\infty} T_n \le \lim_{n\to\infty} \left(n \cdot \frac{1}{n^2} \right)$$
$$0 \le \lim_{n\to\infty} T_n \le \lim_{n\to\infty} \left(\frac{1}{n} \right)$$
$$0 \le \lim_{n\to\infty} T_n \le 0.$$

Hence, by the squeeze theorem and the inequality in (2.47), we find

$$\lim_{n\to\infty} T_n = 0. \tag{2.48}$$

Step 4. Use the triangle inequality [4] for $R_n = \sum_{k=0}^{n-1} r_{n,k}$, and encounter

$$0 \le |R_n| \le \sum_{k=0}^{n-1} |r_{n,k}| \le C|x|^3 \sum_{k=0}^{n-1} \frac{n^3}{(n^2+k)^3} \le \frac{C|x|^3}{n^2}. \tag{2.49}$$

As $n \to \infty$ in (2.49), we have

$$0 \le \lim_{n\to\infty} |R_n| \le \lim_{n\to\infty} \frac{C|x|^3}{n^2} \tag{2.50}$$
$$0 \le \lim_{n\to\infty} |R_n| \le 0.$$

By the squeeze theorem and the inequality in (2.50), we get

$$\lim_{n\to\infty} |R_n| = 0.$$

Since $|R_n| \to 0$; this is equivalent to stating that

$$\lim_{n\to\infty} R_n = 0. \tag{2.51}$$

Step 5. Apply the limit as $n \to \infty$ to both sides of (2.35)

$$\lim_{n\to\infty} \ln P_n(x) = \lim_{n\to\infty} (x \cdot S_n) - \lim_{n\to\infty} \left(\frac{x^2}{2} \cdot T_n \right) + \lim_{n\to\infty} R_n, \tag{2.52}$$
$$\lim_{n\to\infty} \ln P_n(x) = x \cdot \lim_{n\to\infty} S_n - \frac{x^2}{2} \cdot \lim_{n\to\infty} T_n + \lim_{n\to\infty} R_n.$$

Substitute the right-hand sides of (2.46), (2.48), and (2.51) into the right-hand side of (2.52)

$$\lim_{n\to\infty} \ln P_n(x) = x \cdot 1 - \frac{x^2}{2} \cdot 0 + 0 \tag{2.53}$$
$$= x.$$

Taking the exponential of both sides of (2.53) yields

$$\lim_{n\to\infty} P_n(x) = e^x. \tag{2.54}$$

Lastly, substitute the right-hand side of (2.29) into the left-hand side of (2.54)

$$\lim_{n\to\infty}\prod_{k=0}^{n-1}\left(1+\frac{nx}{n^2+k}\right)=e^x. \tag{2.55}$$

Note that (2.55) is identical to (2.28), which is the desired result. □

Theorem 4. *For every complex number z, the following identity holds*

$$e^z=\lim_{n\to\infty}\prod_{k=0}^{n-1}\left(1+\frac{nz}{n^2+k}\right). \tag{2.56}$$

Proof. For each $n\in\mathbb{N}$, the expression

$$P_n(z):=\prod_{k=0}^{n-1}\left(1+\frac{nz}{n^2+k}\right) \tag{2.57}$$

defines a polynomial in z, and therefore an entire function. We will prove that $P_n(z)\to e^z$ uniformly on every compact subset of $\mathbb{C}$, which implies pointwise convergence for all $z\in\mathbb{C}$.

Let $K\subset\mathbb{C}$ be compact. There exists $M>0$ such that $|z|\le M$ for all $z\in K$. Choose $N\in\mathbb{N}$ such that $\frac{M}{N}\le\frac{1}{2}$. For $n\ge N$ and $0\le k\le n-1$, we have

$$\left|\frac{nz}{n^2+k}\right|\le\frac{nM}{n^2}=\frac{M}{n}\le\frac{1}{2}.$$

Thus, all factors lie in the disk of radius $1/2$ centered at 1, where the principal branch of the logarithm is well-defined and holomorphic. In view of this, we can take the logarithm of (2.57), obtaining

$$\ln P_n(z)=\sum_{k=0}^{n-1}\ln\left(1+\frac{nz}{n^2+k}\right). \tag{2.58}$$

We use the Taylor series expansion of $\ln(1+u)$ around $u=0$,

$$\ln(1+u)=u-\frac{u^2}{2}+O(u^3)\quad\text{for } |u|\le\tfrac{1}{2}, \tag{2.59}$$

where the implicit constant in the $O(u^3)$ term is absolute (i.e., bounded by some $C>0$ for $|u|\le 1/2$).

