Problems with Nontrivial Symmetrizations of Hamiltonian Operators

Jouni S. Puuronen

9.5.2026

Abstract

We discover an example of an infinite collection of simple quantum systems that all have the same classical limit, and recognize this as a problem, because the correct quantization of the example classical system is then not known. Our example contradicts Born-Jordan quantization theory, meaning that at least other one must contain a mistake. We discuss the related possible difficulties with the quantum mechanical description of a charged particle in a magnetic field, but eventually conclude that these difficulties maybe are not so severe after all.

In an earlier article *Criticizing Feynman's Path Integrals* [1] we proved the result that if a Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is defined by using an equation

$$H(x,p) \;=\; \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta} \prod_{n=1}^{N} \big(x_n^{\alpha(n)} p_n^{\beta(n)}\big)$$

and some coefficient mapping

$$a : \mathbb{N}^{\{1,2,\ldots,N\}} \times \mathbb{N}^{\{1,2,\ldots,N\}} \to \mathbb{R}, \qquad (\alpha,\beta) \mapsto a_{\alpha,\beta},$$

then an implication relation

$$i\hbar\partial_t\psi(t,x) \;=\; H(M_x, -i\hbar\nabla_x)\psi(t,x) \qquad \Longrightarrow$$
$$\Big(D_t\langle x(t)\rangle \;=\; \langle \nabla_p H(t)\rangle \qquad \text{and} \qquad D_t\langle p(t)\rangle \;=\; -\langle \nabla_x H(t)\rangle\Big)$$

is true. This was interesting, because this result means that the generic Schrödinger's equation implies Hamilton's equations of motion in a very general setting. However, in order to make the proof work, we had to ease the task by using the simplifying assumption

$$\big(\exists\, n \in \{1,2,\ldots,N\} \quad \text{s.t.} \quad \alpha(n) > 0 \quad \text{and} \quad \beta(n) > 0\big) \qquad \Longrightarrow \qquad a_{\alpha,\beta} = 0.$$

Here, let's return to this topic, and see what happens if we don't use this simplifying assumption. Then the first problem is that without the simplifying assumption it's not obvious what the Hamiltonian operator $H(M_x, -i\hbar\nabla_x)$

is supposed to mean. The real variables $x_n \in \mathbb{R}$ and $p_n \in \mathbb{R}$ commute, and they can be ordered in an arbitrary way before the substitutions $x_n \leftarrow M_{x_n}$ and $p_n \leftarrow -i\hbar D_{x_n}$, which seems to imply ambiguity to the definition of the Hamiltonian operator. One possible way of attempting to deal with this problem is that we demand that the Hamiltonian operator is hermitian, i.e.

$$\big(H(M_x, -i\hbar\nabla_x)\big)^\dagger \;=\; H(M_x, -i\hbar\nabla_x).$$

If an operator A is not hermitian, it can always be made hermitian with the projection $A \leftarrow \frac{1}{2}(A + A^\dagger)$. Although this demand is smart, it's not obvious whether it's sufficient to make the Hamiltonian operator unique. Let's try to figure out which way the answer turns out to be by studying an example.

At this point we should recall that we learned earlier [1] that we can prove the identities

$$D_x M_x^j - M_x^j D_x \;=\; jM_x^{j-1} \qquad \text{and} \qquad D_x^k M_x - M_x D_x^k \;=\; kD_x^{k-1}$$

for all $j, k \in \{1, 2, 3, \ldots\}$ by induction. Here D_x is the ordinary differential operator, whose effect can be described by the notation $(D_x f)(x) = f'(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is some test function, and M_x is the multiplication operator, whose effect is $(M_x f)(x) = xf(x)$.

With a finite amount of effort we can calculate that

$$\begin{aligned}
D_x M_x^2 - M_x^2 D_x &= 2M_x \\
D_x^2 M_x^2 - M_x^2 D_x^2 &= 4M_x D_x + 2\text{id} \\
D_x^3 M_x^2 - M_x^2 D_x^3 &= 6M_x D_x^2 + 6D_x \\
D_x^4 M_x^2 - M_x^2 D_x^4 &= 8M_x D_x^3 + 12D_x^2 \\
D_x^5 M_x^2 - M_x^2 D_x^5 &= 10M_x D_x^4 + 20D_x^3 \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
D_x M_x^3 - M_x^3 D_x &= 3M_x^2 \\
D_x^2 M_x^3 - M_x^3 D_x^2 &= 6M_x^2 D_x + 6M_x \\
D_x^3 M_x^3 - M_x^3 D_x^3 &= 9M_x^2 D_x^2 + 18M_x D_x + 6\text{id} \\
D_x^4 M_x^3 - M_x^3 D_x^4 &= 12M_x^2 D_x^3 + 36M_x D_x^2 + 24D_x \\
D_x^5 M_x^3 - M_x^3 D_x^5 &= 15M_x^2 D_x^4 + 60M_x D_x^3 + 60D_x^2 \\
&\vdots
\end{aligned}$$

