From Rényi Entropy to the Feynman Path Integral

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Abstract

In this sequel to [17 - 18], we argue that Rényi Entropy is organically tied to the Feynman Path Integral formalism of Quantum Field Theory (QFT).

Key words: Rényi Entropy, Feynman Path Integral, complex dynamics, non-equilibrium statistical physics.

Introduction

Continuing the exploration developed in [17 - 18], the goal of this brief study is to outline a "first principles" derivation of the Feynman Path-Integral as constrained maximization of Rényi Entropy. The basis of our work is that, unlike Shannon and Boltzmann-Gibbs (BG) Entropies, Rényi Entropy describes non-equilibrium evolution of complex systems, as it naturally encodes *scale-dependent measures* and *anomalous dimensions* [3 - 4].

We focus here on the collective dynamics of *large ensembles of classical oscillators* evolving in near-equilibrium regimes, where macroscopic observables emerge from a vast space of microscopic configurations. The systems of interest are not assumed to be exactly at equilibrium; rather, they operate in conditions where deviations are small enough for equilibrium behavior to remain meaningful, yet close enough to *non-equilibrium conditions*. These conditions are natural for open, complex many-body systems with long relaxation times, weak nonlinearity, and hierarchical organization of dynamic modes [15].

At the microscopic level, the statistical description of equilibrium is governed by the BG distribution, corresponding to the Rényi Entropy index $q=1$. In this limit, entropy is additive, probability weights are exponential functions, and microstates are sampled according to

$$p_i = \frac{e^{-\beta E_i}}{Z}, \qquad Z = \sum_i e^{-\beta E_i}.$$

As is known, *transitions between microstates* in statistical physics are regarded in field theory as *weighted histories in phase space*, with the partition function Z acting as a normalization over all admissible configurations. As a result, the statistical structure of BG formalism is on par with the sum over histories in QFT. In Feynman's Path-Integral formulation, the vacuum-to-vacuum transition amplitude [5, 13],

$$\langle 0_{\text{out}} \mid 0_{\text{in}} \rangle = \int \mathcal{D}\phi \; e^{\frac{i}{\hbar} S[\phi]},$$

is a weighted sum over all field configurations (histories) compatible with the boundary conditions. The exponential weight plays a role analogous to the Boltzmann factor: it assigns relative statistical importance to different microscopic realizations of the system's dynamics. In this correspondence, the partition function Z maps to the generating functional of connected vacuum diagrams. This shared statistical formalism justifies treating Path

Integral as an entropy-weighted sum over configurations and motivates further analysis of *entropy* and *dimensional flows* as follow-up extensions of this work. Our main finding here is that Feynman's formulation of QFT is the analytic continuation of a *generalized exponential distribution* emerging from maximization of Rényi Entropy.

To make the presentation transparent to a large audience, heavy technical jargon is generally avoided, and – whenever possible - the formal content is kept at the minimum necessary. As such, this introductory study is neither fully rigorous nor complete.

1. Entropy as Primitive Object

The Maximum Entropy Principle (MEP), states that the least biased probability distribution $P(x)$ consistent with known constraints maximizes an entropy functional $S[P]$[1 - 2]. Under suitable normalization and energy constraints, MEP yields both the BG and Shannon entropies written as,

$$S_{\mathrm{Sh}}[P] = -\int dx\, P(x) \ln\, P(x), \qquad (1.1)$$

Shannon Entropy implicitly assumes additivity, absolute continuity of the measure and single-scale statistics. However, the majority of systems operating outside equilibrium typically fail to comply with these assumptions, on account of field configurations forming infinite-dimensional spaces, the emergence of singular measures and scale-dependent fluctuations. These deficiencies motivate the introduction of so-called *generalized entropies*, in particular, the Rényi and Tsallis Entropies [3, 8, 18].

2. Rényi Entropy and Multifractal Measures

Rényi Entropy of order q is defined as [3]:

$$S_q[P] = \frac{1}{1-q} \ln \int dx\, P(x)^q, \qquad q \in \mathbb{R}. \qquad (2.1)$$

where Shannon Entropy is recovered in the limit $q \to 1$, that is,

$$\lim_{q \to 1} S_q = S_{\mathrm{Sh}}. \qquad (2.2)$$

Consider a configuration space Ω endowed with measure μ. For a *multifractal set*, the measure scales as [6 – 7, 14, 17]

$$\mu(B_\ell(x)) \sim \ell^{\,\alpha(x)}, \qquad (2.3)$$

where l is the linear resolution, B_ℓ is a ball of size l and $\alpha(x)$ is the local Hölder exponent. By analogy with conventional Statistical Physics, the partition function of the multifractal set can be shown to satisfy

$$\sum_i \mu_i^q \sim \ell^{(q-1)D_q}, \qquad (2.4)$$

where D_q is the *generalized (Rényi) dimension* of the continuous Rényi order $q = (-\infty, +\infty)$ [6, 17].

3. Maximum Rényi Entropy: Variational Derivation

Before proceeding, a word on notation: the following equations (3.1) – (3.3) are written at the level of entropy measures and their transformations. Differential elements are therefore *explicitly omitted*, as the fundamental objects are the measures themselves rather than their coordinate representations. When integrals are evaluated explicitly, the appropriate differential element is always the entropy-weighted measure having the form $\mu(d\phi)$ or $\mu_{\text{phys}}(d[\phi])$.

3.1 Functional Variation of Rényi Entropy

Rényi Entropy is maximized not directly over P, but over the so-called *escort distributions* [4]:

$$p_q(x) = \frac{P(x)^q}{\int P(x)^q}. \qquad (3.1)$$

We next require *constrained maximization* of the following functional

$$\mathcal{L} = S_q[p] - \lambda_0 \left(\int p \ - 1\right) - \lambda \left(\int p\, \mathcal{A}(x) \ - < \mathcal{A} >\right), \qquad (3.2)$$

where $\mathcal{A}(x)$ is a constrained observable with average $< \mathcal{A} >$, and λ_0, λ two scalar multipliers.

