VOLUME OF GENERAL SPHERICAL WEDGES

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Abstract. A sphere cap is constructed by slicing a sphere with a plane. Slicing a sphere cap again with a second plane splits it into a pair of spherical wedges. This work evaluates spherical wedge volumes, i.e., the triple integrals over essentially the intersection of the two sphere caps established by the two cut planes. The integrals are closed forms of square roots and inverse sines as a function of sphere radius, heights of the sphere caps, and dihedral angle between the cut planes.

1. Volume of the Sphere cap

A sphere cap of height h is generated by slicing a sphere of radius R by a plane parallel to the $x-y$-plane where the distance of the plane to the sphere center is $R-h$: Fig. 1. The volume of the cap is the integral over the areas of the small circles inside the cap, $\pi(R^2-z^2)$, at z-coordinates between $R-h$ and R [2, 3.3.4.14]:

(1)
$$V_h = \int_{R-h}^{R} dz \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} dx \int_{-\sqrt{R^2-z^2-x^2}}^{\sqrt{R^2-z^2-x^2}} dy = 2\int_{R-h}^{R} dz \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} dx \sqrt{R^2-z^2-x^2}$$
$$= 2\int_{R-h}^{R} dz \frac{\pi}{2}(R^2-z^2) = \frac{\pi h^2}{3}(3R-h).$$

This formula can be extended to hyperspherical caps [4]. The basic indefinite integral in action here is [3, 2.262.1, 2.261][2, §21.7.2.5]

$$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}. \tag{2}$$

Remark 1. *If $h > R$ the sphere cap includes more than half the sphere (and the equator), the volume is the full sphere volume $4\pi R^3/3$ minus the volume of the south pole cap:*

$$V_h = \frac{4}{3}\pi R^3 - V_{2R-h} = \frac{h^2\pi}{3}(3R-h), \quad R \le h \le 2R. \tag{3}$$

so the formula (1) remains valid.

Definition 1. *(Field of View of the Spherical Sector) The rim of the cap's base covers an angle α measured in the right triangle that contains (i) the sphere center, (ii) a point on the rim of the cap's base, and (iii) the base center:*

$$\cos\alpha \equiv \frac{R-h}{R}. \tag{4}$$

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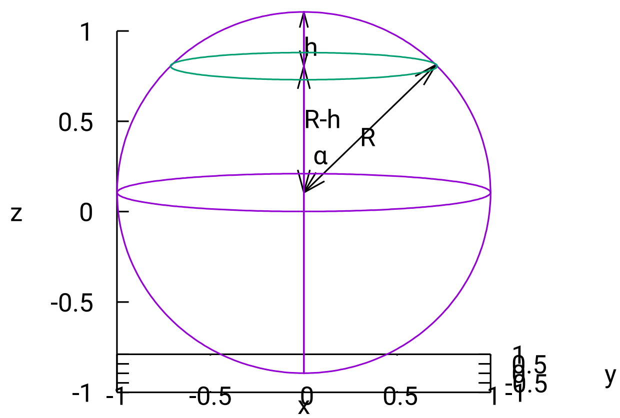

FIGURE 1. A sphere of radius R with magenta outlines (equator and two big circles of constant longitudes) and a sphere cap with a green base and altitude h. Distance of the base plane to the sphere center is $R-h$. Angle between the directions from sphere center to cap apex and cap rim is α.

Definition 2. *(Complementary cap height) The (signed) distance of the cut plane from the sphere center is*

$$\bar{h} \equiv R - h. \tag{5}$$

2. Two Intersecting Caps

2.1. **Different Shapes of Intersections.** The general case of slicing a sphere twice with two different planes of dihedral angle θ defines a configuration with two sphere caps similar to Figure 2.

If the two plane have a line of intersection that cuts through the sphere, the two cuts split the sphere in 4 bodies, the volume of the intersecting caps computed here, and the other three body volumes computed by subtracting it from the sphere cap volumes and from the full sphere volume. If the two cuts split the sphere in 3 bodies, there is nothing new besides (1): there are two cap volumes and their complement with respect to the full sphere volume.

Remark 2. *Sphere cap pairs that have a nonzero volume $V_\cap$ are called connected* [5].

Remark 3. *The case $\theta = 0$ has a well-known result, the spherical layer where the volume is the difference of two caps of common apex* [2, 3.4.4.15].

There are cases where $V_\cap$ equals either V_1 or V_2 because one of the h_i is so large that one sphere cap is entirely inside the other, or because θ is close to zero. There are cases where the intersection is empty because the h_i are small and θ too large to generate a common subvolume. An indicator of these marginal, less interesting cases is that the line of intersection between the two cut planes does not cut through the sphere.

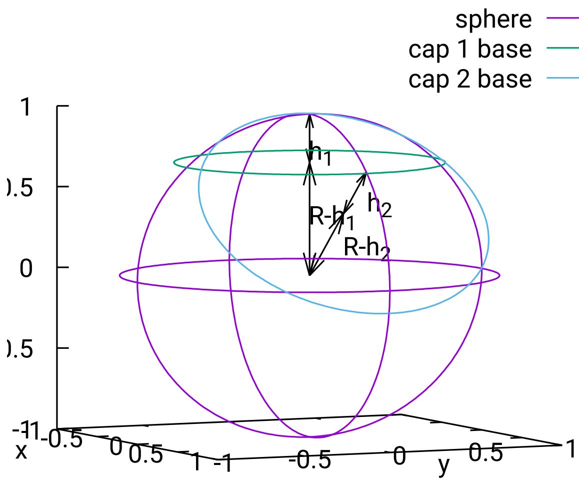

FIGURE 2. A sphere of radius R with magenta outlines, a sphere cap with a green base and altitude h_1, and another sphere cap with a blue base and altitude h_2 at angle of approximately $\theta \approx 45°$ between the two plane normals.

2.2. **Line of Intersection of cut planes.** Without loss of generality we may let point the normal of the first plane to the north pole of the sphere at $(0, 0, R)$, and let point the normal of the second plane to $(R\sin\theta, 0, R\cos\theta)$. Then the points on the circle rim of the cap 1 base are at Cartesian coordinates

$$x^2 + y^2 + (R - h_1)^2 = R^2; \tag{6}$$

$$\Leftrightarrow x^2 + y^2 = 2Rh_1 - h_1^2. \tag{7}$$

The previous equation allows a parametric representation of points of the cap 1 base by azimuths φ_1:

$$x = \sqrt{(2R - h_1)h_1}\cos\varphi_1; \quad y = \sqrt{(2R - h_1)h_1}\sin\varphi_1; \quad 0 \le \varphi_1 \le 2\pi. \tag{8}$$

Remark 4. *With Definition 1* [2, §3.1.6.2]

$$\sqrt{(2R - h_1)h_1} = R\sin\alpha_1. \tag{9}$$

The points on the circle rim of the cap 2 base are obtained by starting with a cap of height h_2 and rotating that small circle by the polar angle θ around the y-axis with the rotation matrix

