On Rényi Entropy and Foundational Physics

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Abstract

Rényi entropy is a generalization of the standard Shannon entropy and plays an important role in understanding complex dynamics of non-equilibrium systems. In line with [7], this brief report discusses the definition and application of Rényi entropy to Quantum Field Theory (QFT) and gravitational regime of primordial cosmology. In particular, we review the connection between Rényi entropy and *entanglement entropy*, on the one hand, and the *maximal entropy principle (MEP)*, on the other. Remarkably, the latter enables derivation of the Higgs and W-boson masses in close agreement with experimental data. Appealing to multifractal analysis and the concept of generalized dimensions, we further bridge the divide between Rényi entropy and the "sum-of-squares" relationship of particle physics, corresponding to maximally entropy at index $q = 0$. We emphasize the deep analogy between QFT entanglement and long-range correlations of complex dynamics and indicate that the MEP aligns with the universal route to chaos described by the Feigenbaum scenario. Finally, we review how Rényi entropy can be used to derive the four dimensionality of classical spacetime for $q = 1/2$.

Key words: Rényi entropy**, g**eneralized dimensions, entanglement entropy, maximal entropy principle, sum-of-squares relationship, spacetime dimensionality.

I. Introduction

Entropy is a central concept in physics, linking statistical mechanics, information theory, and the foundations of quantum theory. While both Boltzmann-Gibbs and Shannon entropies are uniquely defined by axiomatic consistency, they is often insufficient to describe systems exhibiting *complex behavior* (long-range correlations, intermittency, or scale-dependent structure). Such features are ubiquitous in chaotic dynamics, turbulence, critical phenomena, and the non-equilibrium evolution of classical fields.

Rényi entropy generalizes Shannon entropy by introducing a continuous parameter q (named the *Rényi index*), which enables characterization of an entire spectrum of scaling exponents rather than a single averaged quantity.

Viewed in the context of complex dynamics, this generalization is essential for multifractal systems, quantum entanglement, and the spacetime structure above the electroweak scale.

In this work, we argue that Rényi entropy is deeply ingrained in foundational physics. When combined with the Maximal Entropy Principle (MEP), it yields nontrivial predictions for the approach to chaos in nonlinear dynamics, the spectrum of particle masses and spacetime dimensionality at the classical observation level.

II. Rényi Entropy and its Role in Complex Dynamics

The Rényi entropy of order q for a discrete probability distribution $\{p_i\}_{i=1}^{N}$ of the random variable X is defined as [1 – 2, 6]

$$S_q(X) = \frac{1}{1-q} \log\left(\sum_{i=1}^{N} p_i^q\right) \tag{1}$$

where $p_i \geq 0$ is the probability that the random variable lies in microstate i, such that $\sum_i p_i = 1$, $q \in \mathbb{R}$ is the Rényi index, which describes the inhomogeneity of the probability distribution. The limit $q \to 1$ yields the Shannon entropy,

$$S_1 = -\sum_{i=1}^{N} p_i \log p_i \tag{2}$$

ensuring consistency with conventional thermodynamics. While $q > 1$ characterizes dominant (high probability) states, $q < 1$ describes rare events and tails, and $q = 0$ counts the number of accessible states.

Starting from (1), the *generalized dimension* of order q is given by

$$D_q = \lim_{r \to 0} \frac{S_q}{\log r} \tag{3}$$

where r is the normalized "bin" size of microstates. For a self-similar set with equal probabilities $p_i = N^{-1}$, (3) gives $D_q = D_0$, for all real values of q. In this case,

$$D_0 = \frac{1}{q-1} \frac{\log N(N)^{-q}}{\log r} = \frac{\log N}{\log(1/r)} \tag{4}$$

which means that, for $q=0$, the generalized dimension (3) reduces to the definition of the Hausdorff dimension d_H, that is,

$$N = r^{-D_0} = r^{-d_H} \tag{5}$$

If instead of a single "bin" size r, there are r_i "bin" sizes (matching the construction of a self-similar multifractal set), the so called "closure" relationship requires that the following condition holds

$$\sum_{i=1}^{N} p_i^q r_i^{\tau(q)} = 1 \tag{6}$$

in which

$$\tau(q) = (1-q) D_q \tag{7}$$

Since $q=1$ recovers the Shannon entropy, D_1 is called the *information dimension*, playing an important role in quantifying the loss of information as a chaotic system evolves in time. In this sense, when $q=1$, Rényi entropy reduces to the *Kolmogorov (K) entropy* ($S_1 = S_K$).

For $q = 2$, (3) yields the *correlation dimension* D_2. It reflects the correlation properties of a fractal set, that is, the probability of finding another member of the set within a distance r of a given member of the set. It is apparent that correlation properties of a fractal set replicate the attributes of the *entanglement entropy* in the context of quantum physics and QFT.