and

$$
\begin{aligned}
D_x M_x^4 - M_x^4 D_x &= 4M_x^3 \\
D_x^2 M_x^4 - M_x^4 D_x^2 &= 8M_x^3 D_x + 12M_x^2 \\
D_x^3 M_x^4 - M_x^4 D_x^3 &= 12M_x^3 D_x^2 + 36M_x^2 D_x + 24M_x \\
D_x^4 M_x^4 - M_x^4 D_x^4 &= 16M_x^3 D_x^3 + 72M_x^2 D_x^2 + 96M_x D_x + 24\mathrm{id} \\
D_x^5 M_x^4 - M_x^4 D_x^5 &= 20M_x^3 D_x^4 + 120M_x^2 D_x^3 + 240M_x D_x^2 + 120D_x \\
&\vdots
\end{aligned}
$$

Some of these identities become useful below.

Let's have a look at an example, where we set $N = 1$ and

$$H(x,p) \;=\; x^2 p^3.$$

If we do direct substitutions to this, and then project the operator to become hermitian, we get

$$
\begin{aligned}
H(M_x, -i\hbar D_x) &= \frac{1}{2}\big(M_x^2(-i\hbar D_x)^3 \;+\; (-i\hbar D_x)^3 M_x^2\big) \\
&= M_x^2(-i\hbar D_x)^3 \;+\; \frac{(-i\hbar)^3}{2}(6M_x D_x^2 + 6D_x).
\end{aligned}
$$

If we do direct substitutions to

$$H(x,p) \;=\; px^2p^2,$$

and then project the operator to become hermitian, we get

$$
\begin{aligned}
H(M_x, -i\hbar D_x) &= \frac{1}{2}\big((-i\hbar D_x)M_x^2(-i\hbar D_x)^2 \;+\; (-i\hbar D_x)^2 M_x^2(-i\hbar D_x)\big) \\
&= M_x^2(-i\hbar D_x)^3 \;+\; \frac{(-i\hbar)^3}{2}(6M_x D_x^2 + 2D_x),
\end{aligned}
$$

which is different from the previous. This example proves that in generic situations demanding the hermiticity of the Hamiltonian operator is not sufficient to make it unique. If we do direct substitutions to

$$H(x,p) \;=\; xpxp^2,$$

and then project the operator to become hermitian, we get

$$
\begin{aligned}
H(M_x, -i\hbar D_x) &= \frac{1}{2}\big(M_x(-i\hbar D_x)M_x(-i\hbar D_x)^2 \;+\; (-i\hbar D_x)^2 M_x(-i\hbar D_x)M_x\big) \\
&= M_x^2(-i\hbar D_x)^3 \;+\; \frac{(-i\hbar)^3}{2}(6M_x D_x^2 + 4D_x),
\end{aligned}
$$

which is different from the previous ones. If we do direct substitutions to

$$H(x,p) \;=\; xp^2xp,$$

and then project the operator to become hermitian, we get

$$\begin{aligned} H(M_x,-i\hbar D_x) \;&=\; \frac{1}{2}\big(M_x(-i\hbar D_x)^2M_x(-i\hbar D_x) \;+\; (-i\hbar D_x)M_x(-i\hbar D_x)^2M_x\big) \\ &=\; M_x^2(-i\hbar D_x)^3 \;+\; \frac{(-i\hbar)^3}{2}(6M_xD_x^2+2D_x). \end{aligned}$$

If we do direct substitutions to

$$H(x,p) \;=\; pxpxp,$$

we get

$$\begin{aligned} H(M_x,-i\hbar D_x) \;&=\; (-i\hbar D_x)M_x(-i\hbar D_x)M_x(-i\hbar D_x) \\ &=\; M_x^2(-i\hbar D_x)^3 \;+\; (-i\hbar)^3(3M_xD_x^2+D_x). \end{aligned}$$