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ R - h_2 \end{pmatrix}.$$

The representation of the Cartesian coordinates on the base cap 2 circle are the three components of this vector equation:

$$x' = \sqrt{(2R - h_2)h_2}\cos\varphi_2\cos\theta + (R - h_2)\sin\theta; \tag{10}$$

$$y' = \sqrt{(2R - h_2)h_2}\sin\varphi_2; \tag{11}$$

$$z' = -\sqrt{(2R - h_2)h_2}\cos\varphi_2\sin\theta + (R - h_2)\cos\theta. \tag{12}$$

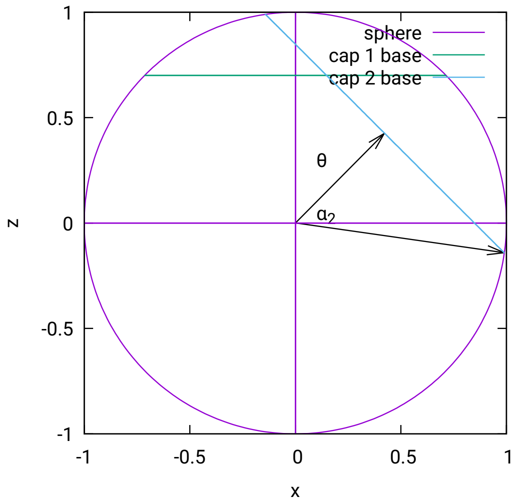

FIGURE 3. The intersection of the two base circles, a projection of Fig. 2 on the $x-z$-plane such that the sphere caps appear as circular segments and the small circles as line segments.

Remark 5. *The dot product of a vector of a point (x', y', z') in a plane multiplied by a vector to the point in the plane closest to the origin is the product of the two lengths by the cosine of the angle α_2 between the vectors* [2, 3.5.1.5.1,3.5.3.4.2]*. The angle α_2 is half the angle of view of the cone with apex at the center and flat face established by the cap's base* [2, Fig. 3.72]*, the angle between the two vectors defining a right triangle in Fig. 3:*

$$\cos\alpha_2 = \bar{h}_2/R. \tag{13}$$

The mid point of the small circle of the base has distance $R-h_2$ to the origin and is at $((R-h_2)\sin\theta, 0, (R-h_2)\cos\theta)$, and the length of (x', y', z') is R, so

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \cdot \begin{pmatrix} (R-h_2)\sin\theta \\ 0 \\ (R-h_2)\cos\theta \end{pmatrix} = R(R-h_2)\cos\alpha_2; \tag{14}$$

$$\Rightarrow x'\sin\theta + z'\cos\theta = R-h_2. \tag{15}$$

This follows also from linear superposition of (10) *and* (12)*.*

The two points where the two base circles intersect are at $x = x'$, $y = y'$, $z = z'$:

$$\sqrt{(2R-h_1)h_1}\cos\varphi_1 = \sqrt{(2R-h_2)h_2}\cos\varphi_2\cos\theta + (R-h_2)\sin\theta; \tag{16}$$

$$\sqrt{(2R-h_1)h_1}\sin\varphi_1 = \sqrt{(2R-h_2)h_2}\sin\varphi_2; \tag{17}$$

$$R-h_1 = -\sqrt{(2R-h_2)h_2}\cos\varphi_2\sin\theta + (R-h_2)\cos\theta. \tag{18}$$

$\cos\varphi_2$ is derived by solving the third equation of this bundle:

$$\cos\varphi_2 = \frac{(R-h_2)\cos\theta - (R-h_1)}{\sqrt{(2R-h_2)h_2}\sin\theta} = \frac{\bar{h}_2\cos\theta - \bar{h}_1}{R\sin\alpha_2\sin\theta}. \tag{19}$$

The two branches of the arccos yield two points/azimuths φ_2 of intersection between the two circles.

Remark 6. *The squared equation* (17) *is*

$$(2R-h_1)h_1(1-\cos^2\varphi_1) = (2R-h_2)h_2(1-\cos^2\varphi_2).$$

Insertion of (19) *and solving for* $\cos\varphi_1$ *yields*

$$\cos\varphi_1 = \frac{(R-h_1)\cos\theta - (R-h_2)}{\sqrt{(2R-h_1)h_1}\sin\theta}. \tag{20}$$

As expected by symmetry (of permuting the two spherical caps) this is (19) *with the indices* $1 \leftrightarrow 2$ *swapped.*

2.3. **Algebraic Criterion for Intersection.** The two small circles don't intersect (and $V_\cap = 0$) if $\cos\varphi_2$ is not in the range $(-1,1)$. So we require for nontrivial intersections of the sphere caps that the squared value (19) is smaller than 1:

$$\left[\frac{(R-h_2)\cos\theta - (R-h_1)}{\sqrt{(2R-h_2)h_2}\sin\theta}\right]^2 \le 1; \tag{21}$$

$$\Rightarrow [(R-h_2)\cos\theta - (R-h_1)]^2 \le (2R-h_2)h_2\sin^2\theta; \tag{22}$$

This is a quadratic inequality for $\cos\theta$:

$$R^2\cos^2\theta - 2(R-h_1)(R-h_2)\cos\theta + R^2 + h_1^2 + h_2^2 - 2R(h_1+h_2) \le 0; \tag{23}$$

Remark 7. *This representation is symmetric/invariant with respect to swapping* $h_1 \leftrightarrow h_2$.

Definition 3. *(Scaled lengths) Unitless lengths scaled with the radius* R *are marked with a roof top or tilde of the variable:*

$$\hat{h}_i \equiv h_i/R; \quad \hat{\bar{h}}_i \equiv \tilde{h} \equiv \bar{h}_i/R; \quad i = 1,2. \tag{24}$$

Dividing the quadratic inequality through R^2 yields

$$\cos^2\theta - 2(1-\hat{h}_1)(1-\hat{h}_2)\cos\theta + 1 + \hat{h}_1^2 + \hat{h}_2^2 - 2(\hat{h}_1+\hat{h}_2) \le 0; \tag{25}$$

The two roots of the quadratic polynomial on the left hand side are

$$\begin{aligned} \cos\theta &= (1-\hat{h}_1)(1-\hat{h}_2) \pm \sqrt{\hat{h}_1\hat{h}_2(2-\hat{h}_1)(2-\hat{h}_2)} \\ &= \tilde{h}_1\tilde{h}_2 \pm \sqrt{(1-\tilde{h}_1^2)(1-\tilde{h}_2^2)}. \end{aligned} \tag{26}$$

The discriminant is positive for all relevant cases because $0 \le \hat{h}_{1,2} \le 2$ for all planes that cut through the sphere, so two real values of $\cos\theta$ exist which establish a minimum and maximum dihedral angle θ at which two sphere caps with given heights $h_{1,2}$ intersect. Given a sphere cap $\hat{h}_1 = 0.3$ and a sphere cap $\hat{h}_2 = 0.4$, Fig. 4 shows in green the cut by plane 1 and in blue four different planes 2 at four dihedral angles: The minimum and maximum dihedral angles are those where a blue line intersects the green line at its left or right end; one intermediate dihedral angle puts the intersection at some positive $x = x'$ and lets the (blue, second) sphere cap still include the north pole of the sphere; the 90° dihedral angle defines the vertical blue line.