It is instructive to note that the multifractal formalism outlined above is in a one-to-one correspondence with the Lagrangian formulation of QFT and the statistical description of equilibrium thermodynamics [10].

It is also instructive to mention that Rényi entropy is a monotonically decreasing function of q, with maximal value reached at $q = 0$ (see fig. 1 below where the generic Rényi entropy is denoted $H_q(X)$).

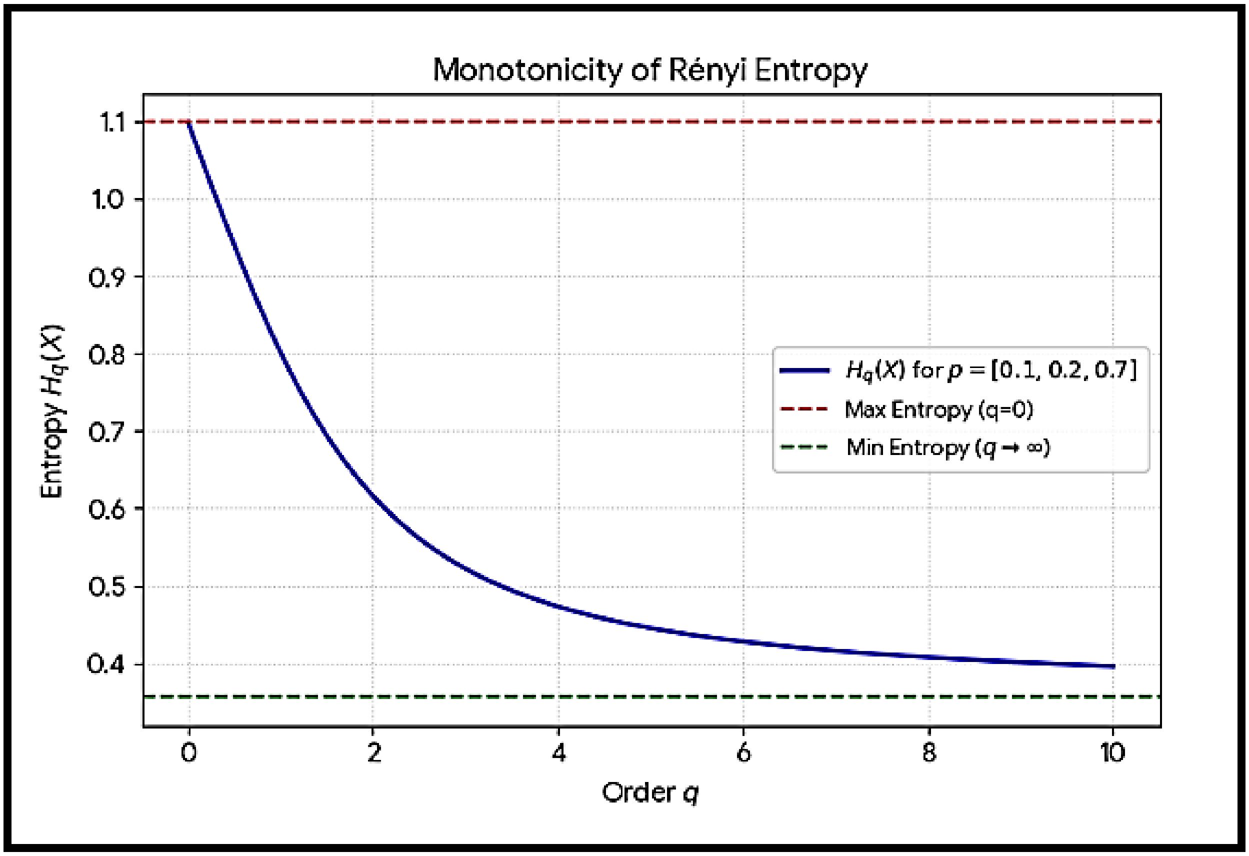

Fig. 1: Rényi Entropy vs. the Rényi index.

III. Rényi Entropy and Entanglement in QFT

Given a quantum system in pure state $|\Psi\rangle$ and a spatial bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the *reduced density matrix* is

$$\rho_A = Tr_B|\Psi\rangle\langle\Psi| \tag{8}$$

The Rényi entanglement entropy is then defined as

$$S_q^{(A)} = \frac{1}{1-q} \log \left(Tr \rho_A^q \right) \tag{9}$$

where $\mathrm{Tr}\left(\rho_A^q\right)$ is the partition function and analytic continuation in q identifies the entanglement entropy with the so-called *von Neumann entropy.*

Remarkably, the use of (9) along with MEP, enable a derivation of Higgs and W-boson masses within current experimental uncertainties [3 - 4].

