General Formula Solution for Quintic Equations

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Abstract: This paper discovers and proves the existence of a general, non-radical-based formulaic solution for the general quintic equation with complex coefficients. The method involves transforming the general form of the equation into a formulaically solvable form— referred to herein as standard form — given by $x^5 - px + 1 = 0$, where p is an arbitrary complex number. When the modulus of p satisfies $|p| \geq 1.65$, the solution is derived using a series expansion involving negative integer powers with its coefficients of an integral series; conversely, when $|p| < 1.65$, a series expansion involving positive powers with its coefficients of a fractional series is employed. These two approaches form a complete and logically closed loop. This method is purely algebraic in nature, requiring neither root searching nor iterative procedures. Furthermore, since any general quintic equation with complex coefficients can invariably be transformed into this standard form, the proposed method possesses universal applicability. Numerical results demonstrate that the method presented in this paper is highly practical and easy to implement.

Keywords: Quintic polynomial equation, Standard form, Algebraic method, Tschirnhaus transformation, Sequence, Series

MSC(2020)：1205

1 Introduction

Finding the roots of polynomial equations is one of the most fundamental problems in mathematics. The development of general methods for solving polynomial equations has long remained a formidable challenge. Although the problem of finding roots for quadratic polynomials was resolved over two millennia ago, no further breakthroughs were achieved for a considerable period thereafter. It was not until the emergence of Cardano's method in the

16th century that techniques for solving cubic and quartic polynomial equations were gradually discovered. Subsequently, the problem of finding roots for quintic polynomials remained intractable; ultimately, Abel (1824) and Galois (1832) provided proofs demonstrating that general polynomial equations of the fifth degree and higher do not possess solutions expressible in terms of radicals [1].

To date, no general formula - non-radical in nature - for solving quintic equations has been reported in the literature. Historically, prominent mathematicians such as Euler, Lagrange, and Gauss have all investigated this problem. Since radical solutions exist only for specific special cases [2], various alternative approaches have been proposed to solve general-form equations; these include de Fériet's Fourier functions [3], hypergeometric functions [4], elliptic modular functions and inverse Lagrange formulas [5]. Due to limitations regarding convergence, these methods often necessitate the application of appropriate transformations [6],[7],[8],[9]. Recent research into solving higher-degree polynomial equations includes Longfellow's work, which - utilizing Tschirnhaus transformations - presented a general solution transformation for sextic equations with real coefficients, as well as solutions involving the inverse regularized beta function [5]; additionally, Wildberger and Rubine have explored the potential of employing hyper-MingAnTu-Catalan series to solve higher-degree polynomial equations with real coefficients [10],[11].

The general quintic equation can be expressed as

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \tag{1}$$

where the coefficient a_i are complex number, and the constant term $a_0 \neq 0$ 。

Historically, during the process of solving quintic equations, various distinct special forms were identified and named [12]. For instance: an equation lacking a quartic term is termed the reduced form; an equation lacking both quartic and cubic terms is termed the principal form; and an equation lacking quartic, cubic, and quadratic terms is termed the Bring-Jerrard form —a special case of which is the Bring form, characterized by a linear term coefficient of 1 and the presence of only a single constant term coefficient:

$$\begin{aligned}
x^5 &\quad + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \\
x^5 &\quad + a_2x^2 + a_1x + a_0 = 0 \\
x^5 &\quad + a_1x + a_0 = 0 \\
x^5 &\quad + x + a_0 = 0
\end{aligned}$$

Furthermore, there is the Brioschi form:

$$x^5 \quad - 10cx^3 \quad + 45c^2x - c^2 = 0$$

and the De Moivre form:

$$x^5 \quad + 5ax^3 \quad + 5a^2x + b = 0$$

as well as the Euler form:

$$x^5 \quad + a_2x^2 \quad + a_0 = 0$$

Among these, the Brioschi form corresponds to solutions via hypergeometric series, while the De Moivre form admits a solution in radicals; the remaining forms represent specific forms derived from the general quintic equation through transformation. For a comprehensive classification of general quintic equations under transformation, please refer to [13, 14].

Based on the various historical forms mentioned above, it can be inferred that—following appropriate transformations—the general quintic equation can be reduced to a limiting case containing only a single coefficient. This observation inspired us to investigate universal solvable forms or "standard forms" that are amenable to solution. The present study demonstrates that the roots of the general quintic equation can be determined via non-radical series formulas, interpreted within the framework of these universal solvable forms or standard forms.

2 General Formulaic Solution for the General Form of a Quintic Equation

2.1 Derivation of the General Solvable Form of Quintic Equations

Through successive second- and fourth-degree Tschirnhaus transformations, a general quintic equation can always be transformed into a single-parameter standard form [12],[15],[16].

2.1.1 Transformation of the General Quintic Equation into the Principal Form Free of Quartic and Cubic Terms

For the general quintic equation (1), let the quadratic form of the Tschirnhaus transformation be

$$T_2 = x^2 + \beta x + \alpha + y$$

where y represents the new variable. By computing the resultant, an equation in terms of the variable y —obtained by eliminating the original variable x —can be derived; the degree of the equation remains unchanged. The transformation parameters can be determined through the transformation process.