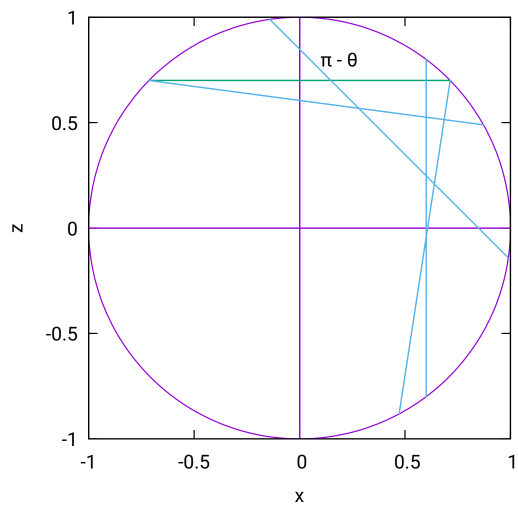

FIGURE 4. The intersection of the base circles, a projection of Fig. 2 on the $x-z$-plane, for four different (blue) dihedral angles θ of the second base. $\pi-\theta$ is the internal angle within the body of intersecting caps.

2.4. **Volume.** The subject of this work is to calculate the volume $V_\cap$ delimited in Fig. 3 by the (off-axis) spherical wedge which includes the north pole, the right part of the green line (base of cap 1) and the upper part of the blue line (base of cap 2). The spherical wedge is a convex body with three faces: two planar faces which are circular segments of the cap bases, and one spherical lune. We know from (16), (20) and (18) the x and z coordinate of the intersection of the two lines,

$$x = x' = \frac{(R-h_1)\cos\theta - (R-h_2)}{\sin\theta}; \quad z = z' = R - h_1. \tag{27}$$

The test of in/exclusion of the north pole by cap 2 means to test the sign of $\theta-\alpha_2$, α_2 as in Defn. 1 and (13): Look in Fig. 3 in the direction of the cap 2 apex, turn counter-clock-wise by the angle α_2, and end up to the right or left from the vertical; if $\theta-\alpha_2<0$, the north pole is included, if positive it is not.

Remark 8. *Alternatively, one can look at the decomposition of the North Pole coordinates in a vector along the plane's normal and perpendicular to it; the y-component is zero. Walking from the sphere center to the base plane, some distance* $t_\parallel$ *inside the plane and then some distance* $t_\perp$ *perpendicular to the plane to the North Pole:*

$$\begin{pmatrix} 0 \\ R \end{pmatrix} = \begin{pmatrix} \bar h_2 \sin\theta \\ \bar h_2\cos\theta \end{pmatrix} + t_\parallel \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} + t_\perp \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{28}$$

Computing the dot product of this equation with the normal vector $(\sin\theta, \cos\theta)$ *yields*

$$R\cos\theta = \bar h_2 + t_\perp. \tag{29}$$

If $t_\perp = R\cos\theta - \bar h_2 > 0$*, the third step of the walk leaves the plane in the direction of the normal, so the north pole is included. [The sign of* $t_\perp$ *is the sign of* $\cos\theta -$

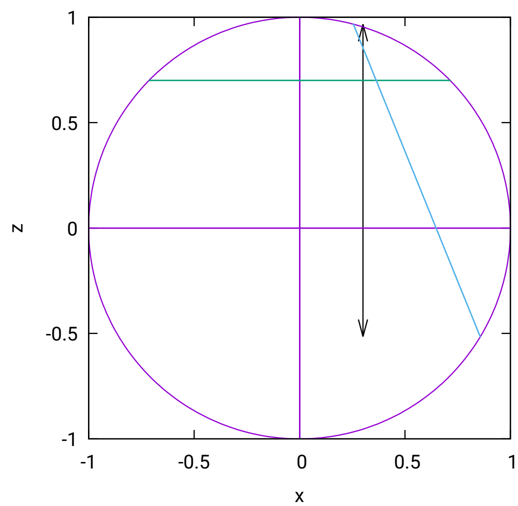

FIGURE 5. The intersection of the base circles, a projection of Fig. 2 on the $x-z$-plane, the base of cap 1 in green and the base of cap 2 in blue. Here cap 2 does not contain the North Pole. The arrows mark the z-range of $z_- \le z \le z_+$ covered by cap 2.

$\bar{h}_2/R = \cos\theta - \cos\alpha_2$, *and opposite to the sign of* $\theta - \alpha_2$ *because (in the main branch) the cosine is a decreasing function.]*

2.4.1. *Without North Pole.* One case of geometry is where the sphere cap 2 does not include the north pole (because either θ is too large or because h_2 is too small) like in Fig. 5. Where the z-coordinate of the base of cap 2 is maximum (where the upper section of the blue line intersects the sphere), $x = R\sin(\theta-\alpha_2)$, $z = R\cos(\theta-\alpha_2)$. The volume of the intersection of the caps is above the green base and right from the blue line. The linear relation between x' and z' along the blue line is (15).

$$V_\cap = \int_{R-h_1}^{R\cos(\theta-\alpha_2)} dz \int_{(R-h_2-z\cos\theta)/\sin\theta}^{\sqrt{R^2-z^2}} dx \int_{-\sqrt{R^2-x^2-z^2}}^{\sqrt{R^2-x^2-z^2}} dy. \tag{30}$$

The technical details of reducing this to square roots and inverse sines are written down in App. A. The difference to the work by Strobl et al [7] is that the integral is not divided into two subvolumes of "regularized" spherical wedges.

2.4.2. *With North Pole.* Another case of geometry is where the sphere cap 2 includes the north pole, like in Fig. 3. Where the z-coordinate of the base of cap 2 is maximum (where the upper section of the blue line intersects the sphere), $x = R\sin(\theta-\alpha_2) < 0$, $z = R\cos(\theta-\alpha_2)$. If we integrate over z only up to $R\cos(\theta-\alpha_2)$, the volume of the spherical cap above that limit up to the North Pole would be missing, so

$$V_\cap = V_{R-R\cos(\theta-\alpha_2)} + \int_{R-h_1}^{R\cos(\theta-\alpha_2)} dz \int_{(R-h_2-z\cos\theta)/\sin\theta}^{\sqrt{R^2-z^2}} dx \int_{-\sqrt{R^2-x^2-z^2}}^{\sqrt{R^2-x^2-z^2}} dy, \tag{31}$$

where the triple integral is again I_0 of the Appendix.

3. Summary

The recipe to compute the volume of the intersection of two sphere caps is:

- Normalize the sphere caps heights by the sphere radius and figure out with (25) whether the bases of the sphere caps intersect inside the sphere. If they do not intersect, either one cap is entirely inside the other, or the wedge volume is zero.
- Calculate the intersections $[z_-, z_+]$ of sphere cap with the sphere via (36).
- Calculate I_0 between the limits set in (33), where the upper limit is $\cos(\theta - \alpha_2) = z_+$.
- Define the angle of view α_2 as in Remark 5 and use the criterion in Remark 8 to check whether the North Pole is part of cap 2. If it is, add the residual sphere cap to the I_0 volume according to (31).