Let's have a look at the operators $\partial_x H(M_x,-i\hbar D_x)$ and $\partial_p H(M_x,-i\hbar D_x)$ too. If we do direct substitutions to

$$\partial_x H(x,p) \;=\; 2xp^3,$$

and then project the operator to become hermitian, we get

$$\begin{aligned} \partial_x H(M_x,-i\hbar D_x) \;&=\; \frac{2}{2}\big(M_x(-i\hbar D_x)^3 \;+\; (-i\hbar D_x)^3M_x\big) \\ &=\; 2M_x(-i\hbar D_x)^3 \;+\; 3(-i\hbar)^3D_x^2. \end{aligned}$$

If we do direct substitutions to

$$\partial_x H(x,p) \;=\; 2pxp^2,$$

and then project the operator to become hermitian, we get

$$\begin{aligned} \partial_x H(M_x,-i\hbar D_x) \;&=\; \frac{2}{2}\big((-i\hbar D_x)M_x(-i\hbar D_x)^2 \;+\; (-i\hbar D_x)^2M_x(-i\hbar D_x)\big) \\ &=\; 2M_x(-i\hbar D_x)^3 \;+\; 3(-i\hbar)^3D_x^2 \end{aligned}$$

again.

If we do direct substitutions to

$$\partial_p H(x,p) \;=\; 3x^2p^2,$$

and then project the operator to become hermitian, we get

$$\begin{aligned} \partial_p H(M_x,-i\hbar D_x) \;&=\; \frac{3}{2}\big(M_x^2(-i\hbar D_x)^2 \;+\; (-i\hbar D_x)^2M_x^2\big) \\ &=\; 3M_x^2(-i\hbar D_x)^2 \;+\; \frac{3(-i\hbar)^2}{2}(4M_xD_x+2\mathrm{id}). \end{aligned}$$

If we do direct substitutions to

$$\partial_p H(x,p) \;=\; 3px^2p,$$

we get

$$\begin{aligned}\partial_p H(M_x, -i\hbar D_x) \;&=\; 3(-i\hbar D_x)M_x^2(-i\hbar D_x)\\ &=\; 3M_x^2(-i\hbar D_x)^2 \;+\; 6(-i\hbar)^2 M_x D_x,\end{aligned}$$

which is different from the previous. If we do direct substitutions to

$$\partial_p H(x,p) \;=\; 3xp^2x,$$

we get

$$\begin{aligned}\partial_p H(M_x, -i\hbar D_x) \;&=\; 3M_x(-i\hbar D_x)^2 M_x\\ &=\; 3M_x^2(-i\hbar D_x)^2 \;+\; 6(-i\hbar)^2 M_x D_x\end{aligned}$$

again. If we do direct substitutions to

$$\partial_p H(x,p) \;=\; 3xpxp,$$

and then project the operator to become hermitian, we get

$$\begin{aligned}\partial_p H(M_x, -i\hbar D_x) \;&=\; \frac{3}{2}\big(M_x(-i\hbar D_x)M_x(-i\hbar D_x) \;+\; (-i\hbar D_x)M_x(-i\hbar D_x)M_x\big)\\ &=\; 3M_x^2(-i\hbar D_x)^2 \;+\; \frac{3(-i\hbar)^2}{2}(4M_x D_x + \mathrm{id}),\end{aligned}$$

which is different from the previous ones.

We see that the example $H(x,p) = x^2p^3$ has the properties that the operators $H(M_x, -i\hbar D_x)$ and $\partial_p H(M_x, -i\hbar D_x)$ are ambiguous, but the operator $\partial_x H(M_x, -i\hbar D_x)$ is not ambiguous. Let's continue by defining collections of operators by setting

$$H_\alpha \;=\; (-i\hbar)^3(M_x^2 D_x^3 + 3M_x D_x^2 + \alpha D_x),$$

where $\alpha \in \mathbb{R}$ is some real constant, and

$$\partial_p H_\beta \;=\; (-i\hbar)^2(3M_x^2 D_x^2 + 6M_x D_x + \beta\mathrm{id}),$$

where $\beta \in \mathbb{R}$ is some real constant. The special values $\alpha = 1,2,3$ and $\beta = 0, \frac{3}{2}, 3$ correspond to the operators we obtained from the different orderings of the parameters x and p. Since $(-i\hbar)^3 D_x$ and $(-i\hbar)^2\mathrm{id}$ are hermitian, H_α and $\partial_p H_\beta$ will be some hermitian operators for all values of $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

For all values of $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, it is possible to interpret the operators H_α and $\partial_p H_\beta$ as weighted averages (possibly with some negative weights) of the operators that were obtained from the different orderings of the parameters x and p.