IV. Applying Rényi Entropy to Foundational Physics

A. Derivation of four spacetime dimensions [9]

Geodesic trajectories in General Relativity arise from the interval equation

$$\sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 1 \tag{10}$$

subject to the constraint

$$\sum_{\nu=0}^{3} g^{\mu\nu} g_{\nu\rho} = \delta_{\rho}^{\mu} = \begin{cases} 1, & \mu = \rho \\ 0, & \mu \neq \rho \end{cases} \tag{11}$$

Consider the limit of low four-velocities

$$\frac{dx^{\mu}}{ds} < 1, \quad \mu = 0,1,2,3 \tag{12}$$

and compare (10) - (12) to (6). One is led to the following mapping

$$p_i \Rightarrow g^{\mu\nu} g_{\nu\rho}, \quad g^{\mu\nu} \Rightarrow p_i^{1/2} \Rightarrow q = \frac{1}{2}, \quad r_i \Rightarrow \frac{dx^{\mu}}{ds}, \quad \tau(q) = 2 \tag{13a}$$

The mapping for Euclidean metric takes the form

$$\frac{1}{4}\sum_i p_i = 1 \Rightarrow \frac{1}{4}\sum_{\mu} (g^{\mu\mu})^2 = 1 \tag{13b}$$

In the stochastic gravitation regime of primordial spacetime, the metric coefficients $g^{\mu\nu}$ may be interpreted as analogs of multifractal "probability amplitudes" and the components of the four-velocity as analogs of normalized "bin" sizes.

Note that it is the *formal structure of the metric* that determines the parameter values of (13). Specifically, $q = 1/2$ follows from (11), whereas $\tau(q) = 2$ follows from the quadratic form of the product of four-velocities, which are interpreted as "bin" sizes. The condition (12) is so chosen as to reflect a scaling operation $r < 1$, which means a "contraction" as opposed to a "dilation" operation for $r > 1$. This is the pedagogical way the idea of unidimensional Cantor sets is introduced in standard textbooks.

Replacing (13) in (6) recovers the four-dimensionality of classical spacetime in the form

$$\boxed{D_{1/2} = 4} \tag{14}$$

It is important to understand that the derivation of (14) *does not amount* to a circular argument. This is because the metric tensor is a rank 2 tensor. Stated differently, one can equally well start from the differential of the line element in $d > 4$ dimensions and arrive at the same result, since each term of (10) in higher dimension spacetime is still the product of metric coefficient $g^{\mu\nu}$ with the square of the ratio in differential coordinates.

B. Derivation of the Sum-of-Squares Relationship [8]

The "sum-of-squares" relationship of the Standard Model links the square of elementary particle masses to the square of the Fermi scale viz.

$$\boxed{m_W^2 + m_Z^2 + m_H^2 + \sum_f m_f^2 = v^2} \tag{15}$$

where W, Z and H stand for the electroweak and the Higgs bosons, respectively, and the sum in the left-hand side is taken over the whole spectrum of SM fermions. The contribution of bosons and fermions in (15) is split in nearly equal shares, that is,

$$\sum_b m_b^2 \approx \sum_f m_f^2 \approx \frac{v^2}{2} \tag{16}$$

Taking $r_i = \frac{m_i}{v}$ to represent "bin" sizes in a multifractal set and accounting for the MEP condition $q = 0$, recovers (6) in the form

$$\sum_i r_i^2 = 1 \tag{17}$$

This sum-of-squares relationship (17) indicates that the Standard Model is a self-contained multifractal set whose configuration reflects the geometry of a unit hypersphere in parameter space.

C. Maximal Entropy Principle and the Feigenbaum Scenario

The Feigenbaum route to chaos exhibits universal scaling behavior due to entropy maximization under repeated bifurcations. The accumulation of scales corresponds to flattening of probability distributions — precisely the condition enforced by MEP. Entropy saturation coincides with the onset of chaotic attractors, as discussed in [5, 11 - 12].

References

1. A. Rényi, *On Measures of Entropy and Information*, Proc. Fourth Berkeley Symp. Math. Stat. Prob. (1961).

2. C. J. G. Evertsz and B. M. Mandelbrot, *Multifractal Measures*, in Chaos, Solitons and Fractals, Springer Verlag, (1992).

3. https://arxiv.org/pdf/1408.0827

4. https://arxiv.org/pdf/2511.17321,

5. M. J. Feigenbaum, Phys. Rev. Lett. **19**, 780 (1978).

6. C. Beck and F. Schögl, *Thermodynamics of Chaotic Systems*, Cambridge University Press (1993).

7. https://doi.org/10.13140/RG.2.2.20005.87527/1

8. 10.13140/RG.2.2.23537.94562

9. 10.13140/RG.2.2.10553.07528

10. 10.20944/preprints202011.0518.v1

11. https://doi.org/10.13140/RG.2.2.30160.21760/1

12. https://doi.org/10.32388/O04HOC