Appendix A. Core Triple Integral

The essential volume integral for the spherical wedge starts by performing the trivial integral over y in (30) and utilizing the integral (2) to integrate over x:

$$I_0 = 2\int_{R-h_1}^{R\cos(\theta-\alpha_2)} dz \int_{(R-h_2-z\cos\theta)/\sin\theta}^{\sqrt{R^2-z^2}} dx \sqrt{R^2-x^2-z^2} \tag{32}$$

$$= 2\int_{R-h_1}^{R\cos(\theta-\alpha_2)} dz \left[\frac{x}{2}\sqrt{R^2-x^2-z^2} + \frac{R^2-z^2}{2}\arcsin\frac{x}{\sqrt{R^2-z^2}}\right] |_{(R-h_2-z\cos\theta)/\sin\theta}^{\sqrt{R^2-z^2}}$$

$$= \int_{R-h_1}^{R\cos(\theta-\alpha_2)} dz \left[x\sqrt{R^2-x^2-z^2} + (R^2-z^2)\arcsin\frac{x}{\sqrt{R^2-z^2}}\right] |_{(R-h_2-z\cos\theta)/\sin\theta}^{\sqrt{R^2-z^2}}$$

Remark 9. *One might replace* arcsin *by* arctan *via* [1, 4.3.45]

$$\arcsin u = \arctan\frac{u}{\sqrt{1-u^2}}.$$

$$(33)\quad I_0 = \int_{\bar{h}_1}^{R\cos(\theta-\alpha_2)} dz\big[(\sqrt{R^2-z^2})\sqrt{R^2-(\sqrt{R^2-z^2})^2-z^2}$$
$$+ (R^2-z^2)\arcsin\frac{\sqrt{R^2-z^2}}{\sqrt{R^2-z^2}} - \frac{\bar{h}_2 - z\cos\theta}{\sin\theta}\sqrt{R^2-(\frac{\bar{h}_2-z\cos\theta}{\sin\theta})^2-z^2}$$
$$-(R^2-z^2)\arcsin\frac{(\bar{h}_2-z\cos\theta)/\sin\theta}{\sqrt{R^2-z^2}}\big]$$
$$= \int_{\bar{h}_1}^{R\cos(\theta-\alpha_2)} dz\big[-\frac{\bar{h}_2-z\cos\theta}{\sin\theta}\sqrt{R^2-(\frac{\bar{h}_2-z\cos\theta}{\sin\theta})^2-z^2}$$
$$-(R^2-z^2)\arcsin\frac{\bar{h}_2-z\cos\theta}{\sin\theta\sqrt{R^2-z^2}} + (R^2-z^2)\frac{\pi}{2}\big]$$
$$= R\int_{\tilde{h}_1}^{\cos(\theta-\alpha_2)} dz\big[-\frac{R\tilde{h}_2-Rz\cos\theta}{\sin\theta}R\sqrt{1-(\frac{\tilde{h}_2-z\cos\theta}{\sin\theta})^2-z^2}$$
$$-R^2(1-z^2)\arcsin\frac{R(\tilde{h}_2-z\cos\theta)}{\sin\theta R\sqrt{1-z^2}} + R^2(1-z^2)\frac{\pi}{2}\big]$$
$$\equiv R^3[-\frac{1}{\sin^2\theta}I_1(z) - I_2(z) + I_3(z)]\,\Big|_{z=\tilde{h}_1}^{\cos(\theta-\alpha_2)}\ .$$

A.1. I_2. The second integral is with partial integration and $(d/dx)\arcsin x = 1/\sqrt{1-x^2}$

$$(34)\quad I_2 = \int dz(1-z^2)\arcsin\frac{\tilde{h}_2-z\cos\theta}{\sin\theta\sqrt{1-z^2}}$$
$$= \frac{1}{3}z(3-z^2)\arcsin\frac{\tilde{h}_2-z\cos\theta}{\sin\theta\sqrt{1-z^2}}$$
$$-\frac{1}{3}\int dzz(3-z^2)\frac{1}{\sqrt{1-[\frac{\tilde{h}_2-z\cos\theta}{\sin\theta\sqrt{1-z^2}}]^2}}[-\frac{\cos\theta}{\sin\theta\sqrt{1-z^2}}+\frac{\tilde{h}_2-z\cos\theta}{\sin\theta}(-2z)\frac{-1}{2(1-z^2)^{3/2}}]$$
$$= \frac{1}{3}z(3-z^2)\arcsin\frac{\tilde{h}_2-z\cos\theta}{\sin\theta\sqrt{1-z^2}} - \frac{1}{3}I_4.$$

To condensate the notation we have defined I_4 as

(35)
$$I_4 = \int dz z(3-z^2)\frac{1}{\sqrt{1-[\frac{\tilde h_2 - z\cos\theta}{\sin\theta\sqrt{1-z^2}}]^2}}\left[-\frac{\cos\theta}{\sin\theta\sqrt{1-z^2}}+\frac{\tilde h_2 - z\cos\theta}{\sin\theta}\frac{z}{(1-z^2)^{3/2}}\right]$$
$$= \int dz z(3-z^2)\frac{1}{\sqrt{\sin^2\theta-[\frac{\tilde h_2 - z\cos\theta}{\sqrt{1-z^2}}]^2}}\left[-\frac{\cos\theta}{\sqrt{1-z^2}}+(\tilde h_2 - z\cos\theta)\frac{z}{(1-z^2)^{3/2}}\right]$$
$$= \int dz z(3-z^2)\frac{1}{\sqrt{\sin^2\theta(1-z^2)-(\tilde h_2 - z\cos\theta)^2}}\left[-\cos\theta+(\tilde h_2 - z\cos\theta)\frac{z}{1-z^2}\right]$$
$$= \int dz z(3-z^2)\frac{1}{\sqrt{-\left[z^2-2z\cos\theta\tilde h_2+\tilde h_2^2-\sin^2\theta\right]}}\left[-\cos\theta+(\tilde h_2 - z\cos\theta)\frac{z}{1-z^2}\right].$$