Next, we are interested in the question that if we assume that

$$i\hbar\partial_t\psi(t,x) \ = \ H_\alpha\psi(t,x),$$

will it imply that

$$D_t\langle x(t)\rangle \ = \ \langle\partial_p H_\beta(t)\rangle \qquad \text{and} \qquad D_t\langle p(t)\rangle \ = \ -\langle\partial_x H(t)\rangle?$$

Let's start calculating $D_t\langle x(t)\rangle$, and see how it turns out:

$$\begin{aligned}
D_t\langle x(t)\rangle \ &= \ D_t\int\limits_{-\infty}^{\infty} x|\psi(t,x)|^2dx \\
&= \int\limits_{-\infty}^{\infty} x\big((\partial_t\psi(t,x))^*\psi(t,x) \ + \ (\psi(t,x))^*\partial_t\psi(t,x)\big)dx \\
&= \int\limits_{-\infty}^{\infty} x\Big(\Big(-\frac{i}{\hbar}H_\alpha\psi(t,x)\Big)^*\psi(t,x) \ + \ \psi^*(t,x)\Big(-\frac{i}{\hbar}H_\alpha\psi(t,x)\Big)\Big)dx \\
&= \int\limits_{-\infty}^{\infty} x\Big(\big(-(-i\hbar)^2(x^2\partial_x^3 + 3x\partial_x^2 + \alpha\partial_x)\psi(t,x)\big)^*\psi(t,x) \\
&\qquad\qquad + \ \psi^*(t,x)\big(-(-i\hbar)^2(x^2\partial_x^3 + 3x\partial_x^2 + \alpha\partial_x)\psi(t,x)\big)\Big)dx \\
&= \ -(-i\hbar)^2\int\limits_{-\infty}^{\infty} \Big(\big((x^3\partial_x^3 + 3x^2\partial_x^2 + \alpha x\partial_x)\psi^*(t,x)\big)\psi(t,x) \\
&\qquad\qquad + \ \psi^*(t,x)\big((x^3\partial_x^3 + 3x^2\partial_x^2 + \alpha x\partial_x)\psi(t,x)\big)\Big)dx \\
&= \ -(-i\hbar)^2\int\limits_{-\infty}^{\infty} \big(\psi^*(t,x)(-D_x^3M_x^3 + 3D_x^2M_x^2 - \alpha D_xM_x)\psi(t,x) \\
&\qquad\qquad + \ \psi^*(t,x)(x^3\partial_x^3 + 3x^2\partial_x^2 + \alpha x\partial_x)\psi(t,x)\big)dx \\
&= \ -(-i\hbar)^2\int\limits_{-\infty}^{\infty} \big(\psi^*(t,x)(-M_x^3D_x^3 - 9M_x^2D_x^2 - 18M_xD_x - 6\mathrm{id} \\
&\qquad\qquad\qquad + 3M_x^2D_x^2 + 3\cdot 4M_xD_x + 3\cdot 2\mathrm{id} \\
&\qquad\qquad\qquad - \alpha M_xD_x - \alpha\mathrm{id})\psi(t,x) \\
&\qquad\qquad + \ \psi^*(t,x)(x^3\partial_x^3 + 3x^2\partial_x^2 + \alpha x\partial_x)\psi(t,x)\big)dx
\end{aligned}$$

$$= (-i\hbar)^2 \int_{-\infty}^{\infty} \psi^*(t,x)(3x^2\partial_x^2 + 6x\partial_x + \alpha\mathrm{id})\psi(t,x)dx$$

$$= \langle \partial_p H_\beta(t)\rangle \ + \ (\beta-\alpha)\hbar^2 \int_{-\infty}^{\infty} |\psi(t,x)|^2 dx$$

We see that the Hamilton's equation of motion $D_t\langle x(t)\rangle = \langle \partial_p H_\beta(t)\rangle$ is true, if $\alpha = \beta$.