Substitute $u=\frac{nz}{n^2+k}$ into (2.59) and obtain

$$\ln\left(1+\frac{nz}{n^2+k}\right)=\frac{nz}{n^2+k}-\frac{1}{2}\left(\frac{nz}{n^2+k}\right)^2+r_{n,k}(z), \tag{2.60}$$

where the remainder satisfies $|r_{n,k}(z)|\le C\left|\frac{nz}{n^2+k}\right|^3$.

Sum both sides of (2.60) over k from 0 to $n-1$

$$\sum_{k=0}^{n-1}\ln\left(1+\frac{nz}{n^2+k}\right)=z\sum_{k=0}^{n-1}\frac{n}{n^2+k}-\frac{z^2}{2}\sum_{k=0}^{n-1}\left(\frac{n}{n^2+k}\right)^2+\sum_{k=0}^{n-1}r_{n,k}(z). \tag{2.61}$$

Substitute the left-hand side of (2.57) into the left-hand side of (2.61) and denote each sum on the right-hand side by S_n, T_n and $R_n(z)$, respectively, as shown below

$$\ln P_n(z) = z \underbrace{\sum_{k=0}^{n-1} \frac{n}{n^2+k}}_{S_n} - \frac{z^2}{2} \underbrace{\sum_{k=0}^{n-1} \left(\frac{n}{n^2+k}\right)^2}_{T_n} + \underbrace{\sum_{k=0}^{n-1} r_{n,k}(z)}_{R_n(z)}. \tag{2.62}$$

Note that S_n and T_n are real sums independent of z. Their limits are

$$\lim_{n\to\infty} S_n = 1, \qquad \lim_{n\to\infty} T_n = 0. \tag{2.63}$$

This was shown via integral comparisons in the real case; the proof is independent of z and remains valid. See the body of the proof of Theorem 3 for further details, specifically equations (2.46) and (2.48).

We now estimate $R_n(z)$ uniformly for $z \in K$. Using the triangle inequality, we find

$$0 \le |R_n(z)| \le C \sum_{k=0}^{n-1} \left(\frac{n|z|}{n^2+k}\right)^3 \le CM^3 \sum_{k=0}^{n-1} \frac{n^3}{(n^2+k)^3} \le CM^3 \cdot n \cdot \frac{n^3}{n^6} = \frac{CM^3}{n^2}. \tag{2.64}$$

As $n \to \infty$ in (2.64), we have

$$\begin{aligned} 0 &\le \lim_{n\to\infty} |R_n(z)| \le \lim_{n\to\infty} \frac{CM^3}{n^2}, \\ 0 &\le \lim_{n\to\infty} |R_n(z)| \le 0. \end{aligned} \tag{2.65}$$

By the squeeze theorem and the inequality in (2.65), we get

$$\lim_{n\to\infty} |R_n(z)| = 0. \tag{2.66}$$

Since $|R_n| \to 0$ and (2.66); this is equivalent to stating that

$$\lim_{n\to\infty} R_n(z) = 0. \tag{2.67}$$

Therefore, for $z \in K$, the limit as $n \to \infty$ of (2.62) is given by

$$\begin{aligned} \lim_{n\to\infty} \ln P_n(z) &= \lim_{n\to\infty} (z \cdot S_n) - \lim_{n\to\infty} \left(\frac{z^2}{2} \cdot T_n\right) + \lim_{n\to\infty} [R_n(z)] \\ &= z \cdot \lim_{n\to\infty} S_n - \frac{z^2}{2} \cdot \lim_{n\to\infty} T_n + \lim_{n\to\infty} [R_n(z)]. \end{aligned} \tag{2.68}$$

Substitute the right-hand sides of (2.63) and (2.67) into the right-hand side of (2.68)

$$\lim_{n\to\infty} \ln P_n(z) = z \cdot 1 - \frac{z^2}{2} \cdot 0 + 0 = z. \tag{2.69}$$

Then, by the uniform continuity on compact sets of the exponential function, we can exponentiate (2.69), yielding

$$\lim_{n\to\infty} P_n(z) = e^{\lim_{n\to\infty} \ln P_n(z)}. \tag{2.70}$$

Substitute the right-hand side of (2.69) into the right-hand side of (2.70)

$$\lim_{n\to\infty} P_n(z) = e^z, \tag{2.71}$$

uniformly on K. Since K is arbitrary, the convergence holds pointwise for all $z \in \mathbb{C}$.