The two roots $z_\pm$ of the quadratic denominator polynomial $z^2-2z\cos\theta\tilde h_2 - \sin^2\theta+\tilde h_2^2 = (z-z_+)(z-z_-)$ are real-valued:

$$z_\pm \equiv \cos\theta\tilde h_2 \pm \sin\theta\sqrt{1-\tilde h_2^2} = \tilde h_2\cos\theta \pm \sin\theta\sqrt{1-\tilde h_2^2}$$
$$= \cos\alpha_2\cos\theta\pm\sin\theta\sin\alpha_2 = \cos(\theta\mp\alpha_2); \tag{36}$$

$$z_+ + z_- = 2\cos\theta\tilde h_2; \tag{37}$$

$$z_+ - z_- = 2\sin\theta\sqrt{1-\tilde h_2^2}. \tag{38}$$

If the cuts are orthogonal:

$$\theta = 90^\circ;\ \cos\theta = z_+ + z_- = 0;\ z_+ - z_- = 2z_+;\ 1-z_+^2 = \tilde h_2^2.$$

$$I_4 = \int dz z(3-z^2)\frac{1}{\sqrt{-(z-z_+)(z-z_-)}}\left[-\cos\theta+(\tilde h_2 - z\cos\theta)\frac{z}{1-z^2}\right]$$
$$= \int dz z(3-z^2)\frac{1}{\sqrt{-(z-z_+)(z-z_-)}}\frac{z\tilde h_2-\cos\theta}{(1+z)(1-z)}. \tag{39}$$

The partial fractions that disperse the rational factors in the integral kernel are

$$\frac{z(3-z^2)}{(1+z)(1-z)} = z - \frac{1}{1+z}+\frac{1}{1-z}. \tag{40}$$

$$\frac{z^2(3-z^2)}{(1+z)(1-z)} = z^2 - 2 + \frac{1}{1+z}+\frac{1}{1-z}. \tag{41}$$

$$I_4 = -\cos\theta\int dz\left[z-\frac{1}{1+z}+\frac{1}{1-z}\right]\frac{1}{\sqrt{-(z-z_+)(z-z_-)}}$$
$$+\tilde h_2\int dz\left[z^2-2+\frac{1}{1+z}+\frac{1}{1-z}\right]\frac{1}{\sqrt{-(z-z_+)(z-z_-)}}$$
$$= -\cos\theta\left[I_5^{(1)}-I_5^{+}+I_5^{(-)}\right]+\tilde h_2\left[I_5^{(2)}-2I_5^{(0)}+I_5^{+}+I_5^{(-)}\right]. \tag{42}$$

This defines a family of 5 integrals $I_5^{(0,1,2,+,-)}$: [3, 2.261][2, 21.7.2.8.241][1, 3.3.27]

$$I_5^{(0)} \equiv \int dz \frac{1}{\sqrt{-(z-z_+)(z-z_-)}} = \arcsin \frac{z-(z_+ + z_-)/2}{(z_+ - z_-)/2}. \tag{43}$$

Remark 10. *For orthogonal cuts* $I_5^{(0)} \overset{\cos\theta\to 0}{\longrightarrow} \arcsin(z/z_+)$.

Via [3, 2.264.2][2, 21.7.2.8.249]

$$I_5^{(1)} \equiv \int dz \frac{z}{\sqrt{-(z-z_+)(z-z_-)}} = -\sqrt{-(z-z_+)(z-z_-)} + \frac{z_+ + z_-}{2} I_5^{(0)}. \tag{44}$$

Remark 11. *For orthogonal cuts* $I_5^{(1)}$ *is irrelevant because the factor* $\cos\theta$ *in* (42) *is zero.*

Via [3, 2.264.3][2, 21.7.2.8.252]

$$\begin{aligned} I_5^{(2)} &\equiv \int dz \frac{z^2}{\sqrt{-(z-z_+)(z-z_-)}} \\ &= -\frac{1}{2}\left(z + 3\frac{z_+ + z_-}{2}\right)\sqrt{-(z-z_+)(z-z_-)} + \frac{1}{8}\left(3z_+^2 + 3z_-^2 + 2z_+z_-\right) I_5^{(0)}. \end{aligned} \tag{45}$$

Remark 12. *For orthogonal cuts*

$$I_5^{(2)} \overset{\cos\theta\to 0}{\longrightarrow} -\frac{1}{2} z\sqrt{z_+^2 - z^2} + \frac{1}{2} z_+^2 \arcsin(z/z_+).$$

.

For $I_5^{(\pm)}$ substitute $u \equiv 1 \pm z$, $z = \pm(u-1)$

$$I_5^{(\pm)} = \int dz \frac{1}{(1\pm z)\sqrt{-(z-z_+)(z-z_-)}} = \pm \int du \frac{1}{u\sqrt{a+bu+cu^2}} \tag{46}$$

such that the coefficients of the u-polynomial are
$a = -(z_+ \pm 1)(z_- \pm 1)$, $b = 2 \pm (z_+ + z_-)$, $c = -1$. Then [3, 2.266][2, 21.7.2.8.258]

$$\begin{aligned} I_5^{(\pm)} &= \pm \frac{1}{\sqrt{(z_+ \pm 1)(z_- \pm 1)}} \arcsin \frac{-2(z_+ \pm 1)(z_- \pm 1) + [2 \pm (z_+ + z_-)]u}{u(z_+ - z_-)} \\ &= \pm \frac{1}{\sqrt{(z_+ \pm 1)(z_- \pm 1)}} \arcsin \frac{-2(z_+ \pm 1)(z_- \pm 1) + [2 \pm (z_+ + z_-)](1\pm z)}{(1\pm z)(z_+ - z_-)}. \end{aligned} \tag{47}$$

Remark 13. *For orthogonal cuts the first term in* (42) *vanishes due to* $\cos\theta = 0$, *and the second term needs only the sum* [3, 2.284][2, 21.7.2.8.267]

(48)

$$I_5^{(+)} + I_5^{(-)} = 2\int dz \frac{1}{(1-z^2)\sqrt{-(z^2 - z_+^2)}} = \frac{2}{\sqrt{1-z_+^2}} \arctan \frac{z\sqrt{1-z_+^2}}{\sqrt{-(z^2-z_+^2)}}.$$

So we have closed expressions for all $I_5^{0,1,2,\pm}$ which plugged into (42) yield I_4 and eventually I_2 via (34).

A.2. I_1.

$$I_1 = \int dz(\tilde h_2 - z\cos\theta)\sqrt{\sin^2\theta - (\tilde h_2 - z\cos\theta)^2 - z^2\sin^2\theta} \tag{49}$$
$$= \int dz(\tilde h_2 - z\cos\theta)\sqrt{-[z^2 - 2z\cos\theta\tilde h_2 + \tilde h_2^2 - \sin^2\theta]}$$
$$= \int dz(\tilde h_2 - z\cos\theta)\sqrt{-(z-z_+)(z-z_-)}$$
$$= \tilde h_2 \int dz\sqrt{-(z-z_+)(z-z_-)} - \cos\theta\int dzz\sqrt{-(z-z_+)(z-z_-)}.$$

