Integral Reduction Formulas
Enhanced With a TI84

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Abstract

The TI84 has both a definite integral function and a recursive list generator. We explore whether the combination can be used to solve single and double integral problems that reference reduction formulas for integral evaluations.

Introduction

Thomas's Calculus text [1] has an example problem that references a reduction formula for $\cos^n$ integrands: Find the moment of inertia, about the y-axis, of the area enclosed by the cardioid $r = a(1 - \cos(\theta)$, page 557.

With a few manipulations and references to formulas, a double integral distills to solving the single integral

$$\int_0^{2\pi} \frac{a^4}{4} \cos^2(1 - \cos(\theta))^4 \, d\theta. \tag{1}$$

There are several instances of powers of $\cos$ in the integrand: $\cos^2$, $\cos^3$, $\cos^4$, $\cos^5$, and $\cos^6$. Each has a coefficient. These coefficients can be stored in a TI84 list.

The evaluation of the indefinite integral of $\cos^n$ can proceed with an integration by parts. A reduction formula pops out; in simplified form it is

$$\int \cos^n = \frac{\cos^{n-1} \sin}{n} + \frac{n-1}{n} \int \cos^{n-2} . \tag{2}$$

If the definite integrals have upper and lower bounds (here 2π and 0) that make the first term on the right 0 (typical, as in this problem), (2) further reduces to

$$\int \cos^n = \frac{n-1}{n} \int \cos^{n-2} .$$

Although this is termed a reduction formula, it really works through recursion. Let $F(n)$ give the nth case in the formula. Then once

$$F(0) = \int_a^b \cos^0 = \int_a^b 1 = b-a \text{ and } F(1) = \int_a^b \cos = |_a^b \sin = \sin(b)-\sin(a)$$

are determined (easy) we can crunch others: for example,

$$F(2,3,4,5,6) = \frac{1}{2}F(0), \frac{2}{3}F(1), \frac{3}{4}F(2), \frac{4}{5}F(3), \frac{5}{6}F(4),$$

all the integrals in (1).

We will show that the TI84 recursive sequence feature can generate a list with all these integral evaluations. When this list is multiplied by a list of coefficients, $\{0, 1, -4, 6, -4, 1\}$ a summed dot product gives the value of the integral. With a little trick, π (masked as i) can be incorporated into the evaluation for an exact answer. Without the $a^4/4$ factor in the integrand of (1), the exact answer is $49\pi/8$.

Details

There are some features desirable for crunching these integrals that the TI84 doesn't have. In Figure 1a) the form (call it) can't be filled in programmatically; the variable assignment "$(n-1)/n * u(n-2)$" $\rightarrow u$ works inside a program, but there is no way to assign $u(0)$ and $u(1)$ outside this form.

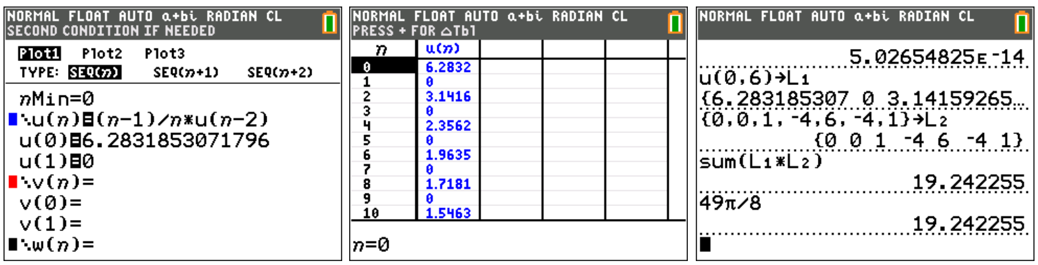

Figure 1: a) Set mode to SEQ and hit the Y1 key; b) Table shows integral values; c) Make lists using u and coefficients then take the sum of the dot product.

A TI84 isn't a CAS. The symbol π is always converted to a decimal approximation. In Figure 1a), $u(0) = 2\pi$ is replaced by such an approximation. A trick that almost works is to use $u(0) = 2i$, but alas this form rejects that type of argument. Figure 2 gives a work around. All the terms with a multiple of π are converted to the same multiple only with i via another list, L3 in Figure 2b).