Finally, substitute the right-hand side of (2.57) into the left-hand side of (2.71)

$$\lim_{n\to\infty}\prod_{k=0}^{n-1}\left(1+\frac{nz}{n^2+k}\right)=e^z. \tag{2.72}$$

Compare and note that (2.72) is identical to (2.56). This completes the proof. □

Theorem 5. *Let* $z\in\mathbb{C}$ *such that* $z\neq 0,-1,-2,\dots$ *Then*

$$\Gamma(z)=\frac{1}{z}\lim_{n\to\infty}\prod_{k=1}^{n^2-1}\left(\frac{z+k}{k}\right)^{\operatorname{sgn}(k-n)}, \tag{2.73}$$

where sgn *denotes the sign function, with the convention* $\operatorname{sgn}(0)=0$.

Proof. Step 1. Note that the sign function in (2.73) is

$$\operatorname{sgn}(k-n):=\begin{cases}-1, & \text{for } k<n,\\ 0, & \text{for } k=n,\\ +1, & \text{for } k>n.\end{cases} \tag{2.74}$$

Replace n^2-1 by N in the finite product in (2.73) and obtain

$$\prod_{k=1}^{N}\left(\frac{z+k}{k}\right)^{\operatorname{sgn}(k-n)}=\prod_{k=1}^{n-1}\left(\frac{k}{z+k}\right)\cdot\prod_{k=n+1}^{N}\left(\frac{z+k}{k}\right), \tag{2.75}$$

observe that the term $k=n$ contributes 1 to the product, because the exponent is zero, and it can be omitted. See the definition in (2.74).

For the first finite product in (2.75), transforming it into gamma functions, we have

$$\prod_{k=1}^{n-1}\frac{k}{z+k}=\frac{(n-1)!}{(z+1)\cdots(z+n-1)}=\frac{\Gamma(n)\,\Gamma(z+1)}{\Gamma(z+n)}. \tag{2.76}$$

Considering $n+1\le k\le N$, for the second finite product in (2.75), transforming it into gamma functions, we find

$$\prod_{k=n+1}^{N}\frac{z+k}{k}=\frac{\Gamma(z+N+1)/\Gamma(z+n+1)}{\Gamma(N+1)/\Gamma(n+1)}=\frac{\Gamma(z+N+1)\,\Gamma(n+1)}{\Gamma(z+n+1)\,\Gamma(N+1)}. \tag{2.77}$$

Step 2. Define the following function $F_n(z)$, originating from (2.73)

$$F_n(z):=\frac{1}{z}\prod_{k=1}^{n^2-1}\left(\frac{z+k}{k}\right)^{\operatorname{sgn}(k-n)}. \tag{2.78}$$

Substitute the right-hand side of (2.75) into the right-hand side of (2.78)

$$F_n(z):=\frac{1}{z}\prod_{k=1}^{n-1}\left(\frac{k}{z+k}\right)\cdot\prod_{k=n+1}^{N}\left(\frac{z+k}{k}\right). \tag{2.79}$$

Step 3. Multiply (2.76) by (2.77) and by the external factor $1/z$, discovering

$$\frac{1}{z}\prod_{k=1}^{n-1}\left(\frac{k}{z+k}\right)\cdot\prod_{k=n+1}^{N}\left(\frac{z+k}{k}\right)=\frac{\Gamma(z+1)}{z}\cdot\frac{\Gamma(n)\,\Gamma(n+1)\,\Gamma(z+N+1)}{\Gamma(z+n)\,\Gamma(z+n+1)\,\Gamma(N+1)}. \tag{2.80}$$

Substitute the left-hand side of (2.79) into the left-hand side of (2.80) and use the relation $\Gamma(z+1)/z = \Gamma(z)$ on the right-hand side of (2.80), obtaining

$$F_n(z) = \Gamma(z) \cdot \frac{\Gamma(n)\,\Gamma(n+1)\,\Gamma(z+N+1)}{\Gamma(z+n)\,\Gamma(z+n+1)\,\Gamma(N+1)}, \qquad N = n^2 - 1. \tag{2.81}$$