3.2 Derivation of the q-Exponential Distribution

Computing the functional derivative of Rényi Entropy yields:

$$\delta S_q = \frac{q}{1-q}\frac{\int p^{q-1}\delta p}{\int p^q}. \qquad (3.3)$$

Setting $\delta\mathcal{L} = 0$ leads to

$$p(x)^{q-1} \propto 1 - (1-q)\,\lambda\,\mathcal{A}(x). \qquad (3.4)$$

Hence,

$$p(x) = \frac{1}{Z_q}[1 - (1-q)\,\lambda\,\mathcal{A}(x)]^{\frac{1}{1-q}}, \qquad (3.5)$$

which represents a q-exponential distribution.

4. Identification of the Action Functional

4.1 Configuration Space in QFT

As stated in the Introduction, microstates of statistical physics are field histories $\phi(x)$ in QFT, whose configuration space is infinite-dimensional:

$$x \;\to\; \phi(\cdot). \qquad (4.1)$$

We can now identify the constraint:

$$\mathcal{A}[\phi] \equiv S[\phi], \qquad (4.2)$$

where $S[\phi]$ is the action functional of field theory.

4.2 Probability Measure on Histories

By (3.5) to (4.2), the entropy-maximizing distribution becomes

$$\boxed{p_q[\phi] = \frac{1}{Z_q}[1 - (1 - q)\, \lambda\, S[\phi]\,]^{\frac{1}{1-q}}}. \qquad (4.3)$$

which can be thought of as a multifractal measure defined on the configuration space.

5. Emergence of the Feynman Weight

5.1 The Shannon Entropy Limit

Taking $q \to 1$ gives

$$\boxed{\lim_{q \to 1} [1 - (1 - q)\, \lambda\, S]^{\frac{1}{1-q}} = e^{-\lambda S}} \qquad (5.1)$$

which means that constrained maximization of Rényi Entropy yields an exponential weight. We next proceed to identify the equivalent of the scalar multiplier in the field theory framework.

5.2 Analytic Continuation to Unitary Dynamics

To obtain quantum amplitudes rather than probabilities, we analytically continue according to

$$\lambda \rightarrow \frac{i}{\hbar}. \qquad (5.2)$$

A legitimate question raised by this identification is: Why is Planck's constant part of (5.2)? To answer this question, we first recall that the focus of this work are large systems of oscillators in near equilibrium conditions; Secondly, referring to [9-10], the long-time evolution of these systems leads to action quantization as result of *Arnold diffusion*, a process describing the instability of nearly integrable Hamiltonian systems with more than two degrees of freedom [11 – 12].

It follows from these considerations that,

$$p[\phi] \;\rightarrow\; \mathcal{A}[\phi] = e^{\frac{i}{\hbar}S[\phi]}. \qquad (5.3)$$

In different words, *Feynman formulation of QFT arises from constrained maximization of Rényi Entropy, followed by analytic continuation.*

6. Path Integral as Entropy Functional

The partition functional becomes

$$Z = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}S[\phi]}, \qquad (6.1)$$

which recovers the Path Integral introduced in [5, 13]. In our context,

$$\mathcal{D}\phi \;<=>\; d\mu_q[\phi], \qquad (6.2)$$

which plays the role of a *multifractal measure* with scale-dependent weights.

7. Feynman Diagrams as Entropy Expansion

Expanding the action around a stationary configuration leads to

$$S[\phi] = S[\phi_{\text{cl}}] + \frac{1}{2}\delta\phi\,\hat{S}''\,\delta\phi + \cdots \qquad (7.1)$$

meaning that each loop correction corresponds to integration over entropy fluctuations. A Feynman diagram Γ generates a contribution

$$\mathcal{A}_\Gamma \sim e^{\frac{i}{\hbar}S_\Gamma}, \qquad (7.2)$$

which corresponds to a local extremum of Rényi Entropy, under certain topological constraints. In this interpretation, loop diagrams encode overcounting of multifractal microstates and breakdown of the mean-field entropy.

8. Classical Limit as Saddle Point of Rényi Entropy

The limit $\hbar \to 0$ recovers the classical regime,

$$p[\phi] \to \delta(\phi - \phi_{\text{cl}}), \qquad (8.1)$$

which corresponds to the collapse of multifractal spectrum ($\alpha(x) \to \alpha$) and dominance of a *single* Hölder exponent. In this interpretation,

$$\delta S[\phi] = 0 \qquad (8.2)$$

can be viewed as the entropy saddle point.

9. Dimensional Flow and Rényi Index

The effective Rényi dimension can be expressed in terms of multifractal measures as [4, 6, 17]

$$D_q = \frac{1}{q-1} \lim_{\ell \to 0} \frac{\ln \sum_i \mu_i^q}{\ln \ell}. \qquad (9.1)$$

In this context, quantum fluctuations correspond to $q \neq 1$; classical spacetime emerges in the Shannon Entropy limit $q=1$. By the same token, Renormalization Group flow reflects the running of the Rényi index q with the observation scale.

10. Conclusions

We have argued here that,

1. The Feynman Path Integral is the constrained maximum of Rényi Entropy,
2. The action functional results from a maximization constraint, and is not necessarily a primitive quantity,
3. Quantum amplitudes arise via analytic continuation of the maximized Rényi Entropy,
4. Renormalization Group flow represents the running of the Rényi index with the observation scale,
5. Classical physics corresponds to the collapse of the multifractal spectrum to a single Hölder exponent.

Follow up results of this paper are currently in progress [16].

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