These are two tabulated integrals: [3, 2.262.1]

$$\int dz\sqrt{-(z-z_+)(z-z_-)} = \frac{2z-(z_+ + z_-)}{4}\sqrt{-(z-z_+)(z-z_-)} + \frac{1}{8}(z_+ - z_-)^2 I_5^{(0)} \tag{50}$$

and [3, 2.262.2]

$$\int dzz\sqrt{-(z-z_+)(z-z_-)} = -\frac{1}{3}[-(z-z_+)(z-z_-)]^{3/2} + \frac{(2z-(z_+ + z_-))(z_+ + z_-)}{8}\sqrt{-(z-z_+)(z-z_-)} + \frac{(z_+ + z_-)(z_+ - z_-)^2}{16} I_5^{(0)}. \tag{51}$$

$$I_1 = \frac{1}{3}\cos\theta[-(z-z_+)(z-z_-)]^{3/2} + (2\tilde h_2 - \cos\theta(z_+ + z_-))\frac{2z-(z_+ + z_-)}{8}\sqrt{-(z-z_+)(z-z_-)} + [2\tilde h_2 - \cos\theta(z_+ + z_-)]\frac{(z_+ - z_-)^2}{16} I_5^{(0)}. \tag{52}$$

A.3. I_3. The last integral in (33) is

$$I_3(z) = \int dz(1-z^2)\frac{\pi}{2} = \frac{\pi}{6}z(3-z^2). \tag{53}$$

Appendix B. Program Listing

The C++ program `spherewedge` in this Appendix computes the volume of two intersecting sphere caps. It is compiled either with

```
g++ -o spherewedge -O2 *.cxx
```

or if the autotool packages are available with

```
autoreconf -i -f -s
configure
make
```

B.1. Makefile.am.

```
AUTOMAKE_OTPIONS =
ACLOCAL_AMFLAGS = ${ACLOCAL_FLAGS} -I m4

lib_LTLIBRARIES =

AM_LIBTOOLFLAGS=--silent

bin_PROGRAMS = spherewedge

dist_man_MANS = spherewedge.1

info_TEXINFOS = spherewedge.texi

spherewedge_SOURCES = spherewedge.cxx SphereWedge.h SphereWedge.cxx

spherewedge_CXXFLAGS = $(CXXFLAGS) $(OPENMP_CXXFLAGS)
spherewedge_LDADD =
spherewedge_LDFLAGS = $(OPENMP_CFLAGS)
spherewedge_DEPENDENCIES =
```

B.2. configure.ac.

```
#                                               -*- Autoconf -*-
# Process this file with autoconf to produce a configure script.

AC_PREREQ([2.68])
AC_INIT([spherewedge], [0.125], [mathar@mpia.de])
AC_CONFIG_MACRO_DIR([m4])
AM_INIT_AUTOMAKE([no-define foreign])

AC_CONFIG_SRCDIR([spherewedge.cxx])
AC_CONFIG_HEADERS([config.h])

LT_INIT
AC_PROG_INSTALL

AC_LANG(C++)
AC_PROG_CXX

AC_CHECK_HEADERS([unistd.h])

# AC_CHECK_FUNCS([sincos])

LT_INIT

AC_PROG_MAKE_SET

AC_CONFIG_FILES([Makefile])
AC_OUTPUT
```

B.3. Main program spherewedge.cxx.

```
#include <unistd.h>
#include <iostream>
```

```
#include <sstream>
#include <cmath>

#include "SphereWedge.h"

/** Print a usage syntax message detailing the command line
* @param[in] argv0 The name of the main program.
* @since 2026-05-05
*/
void usage(char *argv0)
{
    cout << "Usage:" << endl ;
    cout << argv0 << " [-R Radius] -t thetadegrees -1 capheight1 -2 capheight2 [-N NRiemann] " << endl
} /* usage */

/** Main executable spherewedge
* Usage: spherewedge
*/
int main(int argc, char *argv[])
{
    /* option character
    */
    char oc ;

    /* the number of points of a Rieman approximation of the volume integral
    */
    int N(0) ;

    /* angle between the two cap axis directions (radians)
    * default 90 degrees = pi/2
    */
    double theta(M_PI_2) ;

    /* cap heights 1 and 2
    */
    double heights[2] = {0.5, 0.5};

    /* sphere radius
    */
    double R(1.) ;

    while (  (oc=getopt(argc,argv,"R:t:1:2:N:h")) != -1 )
    {
        switch(oc)
        {
        case 'R' :
            R = atof(optarg) ;
            break ;
        case 't' :
            /* argument degrees turned into radians
            */
            theta = atof(optarg)*M_PI/180.0 ;
            break ;
```

```
        case '1' :
            heights[0] = atof(optarg) ;
            break ;
        case '2' :
            heights[1] = atof(optarg) ;
            break ;
        case 'N' :
            N = atoi(optarg) ;
            break ;
        case 'h' :
            usage(argv[0]) ;
            break ;
        case '?' :
            cerr << "Invalid command line option " << optopt << endl ;
            usage(argv[0]) ;
            break ;
        }
    }

    /* reasonable parameters ?
    */
    if ( heights[0] <=0 || heights[1] <=0)
    {
        cerr << "height " << heights[0] << " or " << heights[1] << " negative" << endl ;
        return 1;
    }
    if ( heights[0] >= 2.*R || heights[1] >=2.*R)
    {
        cerr << "height " << heights[0] << " or " << heights[1] << " larger than diameter" << endl ;
        return 1;
    }
    if ( theta <0|| theta > M_PI)
    {
        cerr << "angle " << theta*180./M_PI << " not in range [0..180]" << endl ;
        return 1;
    }

    mpia::rjm::SphereWedge w(R,heights[0],heights[1],theta) ;
    double V = w.vol(N) ;
    cout << " volume " << V << endl ;

    return 0 ;
} /* main */
```

B.4. **Class header SphereWedge.h.**

```
#pragma once

#include <string>
#include <vector>
#include <iostream>
#include <ostream>

using namespace std ;
```

```
namespace mpia
{
    namespace rjm
    {
        /** A sphere wedge defined as the intersection of 2 sphere caps
        * (or a sphere cap cut by another plane)
        * @author Richard J. Mathar
        */
        class SphereWedge
        {
        public:
            /* the 2 caps heights
            */
            double h[2] ;

            /* the sphere radius
            */
            double R ;

            /* the angle (dihedral) between the two caps axis directions (radians)
            */
            double theta ;

                SphereWedge(double R, double h1, double h2, double theta) ;

            double vol(int N) const ;
            double volN(int N) const ;

            double volCap(double h) const ;
            bool doInters() const ;
            bool cap2wNP() const ;
            void zrange(double zpm[2]) const ;
            void ctrange(double ctlim[2]) const ;

        protected:
            double I0(int N) const ;
            double I1(double z, const double zpm[2]) const ;
            double I2(double z, const double zpm[2]) const ;
            double I3(double z) const ;
            double I4(double z, const double zpm[2]) const ;
            double I50(double z, const double zpm[2]) const ;
            double I51(double z, const double zpm[2]) const ;
            double I52(double z, const double zpm[2]) const ;
            double I5pm(double z, const double zpm[2], const int sig) const ;


        private:
            /* the cosine and sine of theta
            */
            double cs[2] ;

            /* the 2 cap base distances from sphere origin divided/scaled by radius
```

```
            */
            double htilde[2] ;
        } ; /* SphereWedge */

    }
}
```