Step 4. We will use the following asymptotic approximation

$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a. \tag{2.82}$$

Apply (2.82) to the ratios of gamma functions in (2.81), for fixed z (away from poles),

$$\frac{\Gamma(n)}{\Gamma(z+n)} \sim n^{-z}, \tag{2.83}$$

$$\frac{\Gamma(n+1)}{\Gamma(z+n+1)} \sim n^{-z}, \tag{2.84}$$

and

$$\frac{\Gamma(z+N+1)}{\Gamma(N+1)} \sim \frac{\Gamma(z+n^2-1+1)}{\Gamma(n^2-1+1)} \sim \frac{\Gamma(z+n^2)}{\Gamma(n^2)} \sim n^{2z}. \tag{2.85}$$

Step 5. Now, substitute the right-hand sides of (2.83), (2.84), and (2.85) into the right-hand side of (2.81) to obtain the asymptotic approximation

$$F_n(z) \sim \Gamma(z) \cdot n^{-z} \cdot n^{-z} \cdot n^{2z} = \Gamma(z) \cdot 1 = \Gamma(z). \tag{2.86}$$

As n approaches infinity on both sides of (2.86), we have

$$\lim_{n\to\infty} F_n(z) = \Gamma(z). \tag{2.87}$$

Put back the right-hand side of (2.78) into the left-hand side of (2.87)

$$\lim_{n\to\infty} \frac{1}{z} \prod_{k=1}^{n^2-1} \left(\frac{z+k}{k}\right)^{\operatorname{sgn}(k-n)} = \Gamma(z),$$

$$\frac{1}{z} \lim_{n\to\infty} \prod_{k=1}^{n^2-1} \left(\frac{z+k}{k}\right)^{\operatorname{sgn}(k-n)} = \Gamma(z). \tag{2.88}$$

Note that the identity in (2.88) is identical to (2.73). This confirms the identity of the statement of this theorem for all $z \in \mathbb{C}$, except at the poles $z = 0, -1, -2, \ldots,$ where both sides coincide as meromorphic functions. □

Theorem 6. *For all $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have*

$$\frac{1}{z\,\Gamma(z)} = \lim_{n\to\infty} \left[\left(\prod_{k=1}^{n-1} \frac{k+z}{k} \right) \left(\prod_{k=n}^{n^2-1} \frac{k}{k+z} \right) \right]. \tag{2.89}$$

Moreover, the function defined by the right-hand side is entire and equals $1/\Gamma(z+1)$ *for all* $z \in \mathbb{C}$.

Proof. Step 1. We recall that, for a positive integer n, we can express the following finite products in terms of the gamma function

$$\prod_{k=1}^{n-1}(k+z)=\frac{\Gamma(n+z)}{\Gamma(1+z)} \tag{2.90}$$

and

$$\prod_{k=1}^{n-1}k=\Gamma(n). \tag{2.91}$$

Analogously, we have

$$\prod_{k=n}^{n^2-1}k=\frac{\Gamma(n^2)}{\Gamma(n)} \tag{2.92}$$

and

$$\prod_{k=n}^{n^2-1}(k+z)=\frac{\Gamma(n^2+z)}{\Gamma(n+z)}. \tag{2.93}$$

Define the product $P_n(z)$, originating from (2.89)

$$P_n(z):=\left(\prod_{k=1}^{n-1}\frac{k+z}{k}\right)\left(\prod_{k=n}^{n^2-1}\frac{k}{k+z}\right). \tag{2.94}$$

Substitute the right-hand sides of (2.90), (2.91), (2.92), and (2.93) into (2.94) as follows

$$\begin{aligned} P_n(z)&=\left(\prod_{k=1}^{n-1}\frac{k+z}{k}\right)\left(\prod_{k=n}^{n^2-1}\frac{k}{k+z}\right)\\ &=\frac{\prod_{k=1}^{n-1}(k+z)}{\prod_{k=1}^{n-1}k}\cdot\frac{\prod_{k=n}^{n^2-1}k}{\prod_{k=n}^{n^2-1}(k+z)}\\ &=\frac{\Gamma(n+z)/\Gamma(1+z)}{\Gamma(n)}\cdot\frac{\Gamma(n^2)/\Gamma(n)}{\Gamma(n^2+z)/\Gamma(n+z)}\\ &=\frac{1}{\Gamma(1+z)}\left[\frac{\Gamma(n+z)}{\Gamma(n)}\right]^2\frac{\Gamma(n^2)}{\Gamma(n^2+z)} \end{aligned} \tag{2.95}$$