The calculation of $D_t\langle p(t)\rangle$ turns out as:

$$\begin{aligned}
D_t\langle p(t)\rangle \ &= \ D_t \int_{-\infty}^{\infty} (\psi(t,x))^*(-i\hbar\partial_x)\psi(t,x)dx \\
&= \int_{-\infty}^{\infty} \big((\partial_t\psi(t,x))^*(-i\hbar\partial_x)\psi(t,x) \ + \ (\psi(t,x))^*(-i\hbar\partial_x)\partial_t\psi(t,x)\big)dx \\
&= \int_{-\infty}^{\infty} \Big(\Big(-\frac{i}{\hbar}H_\alpha\psi(t,x)\Big)^*(-i\hbar\partial_x)\psi(t,x) \\
&\qquad + \ \psi^*(t,x)(-i\hbar D_x)\Big(-\frac{i}{\hbar}H_\alpha\psi(t,x)\Big)\Big)dx \\
&= \int_{-\infty}^{\infty} \Big(\Big(-\frac{i}{\hbar}(-i\hbar)^3(x^2\partial_x^3 + 3x\partial_x^2 + \alpha\partial_x)\psi(t,x)\Big)^*(-i\hbar\partial_x)\psi(t,x) \\
&\qquad + \psi^*(t,x)(-i\hbar D_x)\Big(-\frac{i}{\hbar}(-i\hbar)^3(x^2\partial_x^3 + 3x\partial_x^2 + \alpha\partial_x)\psi(t,x)\Big)\Big)dx \\
&= -(-i\hbar)^3 \int_{-\infty}^{\infty} \Big(\big((x^2\partial_x^3 + 3x\partial_x^2 + \alpha\partial_x)\psi(t,x)\big)^*\partial_x\psi(t,x) \\
&\qquad + \psi^*(t,x)\big(D_x(M_x^2D_x^3 + 3M_xD_x^2 + \alpha D_x)\big)\psi(t,x)\Big)dx \\
&= -(-i\hbar)^3 \int_{-\infty}^{\infty} \big(\psi^*(t,x)(-D_x^3M_x^2D_x + 3D_x^2M_xD_x - \alpha D_x^2)\psi(t,x) \\
&\qquad + \ \psi^*(t,x)(D_xM_x^2D_x^3 + 3D_xM_xD_x^2 + \alpha D_x^2)\psi(t,x)\big)dx \\
&= -(-i\hbar)^3 \int_{-\infty}^{\infty} \psi^*(t,x)\big(-M_x^2D_x^4 - 6M_xD_x^3 - 6D_x^2 \\
&\qquad\qquad + 3M_xD_x^3 + 3\cdot 2D_x^2 \\
&\qquad\qquad + M_x^2D_x^4 + 2M_xD_x^3 \\
&\qquad\qquad + 3M_xD_x^3 + 3D_x^2\big)\psi(t,x)dx
\end{aligned}$$

$$
\begin{aligned}
&= -(-i\hbar)^3 \int_{-\infty}^{\infty} \psi^*(t,x)(2x\partial_x^3 + 3\partial_x^2)\psi(t,x)dx \\
&= -\langle \partial_x H(t) \rangle
\end{aligned}
$$

Here the constant α cancelled out, and the Hamilton's equation of motion $D_t\langle p(t)\rangle = -\langle \partial_x H(t)\rangle$ is true with all values of α and β.

This means that if we let $\alpha \in \mathbb{R}$ be an arbitrary constant, and then define the operators

$$H_\alpha = (-i\hbar)^3(M_x^2 D_x^3 + 3M_x D_x^2 + \alpha D_x), \quad (1)$$

$$\partial_x H = (-i\hbar)^3(2M_x D_x^3 + 3D_x^2) \quad (2)$$

and

$$\partial_p H_\alpha = (-i\hbar)^2(3M_x^2 D_x^2 + 6M_x D_x + \alpha \mathrm{id}) \quad (3)$$

like this, then these operators define a quantum system whose classical limit is the system defined by the Hamiltonian $H(x,p) = x^2p^3$. In other words, we have found an infinite collection of different quantum systems that all have the same classical limit. Furthermore, all of these quantum systems can be interpreted to arise from attempts to apply the substitutions $x \leftarrow M_x$ and $p \leftarrow -i\hbar D_x$, meaning that they have not been artificially constructed with some phony differences. This is a big problem, because this means that we don't know how to quantize the classical system $H(x,p) = x^2p^3$.

