Eigenvalues of Transpose of a Matrix

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Abstract

A proof without using determinants is given to show that the eigenvalues of a square matrix A are the same as those of its transpose A^T. The proof only uses the fact that the row and column ranks of a matrix are same.

Keywords: Eigenvalues, Transpose, Determinants, Characteristic Equation

1 Proof

The usual way of showing that the eigenvalues of a matrix are the same as those of its transpose is by saying that the characteristic polynomial of the two matrices is the same. This uses the fact that the determinant of a matrix equals that of its transpose. In this short note, a direct proof is given. The proof uses only the fact that the row and column ranks of a matrix are equal.

Assume λ is an eigenvalue of an $n \times n$ matrix A with eigenvector x. Then $Ax = \lambda x$ or $(A - \lambda I)x = 0$

If $x = (x_1, \ldots, x_n)^T$ and $C_1, \ldots, C_n$ are columns of $A - \lambda I$ then the above equation is equivalent to $x_1C_1 + \ldots + x_nC_n = 0$, showing that columns are linearly dependent.

But as the column rank of a matrix is the same as the row rank of the matrix, there are numbers (not all zeroes) $y_1, y_2, \ldots, y_n$ such that $y_1R_1 + \ldots + y_nR_n = 0$ where $R_1, \ldots, R_n$ are rows of $(A - \lambda I)$.

This equation is equivalent to $y^T(A - \lambda I) = 0$, where $y = (y_1, y_2, \ldots, y_n)^T$ or taking the transpose, $(A^T - \lambda I)y^T = 0$.

Thus, λ is also an eigenvalue of A^T.

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