Step 2. For $x\longrightarrow+\infty$ and z fixed, Stirling's formula gives the fundamental limit

$$\frac{\Gamma(x+z)}{\Gamma(x)}\sim x^z, \tag{2.96}$$

that is,

$$\lim_{x\to\infty}\frac{\Gamma(x+z)}{\Gamma(x)\,x^z}=1,$$

uniformly on compact subsets of $\mathbb{C}$, excluding the poles.

Applying (2.96) to the gamma function ratios in (2.95), with $x = n$ and $x = n^2$, as $n \to \infty$, we have

$$\left[\frac{\Gamma(n+z)}{\Gamma(n)}\right]^2 \sim n^{2z} \tag{2.97}$$

and

$$\frac{\Gamma(n^2)}{\Gamma(n^2+z)} \sim (n^2)^{-z} = n^{-2z}. \tag{2.98}$$

Multiply (2.97) by (2.98) and the power of n cancels as follows

$$\left[\frac{\Gamma(n+z)}{\Gamma(n)}\right]^2 \frac{\Gamma(n^2)}{\Gamma(n^2+z)} \sim n^{2z} \cdot n^{-2z} = 1,$$

namely,

$$\lim_{n\to\infty} \left\{ \left[\frac{\Gamma(n+z)}{\Gamma(n)}\right]^2 \frac{\Gamma(n^2)}{\Gamma(n^2+z)} \right\} = 1. \tag{2.99}$$

*Step*3. Therefore, letting $n \to \infty$ in both sides of (2.94), we find

$$\begin{aligned} \lim_{n\to\infty} P_n(z) &= \lim_{n\to\infty} \left\{ \frac{1}{\Gamma(1+z)} \left[\frac{\Gamma(n+z)}{\Gamma(n)}\right]^2 \frac{\Gamma(n^2)}{\Gamma(n^2+z)} \right\} \\ &= \frac{1}{\Gamma(1+z)} \cdot \lim_{n\to\infty} \left\{ \left[\frac{\Gamma(n+z)}{\Gamma(n)}\right]^2 \frac{\Gamma(n^2)}{\Gamma(n^2+z)} \right\}. \end{aligned} \tag{2.100}$$

Substitute the right-hand side of (2.99) into the right-hand side of (2.100)

$$\lim_{n\to\infty} P_n(z) = \frac{1}{\Gamma(1+z)} \cdot 1 = \frac{1}{\Gamma(z+1)}. \tag{2.101}$$

Apply the relation $\Gamma(z+1) = z\,\Gamma(z)$ to the right-hand side of (2.94)

$$\lim_{n\to\infty} P_n(z) = \frac{1}{\Gamma(z+1)} = \frac{1}{z\,\Gamma(z)}. \tag{2.102}$$

Substitute the right-hand side of (2.94) into the left-hand side of (2.102)

$$\lim_{n\to\infty} \left[\left(\prod_{k=1}^{n-1} \frac{k+z}{k} \right) \left(\prod_{k=n}^{n^2-1} \frac{k}{k+z} \right) \right] = \frac{1}{z\,\Gamma(z)}. \tag{2.103}$$

Compare and note that the identity in (2.103) is identical to (2.89). Moreover, the convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{Z}_{\le 0}$, so the limit defines a holomorphic function in this region, which coincides with $1/[z\,\Gamma(z)]$. Since the latter has removable singularities at the non-positive integers and extends to the entire function $1/[z\,\Gamma(z)]$, the identity remains valid in the whole complex plane, interpreting the left-hand side by its limit at those points. This completes the proof. □

3. Exercises

Exercise 7. Prove that

$$e^{xy} = \lim_{n\to\infty} \prod_{k=0}^{yn-1} \left(1 + \frac{nx}{n^2 + k}\right),$$

for $y \geq 0$ and $y \in \mathbb{Q}$, x is any number (real or complex).