An example of a phony difference between quantum systems would be that we add some arbitrary terms, whose expectation values would be proportional to $\hbar, \hbar^2, \hbar^3, \ldots$ and so on, with no good reason just for sake of adding them, and then say that the terms vanish in the classical limit.

We are not the first people to think about these things, and according to (one interpretation of) the mainstream physics, these issues were solved by Born and Jordan in 1925 [2]. According to Born and Jordan, the only way to make Hamilton's equations of motion satisfied as a classical limit with the system

$$H(x,p) = x^j p^k$$

would be to use the Hamiltonian operator

$$H(M_x, -i\hbar D_x) = \frac{1}{k+1}\sum_{\ell=0}^{k}(-i\hbar D_x)^{k-\ell} M_x^j (-i\hbar D_x)^\ell.$$

We have no other option but to notice that the example we discovered above contradicts the claim by Born and Jordan. This is a reason to be skeptical about Born-Jordan quantization.

One possible opinion we could consider, that maybe could resolve the contradiction, would be that we are not allowed to define a quantum system by defining all the three operators H, $\partial_x H$ and $\partial_p H$ by some formulas. Instead, only the H should be defined, and the $\partial_x H$ and $\partial_p H$ should then follow by some rule. An answer to this is that yes it would be interesting to learn what that rule is, and how it is justified. The operators defined in equations (1), (2) and (3) look reasonable, though. They are not blatantly violating the rules of ordinary calculus at least.

Suppose A^μ is some classical vector field, whose dynamics is dictated by some Lagragian density $\mathcal{L}$ that can maybe be $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ or $\mathcal{L} = \frac{1}{2}((\partial_\mu A_\nu)(\partial^\mu A^\nu) - m^2 A_\mu A^\mu)$, or maybe something else; it doesn't matter much in this discussion. Suppose that ρ^μ describes some mass or charge density, and that we want the vector field to interact with this mass or charge density. In Lagrangian formalism the interaction term is supposed to be pointwisely scalar, and the simplest scalar quantity that uses A^μ and ρ^μ is the inner product. Let's make $-A_\mu\rho^\mu$ be the interaction term, because it's a common convention to have a minus sign here. Then, suppose we have constructed a model where a collection of relativistic particles interact via the vector field and the interaction term $\propto -A_\mu\rho^\mu$. Suppose we extract one of the particles as a subject of interest, and consider everything else as a background. When we substitute the four-current of the one particle with Dirac delta function into the interaction term $\propto -A_\mu\rho^\mu$, and keep the constants as simple as possible, we find that the dynamics of the particle are dictated by a Lagrangian function

$$L(t, \mathbf{x}, \dot{\mathbf{x}}) \;=\; -mc^2\sqrt{1 - \frac{\|\dot{\mathbf{x}}\|^2}{c^2}} \;-\; cA^0(t, \mathbf{x}) \;+\; \mathbf{A}(t, \mathbf{x}) \cdot \dot{\mathbf{x}}.$$

If we solve the Hamiltonian that is equivalent with this, the answer turns out to be

$$H(t, \mathbf{x}, \mathbf{p}) \;=\; \sqrt{(mc^2)^2 + c^2\|\mathbf{p} - \mathbf{A}(t, \mathbf{x})\|^2} \;+\; cA^0(t, \mathbf{x}). \tag{4}$$

The big question is that how do we quantize this system of a particle interacting with a background vector field? An obvious idea would be to write

$$i\hbar\partial_t\psi(t, \mathbf{x}) \;=\; \big(\sqrt{(mc^2)^2 + c^2\| - i\hbar\nabla_{\mathbf{x}} - M_{\mathbf{A}}\|^2} \;+\; cM_{A^0}\big)\psi(t, \mathbf{x}),$$

but it's not immediately clear whether this means anything. Here M_{A^0} and $M_{\mathbf{A}}$ are the multiplication operators

$$(M_{A^0}\psi)(t, \mathbf{x}) \;=\; A^0(t, \mathbf{x})\psi(t, \mathbf{x}) \qquad \text{and} \qquad (M_{\mathbf{A}}\psi)(t, \mathbf{x}) \;=\; \mathbf{A}(t, \mathbf{x})\psi(t, \mathbf{x}).$$