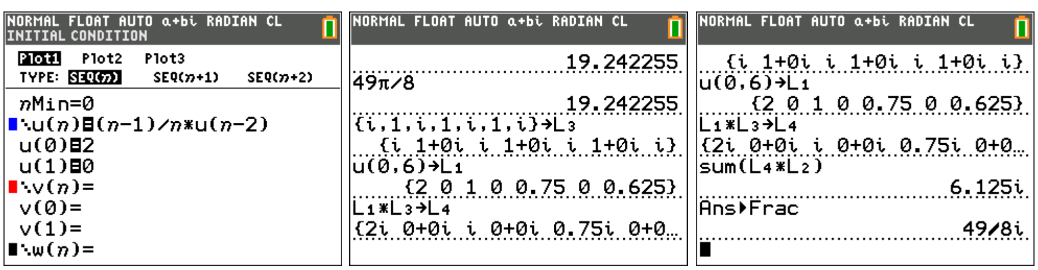

Figure 2: a) Reinitialize the u variable by dropping the π; b) Make a mask for π using i, L_3, reassign L_1, dot this for a new L_3 to make L_4; c) Sum the dot product of L_4 and L_2 for the answer.

Programs

What program might streamline the number crunches involved with such problems? Problems 8 and 9 on page 557 of Thomas can be solved with the program below. The sequence form has to be adjusted with $u(0)$ and $u(1)$ – you could calculate these using $fnInt$, the definite integral function, but the integral evaluations are easy. Ideally assignments of the form $fnInt(1, x, 0, 2\pi) \rightarrow u(0)$ in the program and something similar for $u(1)$ could be used. The other wonderful addition to this great calculator would have a *re-open last program edited* with a single alpha F5 keystroke, say. Figure 3 gives a program that can solve the problem above and exercises 8 and 9 on page 558.

```
001 Seq
002 Disp "ALL N TO MAX"
003 Prompt L1
004 dim(L1)→A
005 A-1→A:u(0,A)→L2
006 L4→L5:A+1→dim(L5)
007 L5*L2→L3
008 sum(L1*L3)→B
009 Disp B▶Frac
010 imag(B)*π→C
011 real(B)+C→D
012 Disp D
013 real(B)▶Frac
014 toString(Ans)+"+"→Str1
015 imag(B)▶Frac
016 toString(Ans)+"π"→Str2
017 Disp Str1+Str2
```

Figure 3: When prompted enter a list of coefficients and use 0s for early powers of cosine not in the integrand. The last non-zero entry gives the dimension of L_1.

Users are prompted to enter the coefficients for the powers of cosine. Typical problems don't have coefficients for powers of cosine of zero, a constant, so the user needs to put a zero in for that coefficient and other placeholder zeros. The program does calculate the number of coefficients entered using the last non-zero coefficient. That said, users need to set up the form for the recursion apart from running the program. The list L_4 is $\{i, 1, i, 1, i, 1, i, 1, i, 1\}$, the mask for the i alias for π trick. This supposes that even powers have multiples of π in them. The assignment of u to a list has a 0 start, line 005, Figure 3.

There is a work around for the limitations mentioned. One can programmatically prompt for $u(0)$ and $u(1)$, in effect, by prompting for A and B variables. Then assign Y_1, a function variable off of the *vars* key (or, more familiarly the *y*=, the *f1* key). The following program, Figure 4 solves the problems mentioned as well as number 6 on page 561: the last involves

$$\int_0^\pi \sin^4\theta \, d\theta.$$

This follows as the reduction formula for powers of sine is the same as that for cosine powers. An elaboration of this program would allow, per the previous program a list assignment of the coefficients of powers of cosine or sine and the crunching of the dot product – now via a list assignment of Y_1 values.

```
Func
Prompt A,B,C
"(X=Ø)A+(X=1)B+(X=2)(1/2)A+(X=3)(2/3)B"→Str1
Str1+"+(X=4)(1/2)(3/4)A+(X=5)(2/3)(4/5)B"→Str1
Str1+"+(X=6)(1/2)(3/4)(5/6)A"→Str1
Str1+"+(X=7)(6/7)(4/5)(2/3)B"→Str1
Str1→Y1
Disp Y1(C)
```

Figure 4: Enter A (alias for $u(0)$) and B (alias for $u(1)$) and the power of cosine or sine, C.

Conclusion

Larson and Stewart, unlike Thomas, don't ask students to do these harder reduction formula problems. It is curious that with this TI84 calculator one can get crunch $\int_{-\pi/2}^{\pi/2} \cos^{60}\theta \, d\theta$ with a table lookup and with a little more program work get the exact answer. These hardest of integrals become the easiest even in combination, as in linear combination.

References

[1] G. B. Thomas, *Calculus and Analytic Geometry*, 4th ed., Addison-Wesley (1968).