Exercise 8. Prove that, in a symmetric manner, we have

$$e^{xy} = \lim_{n\to\infty} \prod_{k=0}^{xn-1} \left(1 + \frac{ny}{n^2 + k}\right),$$

for $x \geq 0$ and $x \in \mathbb{Q}$, y is any number (real or complex).

References

[1] Abramowitz, M., & Stegun, I. A. (Eds.). (1964). Handbook of mathematical functions with formulas, graphs, and mathematical tables (National Bureau of Standards Applied Mathematics Series No. 55). Washington, DC: U.S. Government Printing Office. (Formula 6.1.37, p. 257.)

[2] Wikipedia contributors. (2026, May 2). Harmonic number. In Wikipedia, The Free Encyclopedia. Retrieved 19:36, May 12, 2026, from https://en.wikipedia.org/w/index.php?title=Harmonic_number&oldid=1352172812

[3] Abbott, S. (2015). Understanding analysis (2nd ed.). Springer. https://doi.org/10.1007/978-1-4939-2712-8. (Exercise 2.3.3, *Squeeze Theorem*, p. 54.)

[4] Apostol, T. M. (1974). Mathematical analysis (2nd ed.). Reading, MA: Addison–Wesley. (Theorem 1.22, p. 13.)

Appendix A. Two Claims

The statement below was written after the completion of this article, with the aim of clarifying the reasoning and the origin of inequality (2.41), which was used in the proof of Theorem 3.

Claim 9. For each $n \in \mathbb{N}_{\geq 1}$, define

$$S_n := \sum_{k=0}^{n-1} \frac{n}{n^2 + k}. \tag{A.1}$$

Then

$$n \ln\left(1 + \frac{1}{n}\right) \;\leq\; S_n \;\leq\; \frac{1}{n} + n \ln\left(1 + \frac{1}{n} - \frac{1}{n^2}\right). \tag{A.2}$$

Proof. Define A_n as follows

$$A_n := \sum_{k=0}^{n-1} \frac{1}{n^2 + k}. \tag{A.3}$$

Since n is a constant factor, we can compose the following identity, using the left-hand side of (A.3) on the right-hand side of (A.1)

$$S_n = nA_n. \tag{A.4}$$

Henceforth, the strategy we will adopt is simply to obtain estimates for A_n and multiply them by n.

Consider $f(t) = 1/(n^2+t)$ for $t \geq 0$. The function is decreasing on $[0, \infty)$. For any integer k with $0 \leq k \leq n-1$ and all $t \in [k, k+1]$, one has

$$f(k+1) \leq f(t) \leq f(k). \tag{A.5}$$

Integrate each member of (A.5) with respect to t from k to $k+1$

$$\int_k^{k+1} f(k+1)\,dt \leq \int_k^{k+1} f(t)\,dt \leq \int_k^{k+1} f(k)\,dt. \tag{A.6}$$

Since $f(k+1)$ and $f(k)$ are constants on this interval, (A.6) becomes

$$f(k+1) \leq \int_k^{k+1} f(t)\,dt \leq f(k). \tag{A.7}$$

Summing all members of the inequality in (A.7) for $k = 0, 1, \ldots, n-1$ yields

$$\sum_{k=0}^{n-1} f(k+1) \leq \sum_{k=0}^{n-1} \int_k^{k+1} f(t)\,dt \leq \sum_{k=0}^{n-1} f(k),$$

$$\sum_{k=0}^{n-1} f(k+1) \leq \int_0^{n} f(t)\,dt \leq \sum_{k=0}^{n-1} f(k). \tag{A.8}$$

By a change of index, setting $k = j-1$, the first sum becomes $\sum_{j=1}^{n} f(j)$; then renaming j to k gives $\sum_{k=1}^{n} f(k)$. Placing this last sum on the left-hand side of (A.8) yields

$$\sum_{k=1}^{n} f(k) \leq \int_0^{n} f(t)\,dt \leq \sum_{k=0}^{n-1} f(k). \tag{A.9}$$

In the left inequality of (A.9), replace n by $n-1$ to obtain

$$\sum_{k=1}^{n-1} f(k) \leq \int_0^{n-1} f(t)\,dt. \tag{A.10}$$

Add $f(0)$ to both sides of (A.10) and manipulate the expression to find an upper bound

$$f(0) + \sum_{k=1}^{n-1} f(k) \leq f(0) + \int_0^{n-1} f(t)\,dt,$$

$$\sum_{k=0}^{n-1} f(k) \leq f(0) + \int_0^{n-1} f(t)\,dt. \tag{A.11}$$

From the right-hand side of the inequalities in (A.9) and the entire inequality in (A.11), we deduce that

$$\int_0^{n} f(t)\,dt \leq \sum_{k=0}^{n-1} f(k) \leq f(0) + \int_0^{n-1} f(t)\,dt. \tag{A.12}$$