B.5. **Class body SphereWedge.cxx.**

```
#include <string>
#include <cstring>
#include <cmath>
#include <cstdlib>

#include "SphereWedge.h"


using namespace std ;

/** Constructor for the main problem
* @param Radi sphere radius
* @param h1 height cap 1
* @param h2 height cap 2
* @param polan polar angle, the difference in the cap apex directions
* @since 2026-05-05
* @author Richard J. Mathar
*/
mpia::rjm::SphereWedge::SphereWedge(double Radi, double h1, double h2, double polang)
    : R(Radi), theta(polang)
{
    h[0] = h1 ;
    h[1] = h2 ;
    cs[0] = cos(theta) ;
    cs[1] = sin(theta) ;
    for(int i=0 ; i < 2 ; i++)
        htilde[i] = 1.-h[i]/R ;
} /* SphereWedge::SphereWedge */

/** Numerical integration of the volume with N Riemann abscissae
* @param N the number of support points along the z axis.
*   If 0 use the analytical formulas.
* @return the volume (including the R^3 factor, including north cap region if applicable)
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::volN(int N) const
{
    double v= pow(R,3.)*I0(N) ;

    if ( cap2wNP() )
    {
        /* calculate z+, z-, the R-scaled z-values where base of cap 2 intersects the sphere
        */
        double zpm[2] ;
```

```
        zrange(zpm) ;
        /* sphere cap 2 includes the NP so we need to
        * add the volume of the sphere cap above zpm[1].
        * In natural units this is R*zpm[1] equivalent to cap height R(1-zpm[1])
        */
        v += volCap(R*(1.-zpm[1])) ;
    }
    return v ;
} /* SphereWedge::volN */

/** Numerical integration of the volume with N Riemann abscissae for I_0.
* @param N the number of support points along the z axis projection of cap 1
* @return The value of the z-integral I0 (without the R^3 factor, excluding the NP region)
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I0(int N) const
{
    double vol(0.) ;
    /* calculate z+, z-, the R-scaled z-limit-values where base of cap 2 intersects the sphere
    */
    double zpm[2] ;
    zrange(zpm) ;

    /* lower and upper limit of z-integral. Lower limit set by sphere cap 1 base;
    * upper limit by the maximum value of sphere cap 2 base inside the sphere.
    */
    double zlim[2] ;
    zlim[0] = htilde[0] ; /* htilde 1 in manuscript */
    zlim[1] = zpm[1] ; /* z+ in manuscript */

    if ( N > 0)
    {
        /* step width */
        const double dz = (zlim[1]-zlim[0])/N ;
        /* loop over z abscissae */
        for (int i =0 ; i < N ; i++)
        {
            const double z = zlim[0]+(i+0.5)*dz ;
            /* h2 tilde minus z *cos theta
            */
            const double hz = htilde[1] - z*cs[0] ;

            /* contribution to I1
            */
            vol -= hz*sqrt(-(z-zpm[0])*(z-zpm[1]))/pow(cs[1],2) ;

            /* contribution to I2
            */
            vol -= (1-z*z)*asin(hz/(cs[1]*sqrt(1-z*z))) ;

            /* contribution to I3 (1-z^2)*pi/2
            */
```

```
            vol += (1-z*z)*M_PI_2 ;
        }

        /* scale with binning width of the Riemann sum
        */
        vol *= dz ;
    }
    else
    {
        vol = -I1(zlim[1],zpm)/(cs[1]*cs[1]) +I1(zlim[0],zpm)/(cs[1]*cs[1])  ;

        vol += -I2(zlim[1],zpm) +I2(zlim[0],zpm) ;

        vol += I3(zlim[1]) -I3(zlim[0]) ;
    }

    return vol ;
} /* SphereWedge::volN */

/** contribution of analytical value of I_1
* @param z R-scaled z-value
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I1(double z, const double zpm[2]) const
{
    const double zmed = (zpm[0]+zpm[1])/2 ;
    const double zspan = (zpm[1]-zpm[0])/2 ;
    double D = -(z-zpm[0])*(z-zpm[1]);
    if ( D <=0.)
        /* marginal inaccuracies may lead to negative values */
        D=0. ;
    const double hct  = htilde[1]-cs[0]*zmed ;
    if ( D > 0.)
        return cs[0]*pow(D,1.5)/3. +hct*(z-zmed)/2. * sqrt(D) + hct*zspan*zspan/2.* I50(z,zpm) ;
    else
        return hct*zspan*zspan/2.* I50(z,zpm) ;
}

/** contribution of analytical value of I_2
* @param z R-scaled z-value
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I2(double z, const double zpm[2]) const
{
    double D = (htilde[1]-z*cs[0])/(cs[1]*sqrt(1-z*z)) ;
    double asinD ;
    if ( D >= 1.)
        asinD = M_PI_2 ;
    else if ( D <= -1.)
        asinD = -M_PI_2 ;
    else
```

```
        asinD = asin(D) ;
    return z*(1-z*z/3.) *asinD -I4(z,zpm)/3. ;
}

/** contribution of analytical value of I3.
* @param z R-scaled z-value
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I3(double z) const
{
    return z*(1.-z*z/3.)*M_PI_2 ;
}

/** contribution of analytical value of I_4
* @param z R-scaled z-value
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I4(double z, const double zpm[2]) const
{
    /* if costheta=0, avoid numerical jitter....
    */
    double ret(0.) ;
    if ( fabs(cs[0]) > 1.e-6)
    {
        ret = -cs[0]*( I51(z,zpm) -I5pm(z,zpm,1) + I5pm(z,zpm,-1)) ;
    }
    ret += htilde[1]*( I52(z,zpm) -2*I50(z,zpm) + I5pm(z,zpm,1) + I5pm(z,zpm,-1)) ;
    return ret ;
}

/** contribution of analytical value of I_5^0.
* @param z R-scaled z-value
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I50(double z, const double zpm[2]) const
{
    return asin( (2*z-(zpm[0]+zpm[1]))/(zpm[1]-zpm[0]) ) ;
}

/** contribution of analytical value of I_5^1.
* @param z R-scaled z-value
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I51(double z, const double zpm[2]) const
{
    return -sqrt(-(z-zpm[0])*(z-zpm[1]))+(zpm[0]+zpm[1])/2.*I50(z,zpm) ;
}

/** contribution of analytical value of I_5^2.
```

```
* @param z R-scaled z-value
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I52(double z, const double zpm[2]) const
{
    double D = -(z-zpm[0])*(z-zpm[1]) ;
    if ( D <=0.)
        /* marginal inaccuracies may lead to negative values */
        D=0. ;
    return -(z+1.5*(zpm[0]+zpm[1]))/2.*sqrt(D)
        +(3*zpm[0]*zpm[0]+3*zpm[1]*zpm[1]+2*zpm[0]*zpm[1])*I50(z,zpm)/8. ;
}

/** contribution of analytical value of I_5^{+-}.
* @param z R-scaled z-value
* @param sig +1 or -1 depending on the upper sign in the name
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::I5pm(double z, const double zpm[2], const int sig) const
{
    double D = (zpm[0]+sig)*(zpm[1]+sig) ;
    if ( D <=0.)
        /* marginal inaccuracies may lead to negative values */
        D=0. ;
    const double zmed = (zpm[0]+zpm[1])/2 ;
    const double zspan = (zpm[1]-zpm[0])/2 ;
    const double onez = 1+sig*z ;
    /* for theta =90 deg, onez may be zero if z=1 or z=-1
    */
    double D2 = (-D+(1+sig*zmed)*onez) /(onez*zspan);
    double asinD ;
    if ( D2 >= 1.)
        asinD = M_PI_2 ;
    else if ( D2 <= -1.)
        asinD = -M_PI_2 ;
    else
        asinD = asin(D2) ;
    return sig/sqrt(D)*asinD ;
}


/** Numerical or analytical integration of the volume.
* @param N the number of support points along the z axis
*  If zero, analytical formulas, else Riemann sum.
* @since 2026-05-05
* @author Richard J. Mathar
*/
double  mpia::rjm::SphereWedge::vol(int N) const
{
    double v=0. ;
    /* handle degenerate theta=0 or Pi here... wedge is trivial difference ofcaps
```

```
*/
if ( theta <= 0.)
{
    /* parallel bases and same apex at north pole: volume of the smaller cap
    */
    double hmin = std::min(h[0],h[1]) ;
    return volCap(hmin) ;
}
else if ( theta >= M_PI)
{
    /* parallel bases and apex cap 2 at south pole.
    */
    if ( h[0]+h[1] <= 2.*R)
        /* no overlap: bases too far apart
        */
        return 0 ;
    else
    {
        /* zrange with overlap: slice R-h[1] <=z <= -R+h[0], implies h[1]+h[0] >= 2R
        * subtract volume of cap starting at R-h[1] and volume startign at -R+h[0]
        * (the latter meaning height 2*R-h[0])
        */
        return volCap(h[1])-volCap(2*R-h[0]) ;
    }
}

if ( doInters() )
{
    /* standard problem: some of the plane of cap 2 base intersects cap 1 base
    */
    return volN(N) ;
}
else
{
    /* caps 1 and 2 intersect the sphere but not each other */
    double ctlim[2] ;
    ctrange(ctlim) ;
    if (cs[0] < ctlim[0])
    {
        /* cosine theta is smaller than the limit of intersections,
        * which means theta is larger than the limit o fintersecions
        */
        cerr << "non-intersecting caps" << endl ;
        return 0. ;
    }
    else if (cs[0] > ctlim[0])
    {
        /* cosine theta is larger than the limit of intersections,
        * which means theta is smaller than the limit o fintersecions.
        * cap 1 entirely within cap 2
        */
        cerr << "one cap inside the other" << endl ;
        return volCap(h[0]) ;
```

```
        }
    }
    return v;
} /* SphereWedge::volN */

/** volume of a cap of height h
* @param h height of the spherical cap
* @since 2026-05-05
* @author Richard J. Mathar
*/
double mpia::rjm::SphereWedge::volCap(double h) const
{
    return M_PI*h*h*(R-h/3.) ;
} /* SphereWedge::volCap */

/** check whether sphere cap 2 base intersects sphere cap 1
* @return true if an intersection of the two circles of the sphere caps exists on the sphere surface.
*   False if the line of intersection fo the two planes of the caps does not run through the sphere.
* @since 2026-05-05
* @author Richard J. Mathar
*/
bool  mpia::rjm::SphereWedge::doInters() const
{
    double hscal[2] ;
    for (int i=0 ; i < 2 ; i++)
        hscal[i] = h[i]/R ;
    const double D = cs[0]*cs[0]
        - 2*htilde[0]*htilde[1]*cs[0]
        +1 + hscal[0]*hscal[0] + hscal[1]*hscal[1] -2*(hscal[0]+hscal[1])  ;
    return (D<0)? true : false ;
} /* SphereWedge::doInters */

/** check whether sphere cap 2 includes the north pole.
* @return true if the north pole at z=R is at the surface of cap 2.
* @since 2026-05-05
* @author Richard J. Mathar
*/
bool  mpia::rjm::SphereWedge::cap2wNP() const
{
    /* check sign of cos(theta)-htilde2. if >0, the NP is in the cap
    */
    return (cs[0] -htilde[1] >0) ? true : false ;
} /* SphereWedge::cap2wNP */

/** Define scaled values of z where cap 2 intersects sphere
* @param[out] zpm The lower and upper z+, z-
* @since 2026-05-05
* @author Richard J. Mathar
*/
void  mpia::rjm::SphereWedge::zrange(double zpm[2]) const
{
    const double sinalpha2 = sqrt(1-htilde[1]*htilde[1]) ;
    zpm[0] = cs[0]*htilde[1] - cs[1]*sinalpha2 ;
```

```
    zpm[1] = cs[0]*htilde[1] + cs[1]*sinalpha2 ;
} /* SphereWedge::zrange */

/** Compute the range of cos(theta) for given cap heighs which allows base intersections
* @param[out] ctlim The lower and upper value for cos(theta)
* @since 2026-05-05
* @author Richard J. Mathar
*/
void  mpia::rjm::SphereWedge::ctrange(double ctlim[2]) const
{
    const double D = (1-htilde[0]*htilde[0])*(1-htilde[1]*htilde[1]) ;
    ctlim[0] = htilde[0]*htilde[1]-sqrt(D) ;
    ctlim[1] = htilde[0]*htilde[1]+sqrt(D) ;
} /* SphereWedge::ctrange */
```