By setting the coefficient of the quartic term in the transformed equation for y to zero, one can solve for

$$\alpha = \frac{1}{5}(a_4\beta + 2a_3 - a_4^2)$$

Substituting this into the coefficient of the cubic term and setting it to zero yields a quadratic equation in β, from which β can be solved. Consequently, the general quintic equation in x is transformed into a quintic equation in the new variable y

$$y^5 + b_2y^2 + b_1y + b_0 = 0 \tag{2}$$

2.1.2 Second Transformation into the General Solvable Form

To further eliminate the quadratic term, let the quartic Tschirnhaus transformation be

$$T_4 = ey^4 + dy^3 + cy^2 + by + a + z$$

Following the transformation, the equation becomes a quintic equation in the variable z. By setting the coefficient of the quartic term in the transformed equation to zero, we can solve for

$$a = \frac{1}{5}(3b_2d + 4b_1e)$$

Substitute this into the coefficient of the cubic term, simplify, and set the result equal to zero. This yields a linear equation in terms of b, whose linear coefficient is

$$3b_2c + 4b_1d + 5b_0e$$

Setting it to zero will solve for c

$$c = -\frac{1}{3}(4b_1d + 5b_0e)$$

Substituting this into the remaining constant term of the cubic coefficient—which now yields a quadratic equation in terms of d — allows for the determination of d. Next, substituting the values of a, c, and d into the coefficient of the quadratic term of the transformed z-equation and setting it to zero—which now yields a cubic equation in terms of b — allows for the determination of b.

Finally, substituting the determined transformation parameters a, b, c, and d into the constant term of the transformed z-equation and setting it equal to 1 allows for the determination of e.

Thus, the aforementioned quintic equation (2) has been transformed into a generally solvable form or standard form.

$$z^5 - px + 1 = 0 \tag{3}$$

2.2 Root-Finding Formulas Based on General Solvable Forms

For equations of the general solvable or standard form (3), it is evident that the roots of the equation are functions solely of the coefficient p. By appropriately constructing this function, one of the roots of the equation can be obtained. For convenience, a series representation is now selected.

When the modulus of the complex number p satisfies $|p| \geq 1.65 > \frac{5}{4}2^{\frac{2}{5}}$, a series expansion in negative powers is applied to equation (3) , yielding the following formula:

$$z = \sum_{n \geq 0} C_n \; p^{-(1+5n)} \tag{4}$$

where $C_n = \frac{(5n)!}{n!(1+4n)!}$ is an integer sequence composed of fifth-order MingAnTu–Catalan numbers. [14]

When $|p| < \frac{5}{4}2^{\frac{2}{5}} < 1.65$, performing a power series expansion on equation(3) yields one root of the quintic equation as:

$$z = \sum_{n \geq 0} \frac{(-1)^n D_n}{5^{n+[n/5]+[n/25]}} \; p^n \tag{5}$$

where the numerators of the fractional sequence within the coefficients
$D_n = \{-1, 1, 1, 1, 0, 21, 78, 187, 286, 0, 9367, ...\}$.

2.3 Solving for the Root of the General Quintic Equation

By substituting a root obtained from the solvable form of the quintic equation back into the quartic transformation equation, one typically obtains four solutions; from these, a specific solution to equation (2) can be selected. Subsequently, by substituting this solution back into the quadratic transformation equation, one typically obtains two solutions; from these, a specific solution to equation (1) can be selected.

The aforementioned process for solving for a single root of a general quintic equation with complex coefficients involves a quadratic transformation, a quartic transformation, and the computation of one of two series; the back-substitution process entails solving a quartic equation and a quadratic equation. The transformations, solving procedures, and back-substitution steps involved are all deterministic calculations based on algebraic methods. Compared to existing numerical methods, this approach requires neither root localization searches nor iterative procedures, thereby exhibiting superior computational efficiency and stability.

3 Examples of Solving Quintic Equations

Example 1: Consider a quintic polynomial equation with real coefficients

$$x^5 - 2.1x + 1 = 0$$

Since $p = 2.1 \geq 1.65$, Formula (4) is applied, expanding to 20 terms. The calculated result is 0.4895847836; the result obtained using the iterative method is 0.4895847836.

Example 2: Let a quintic polynomial equation with real coefficients

$$x^5 - 1.1x + 1 = 0$$

Since $p = 1.1 < 1.65$， formula (5) is applied, expanding to 20 terms. The calculated result is -1.181188748, while the result obtained using the iterative method is -1.181189099.

Example 3: Let a quintic polynomial equation with complex coefficient be given as

$$x^5 + (1.1 + 2.45I)x + 1 = 0$$

For $p = -(1.1 + 2.45I)$, with the modulus $|p| \geq 1.65$, applying formula (4) and expanding to 20 terms yields the calculated result $-0.1504187777 + 0.3381707909I$,and the result obtained using the iterative method is $-0.150418777703487 + 0.3381707909431631I$.