Now set

$$f(t) = \frac{1}{n^2+t}, \tag{A.13}$$

$$f(k) = \frac{1}{n^2 + k} \tag{A.14}$$

and

$$f(0) = \frac{1}{n^2}. \tag{A.15}$$

Substitute (A.13), (A.14), and (A.15) into (A.12)

$$\int_0^n \frac{1}{n^2 + t}\, dt \le \sum_{k=0}^{n-1} \frac{1}{n^2 + k} \le \frac{1}{n^2} + \int_0^{n-1} \frac{1}{n^2 + t}\, dt. \tag{A.16}$$

Compute the integrals

$$\begin{aligned} \int_0^n \frac{dt}{n^2 + t} &= \ln(n^2 + t)\Big|_0^n \\ &= \ln(n^2 + n) - \ln(n^2) \\ &= \ln\left(1 + \frac{1}{n}\right) \end{aligned} \tag{A.17}$$

and

$$\begin{aligned} \int_0^{n-1} \frac{dt}{n^2 + t} &= \ln(n^2 + t)\Big|_0^{n-1} \\ &= \ln(n^2 + n - 1) - \ln(n^2) \\ &= \ln\left(1 + \frac{n-1}{n^2}\right) \\ &= \ln\left(1 + \frac{1}{n} - \frac{1}{n^2}\right). \end{aligned} \tag{A.18}$$

Substituting the right-hand sides of (A.17) and (A.18) into the left and right sides of (A.16) gives

$$\ln\left(1 + \frac{1}{n}\right) \le \sum_{k=0}^{n-1} \frac{1}{n^2 + k} \le \frac{1}{n^2} + \ln\left(1 + \frac{1}{n} - \frac{1}{n^2}\right). \tag{A.19}$$

Replace the left-hand side of (A.3) into the inequality (A.19)

$$\ln\left(1 + \frac{1}{n}\right) \le A_n \le \frac{1}{n^2} + \ln\left(1 + \frac{1}{n} - \frac{1}{n^2}\right). \tag{A.20}$$

Multiply by n all members of the inequalities in (A.20)

$$n \ln\left(1 + \frac{1}{n}\right) \le nA_n \le \frac{n}{n^2} + n \ln\left(1 + \frac{1}{n} - \frac{1}{n^2}\right),$$

$$n \ln\left(1 + \frac{1}{n}\right) \le nA_n \le \frac{1}{n} + n \ln\left(1 + \frac{1}{n} - \frac{1}{n^2}\right). \tag{A.21}$$

Substitute the left-hand side of (A.4) into (A.21)

$$n \ln\left(1 + \frac{1}{n}\right) \le S_n \le \frac{1}{n} + n \ln\left(1 + \frac{1}{n} - \frac{1}{n^2}\right). \tag{A.22}$$

Note that the inequalities obtained in (A.22) are identical to the inequalities in (A.2). This completes the proof. □

We make this statement below to justify equations (2.82) and (2.96), used in the body of the proof of Theorems 5 and 6 of this paper.

Claim 10. For every fixed real number a, one has

$$\lim_{x\to\infty} \frac{\Gamma(x+a)}{\Gamma(x)\,x^a} = 1, \tag{A.23}$$

or, in asymptotic notation,

$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a, \tag{A.24}$$

when $x \to \infty$ (with x positive real).