The first difficulty is that in attempts to interpret $\| - i\hbar\nabla_{\mathbf{x}} - M_{\mathbf{A}}\|^2$ we encounter products of noncommuting operators, and as we learned earlier with a simpler example, these type of situations can be very nontrivial. The most obvious attempt to interpret this operator is

$$\| -i\hbar\nabla_{\mathbf{x}} - M_{\mathbf{A}}\|^2 \;=\; -\hbar^2\nabla_{\mathbf{x}}^2 \;+\; M_{\mathbf{A}}^2 \;-\; (-i\hbar\nabla_{\mathbf{x}})\cdot M_{\mathbf{A}} \;-\; M_{\mathbf{A}}\cdot(-i\hbar\nabla_{\mathbf{x}}).$$

It is nice that this interpretation produces some hermitian operator, but as we learned earlier with a simpler example, the fact that an operator has been made hermitian somehow is still no guarantee that the operator would be the right one.

There are also difficulties related to the square root, but they are not the topic of this article. The difficulties related to the operator $\|-i\hbar\nabla_{\mathbf{x}} - M_{\mathbf{A}}\|^2$ remain also in the nonrelativistic approximation.

At this point we could consider putting forward a claim that if we don't know how to quantize the system $H(x,p) = x^2p^3$, then we also don't know how to quantize the system defined in Equation (4)?

An interaction of an electrically charged particle with a magnetic field is described with this type of vector potential $\mathbf{A}$, so at this point we could consider a conclusion, that what happens to a quantum mechanically described charged particle in a magnetic field, would still be an open problem?

Let's not give up, but continue studying some simple examples. Suppose $j \in \{1,2,3,\ldots\}$ and $\ell \in \{0,1,2,\ldots,j\}$ are some numbers. If we do direct substitutions to

$$H(x,p) \;=\; x^{j-\ell}px^{\ell},$$

and then project the operator to become hermitian, we get

$$\begin{aligned} H(M_x, -i\hbar D_x) \;&=\; \frac{1}{2}\big(M_x^{j-\ell}(-i\hbar D_x)M_x^{\ell} \;+\; M_x^{\ell}(-i\hbar D_x)M_x^{j-\ell}\big) \\ &=\; \frac{-i\hbar}{2}\big(M_x^{j-\ell}(M_x^{\ell}D_x + \ell M_x^{\ell-1}) \\ &\qquad +\; M_x^{\ell}(M_x^{j-\ell}D_x + (j-\ell)M_x^{j-\ell-1})\big) \\ &=\; M_x^{j}(-i\hbar D_x) \;+\; \frac{j(-i\hbar)}{2}M_x^{j-1}. \end{aligned}$$

It turned out that this operator does not depend on the parameter ℓ.

Let's think about what this looks like. We have now learnt that the system $H(x,p) = x^2p^3$ is very problematic, because firstly, in this example, demanding the hermiticity of the Hamiltonian operator doesn't make it unique, and secondly, demanding the implication of Hamilton's equations of motion also doesn't make it unique. We have also learnt that the systems $H(x,p) = xp$, $H(x,p) = x^2p$, $H(x,p) = x^3p$, $H(x,p) = x^4p$, $H(x,p) = x^5p$, . . . and so on, are not problematic at all. With these, demanding the hermiticity of the Hamiltonian operator does make it unique. These examples support a conclusion that in order for the Hamiltonian to be problematic, there must be a term where both of the exponents on x and p are greater than 1 simultaneously. If one exponent is 1, and the other one is not 0, we may have to use a nontrivial projection, but there are no problems remaining after that. We could consider extrapolating this result into situations where the Hamiltonian is something else than a polynomial. Now it starts to look like that maybe there are no problems remaining with the operator $(-i\hbar\nabla_{\mathbf{x}}) \cdot M_{\mathbf{A}} + M_{\mathbf{A}} \cdot (-i\hbar\nabla_{\mathbf{x}})$ after all. Could it be that we can handle this operator by noting that firstly, it is a hermitian operator, and secondly, the exponents on the differential operators are no greater than 1, and for these two reasons this operator is fine? Maybe the foundations of a quantum mechanically described charged particle in a magnetic field are no more elaborate than this?

References

[1] *Criticizing Feynman's Path Integrals*, J. S. Puuronen, 2025, viXra:2512.0106

[2] *Born-Jordan Quantization and the Equivalence of Matrix and Wave Mechanics*, M. A. de Gosson, 2014, arXiv:1405.2519