Example 4: Let a quintic polynomial equation with real coefficients

$$x^5 + 3x^4 + 5x^3 + 4x^2 + 7x + 11 = 0$$

First, apply a quadratic Tschirnhaus transformation

$$T_2 = x^2 + (\frac{27}{14} + \frac{1}{14}\sqrt{1315}\ I)x + (\frac{19}{14} + \frac{3}{70}\sqrt{1315}\ I) + y$$

Transformed into a quintic equation in the new variable y, with the quartic and cubic terms eliminated

$$y^5 + (89.93002928 - 188.6940969I)y^2 - (278.8915460 + 849.0811460I)y-$$

$$-(1011.330106+615.4764513I)=0$$

Followed by another quartic Tschirnhaus transformation

$$T_4=(0.003221311181+0.002311593544I)y^4+(-0.004444610521-0.002089828457I)y^3+$$

$$+(0.00175211275+0.01915298186I)y^2+(0.8398143141-0.4503294000I)y+$$

$$+0.3750418832-2.313430729I+z$$

then transformed into z-equation in general solvable or standard form

$$z^5-pz+1=0$$

In this case $p=-0.9313284+0.23892910\ I$, the modulus of p is $|p|=0.9614882753<1.65$.

Applying formula (5) and expanding to 20 terms, one root of the quintic equation for z is calculated to be $-0.7764693768-0.06818657410\ I$. Substitute this back into the fourth-order transformation equation T_4 to solve for the four roots. From these, select one root of the y-equation as $-0.7315819320+2.372390672\ I$.

Substitute this back into the transformation equation T_2 and solve for the two roots. Finally, substitute these two roots back into the original given equation to determine one of the roots of the original equation $-1.515906976+(9.984446258\ 10^{-10})\ I$. The result of solving the original equation using the iterative method is -1.51590697506617.

As indicated by the results above, the solutions derived from the general formula are highly consistent with those obtained through iteration. Evidently, the solution accuracy improves as the number of terms in the series summation increases.

4 Proof and Verification of the Convergence of the Root-Finding Formula

By comparing equation (4)—the root-finding formula for integer sequences—with the general term of the harmonic series, it can be deduced that the condition for convergence must be

$$\frac{(5n)!}{n!(1+4n)!}\frac{1}{|q|^{1+5n}}<\frac{1}{n}$$

that is

$$|q|>\sqrt[(1+5n)]{\frac{n\ (5n)!}{n!(1+4n)!}}$$

where $|q|$ denotes the modulus of the complex number. Since its coefficients form the MingAnTu–Catalan integer sequence—a monotonically increasing sequence—we take its limit.

$$\lim_{n\to\infty} \sqrt[(1+5n)]{\frac{n\ (5n)!}{n!(1+4n)!}} = \frac{5}{4}2^{\frac{2}{5}} = 1.649384889$$

Specifically, the convergence condition for the root-finding formula (4) is $|q| > 1.649384889$.

The series defined by the fractional sequence in formula (5) currently lacks a general term formula; however, experimental measurements indicate that its radius of convergence exceeds 1.65. That is, when $|q| < 1.65$, the fractional sequence series of formula (5) converges.

This demonstrates that the convergence domains of the two series defined by formulas (4) and (5) collectively span the entire range of possible values for the modulus of the complex number p; together, they form a complete and logically closed loop.

5 结论与讨论

The results presented in this paper demonstrate that general quintic equations admit a universal, non-radical formulaic solution. The corresponding universally solvable or standard form is given by $x^5 - px + 1 = 0$, where p is a complex number. Specifically, when the modulus of p satisfies $|p| \geq 1.65$, a root-finding formula based on a negative-exponent power series of integer sequences is applied; conversely, when $|p| < 1.65$, a root-finding formula based on a positive-exponent power series of fractional sequences is utilized. These two approaches collectively form a complete and logically closed loop. This method is purely algebraic in nature, requiring neither search algorithms nor iterative procedures. Furthermore, since any general quintic equation with complex coefficients can invariably be transformed into this standard form, the method possesses universal applicability, offers significant practical utility, and is easy to implement.

Corresponding to the aforementioned general solvable form or standard form, there exist other single-parameter forms of quintic equations—such as $px^5 + x + 1 = 0$ and $x^5 + x + r = 0$. Although these forms possess root-finding formulas based on positive-exponent power series, they lack corresponding formulas based on negative-exponent power series to form a complete and logically closed loop. Consequently, when convergence conditions are not met, they cannot be solved directly; thus, neither constitutes a general solvable form. In practical applications, however, both forms can be transformed into a general solvable form to facilitate their solution.

The primary challenge in solving general polynomial equations lies in determining whether there exist universally applicable solution methods and formulas that are easy to implement.

An interesting fact is that, even in the absence of solutions expressible in terms of radicals, the series formulas presented in this paper still enable the straightforward determination of roots for polynomial equations of degree five or lower. Regarding the solution of roots for higher-degree equations, further research and exploration are required to determine whether analogous solution methods exist.

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