Proof. For z real positive and $z \to \infty$, the gamma function [1] has the asymptotic expansion

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z}\left(1 + O\left(\frac{1}{z}\right)\right). \tag{A.25}$$

We apply the approximation in (A.25) to $\Gamma(x+a)$ and $\Gamma(x)$, because as $x \to \infty$, also $x + a \to \infty$

$$\Gamma(x+a) = \sqrt{2\pi}\,(x+a)^{x+a-\frac{1}{2}} e^{-(x+a)}\left(1 + O\left(\tfrac{1}{x}\right)\right) \tag{A.26}$$

and

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x}\left(1 + O\left(\tfrac{1}{x}\right)\right). \tag{A.27}$$

Divide (A.26) by (A.27) and find

$$\frac{\Gamma(x+a)}{\Gamma(x)} = \frac{(x+a)^{x+a-\frac{1}{2}} e^{-x-a}}{x^{x-\frac{1}{2}} e^{-x}} \cdot \frac{1 + O(1/x)}{1 + O(1/x)}. \tag{A.28}$$

Note that the error factor tends to 1, so we rewrite (A.28) as follows

$$\frac{\Gamma(x+a)}{\Gamma(x)} = (x+a)^{x+a-\frac{1}{2}}\, x^{-(x-\frac{1}{2})}\, e^{-a}\left(1 + O(1/x)\right). \tag{A.29}$$

On the other hand, we decompose $(x+a)^{x+a-\frac{1}{2}}$ in this way

$$(x+a)^{x+a-\frac{1}{2}} = (x+a)^a \cdot (x+a)^{x-\frac{1}{2}}. \tag{A.30}$$

Substitute the right-hand side of (A.30) into the right-hand side of (A.29)

$$\begin{aligned}
\frac{\Gamma(x+a)}{\Gamma(x)} &= (x+a)^a \cdot \frac{(x+a)^{x-\frac{1}{2}}}{x^{x-\frac{1}{2}}} \cdot e^{-a}\left(1 + O(1/x)\right) \\
&= (x+a)^a \cdot \left(\frac{x+a}{x}\right)^{x-\frac{1}{2}} \cdot e^{-a}\left(1 + O(1/x)\right) \\
&= (x+a)^a \cdot \left(1 + \frac{a}{x}\right)^{x-\frac{1}{2}} \cdot e^{-a}\left(1 + O(1/x)\right) \\
&= \left[x\left(1 + \frac{a}{x}\right)\right]^a \cdot \left(1 + \frac{a}{x}\right)^{x-\frac{1}{2}} \cdot e^{-a}\left(1 + O(1/x)\right) \\
&= x^a \cdot \left(1 + \frac{a}{x}\right)^a \cdot \left(1 + \frac{a}{x}\right)^{x-\frac{1}{2}} \cdot e^{-a}\left(1 + O(1/x)\right).
\end{aligned} \tag{A.31}$$

Note that

$$\left(1+\frac{a}{x}\right)^{x-\frac{1}{2}} = \left(1+\frac{a}{x}\right)^{x} \cdot \left(1+\frac{a}{x}\right)^{-\frac{1}{2}}. \tag{A.32}$$

We know that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} = e^a$ and $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{-\frac{1}{2}} = 1$. Therefore, the limit of both sides of (A.32), as x approaches infinity, is

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x-\frac{1}{2}} &= \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} \cdot \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{-\frac{1}{2}} \\ &= e^a \cdot 1 = e^a.\end{aligned} \tag{A.33}$$

Combine the like factors in (A.31) and obtain

$$\frac{\Gamma(x+a)}{\Gamma(x)} = x^a \cdot \left(1+\frac{a}{x}\right)^{x+a-\frac{1}{2}} \cdot e^{-a}\left(1+O(1/x)\right). \tag{A.34}$$

Divide both sides of (A.34) by x^a

$$\frac{\Gamma(x+a)}{\Gamma(x)\cdot x^a} = \left[\left(1+\frac{a}{x}\right)^{x+a-\frac{1}{2}} \cdot e^{-a}\right]\left(1+O(1/x)\right). \tag{A.35}$$

Since the exponent $x+a-\frac{1}{2}$ is asymptotically $x-\frac{1}{2}$, we have

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x+a-\frac{1}{2}} \cdot e^{-a} &= \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x-\frac{1}{2}} \cdot e^{-a} \\ &\overset{(A.33)}{=} e^a \cdot e^{-a} \\ &= 1.\end{aligned} \tag{A.35}$$

Observe that the error factor $1+O(1/x)$ also tends to 1. Due to the previous observation and to (A.35); then, (A.35) becomes

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{\Gamma(x)\,x^a} = 1. \tag{A.36}$$

Thus, we have shown that

$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a \qquad (x\to\infty). \tag{A.37}$$

Note that (A.36) and (A.37) are exactly the expressions (A.23) and (A.24) stated above. Here ends the proof. □