B.6. **man page.**

NAME

spherewedge − volume of intersecting sphere caps

SYNOPSIS

spherewedge [**−R** *radius*] [**−1** *heigh1*] [**−2** *height2*] [**−t** *dihedralDegrees*] [**−N** *NRieman*]

DESCRIPTION

spherewedge computes the volume of the intersection of sphere caps of a common sphere.

OPTIONS

−R *radius* defines the radius of the sphere. If the option is not used the radius is assumed to be 1.

−t *dihedralDegrees* defines the angle (in degrees) between the directions from the sphere center to the apexes of the two sphere caps. The value must be in the range 0 to 180. If not used, 90 degrees are used.

−1 *height1* defines the height of the first sphere cap. This should be a positive number between 0 and twice the radius. If not used, the value of 0.5 is assumed.

−2 *height2* defines the height of the second sphere cap. This should be a positive number between 0 and twice the radius. If not used, the value of 0.5 is assumed.

−N *NRiemann* defines a positive number of abscissa points of the integration of the areal slices with a Riemann sum approximation. If the option is not used, no such approximation is used and the integrals are computed in analytic form. With arguments of the order of a few hundred this serves as a test of the validity of the analytical formulas.

EXAMPLES

```
spherewedge −1 0.3 −2 0.4 −t 45 −N 100
spherewedge −1 0.3 −2 0.4 −t 45
# quarter sphere volume Pi/3, avoiding NaN in formulas:
spherewedge −1 1. −2 1. −t 89.99
# octal sphere volume Pi/6
spherewedge −1 1. −2 1. −t 135
```

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