Non-inertial Relativity Theory, Finite Modified Newtonian Potentials and Asymptotic Freedom

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Abstract

After reviewing the basics of Non-inertial relativity theory based on the existence of a maximal proper force b, it allowed to postulate a modified Newtonian attractive gravitational force (and potential) which is $finite$ at the origin : $|F(r=0)| = b$, and which vanishes at $r = \infty$. Secondly, from the modified gravitational potential energy we were able to glean the expression for a running gravitational coupling $G(r)$ which exhibits asymptotic-freedom-like properties : $G(r=0) = 0$, and $G(r=\infty) = G_N$. No quantum corrections were necessary to decrease the strength of gravity at short distances. Thirdly, we found that for very *large* masses m_1, m_2 (compared to $\sqrt{b}$) the *threshold* in the values of r obeying $\kappa r^2 << 1$, where the non-Newtonian regime becomes manifest, becomes larger and larger as m_1, m_2 become larger and larger. Whereas for very *small* masses (compared to $\sqrt{b}$) the *threshold* in the values of r obeying $\kappa r^2 << 1$, where the non-Newtonian regime becomes manifest, becomes smaller and smaller as m_1, m_2 become smaller and smaller. In the $b = \infty$ limit one recovers the Newtonian gravitational force for all values of $r > 0$. These results were all possible by abandoning the weak equivalence principle at short distances.

Keywords : Born Reciprocal Relativity; Non-inertial Relativity; Modified Newtonian dynamics; Asymptotic Freedom; Strings; Quantum Gravity.

1 Introduction : Background

Recents explorations on Non-inertial Relativity theory [5] revealed that deviations from Newtonian dynamics emerged in the Galilean limit ($c = \infty$) while keeping the maximal proper force b finite. The principle behind the concept of "Born reciprocal relativity theory", or non-inertial relativity to be more precise[1], was advocated by [3], [4], [6] and it was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. Hence, a *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [6], [10], [11].

The generalized velocity and force (acceleration) boosts (rotations) transformations of the *flat* $8D$ Phase space coordinates, $X^i, t, E, P^i; i = 1,2,3$, were given by [3] and based on the group $U(1,3)$ which is the Born version of the Lorentz group $SO(1,3)$. Adopting the natural units $\hbar = c = 1$, the $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\ dt \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born-Green line interval in the *flat* $8D$ phase-space

$$(d\omega)^2 \ = \ (dt)^2 \ - \ (dX)^2 \ - \ (dY)^2 \ - \ (dZ)^2 \ +$$

$$\frac{1}{b^2} \ \left((dE)^2 \ - \ (dP_x)^2 \ - \ (dP_y)^2 \ - \ (dP_z)^2 \right), \quad (c = 1) \tag{1.1}$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1,3) = Sp(8,R) \cap O(2,6)$; $U(3,1) = Sp(8,R) \cap O(6,2)$, and $U(2,2) = Sp(8,R) \cap O(4,4)$.

These transformations can be *simplified* drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following phase space line interval

$$(d\omega)^2 \ = \ (dt)^2 \ - \ (dX)^2 \ + \ \frac{(dE)^2 \ - \ (dP)^2}{b^2} =$$

$$(d\tau)^2 \ \left(1 \ + \ \frac{(dE/d\tau)^2 \ - \ (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \ \left(1 \ + \ \frac{\mathcal{F}^2}{F^2_{max}} \right) \ =$$

$$(d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right), \quad \mathcal{F}^2 = -F^2 < 0, \quad F_{max} = b \tag{1.2}$$

where one has factored out the non-vanishing proper time infinitesimal $(d\tau)^2 = dt^2 - dX^2 \neq 0$ in eq-(1.2). The numerical quantity F^2 is positive by definition. The proper force-squared on a massive particle is $\mathcal{F}^2 = m^2 a^2$, where a^2 is the proper acceleration-squared $a^2 = a_\mu a^\mu$, and m is the rest mass. We refrained from factoring out $(dt)^2$ in (1.2) because it is not Lorentz invariant, whereas $(d\tau)^2$ is Lorentz invariant.

[1]We thank one of the referees of a previous article for highlighting this fact in order to clarify the point that Born did not propose a reciprocal relativity theory

Due to the orthogonality condition $u_\mu a^\mu = 0$ resulting from differentiating the normalization condition $u_\mu u^\mu = \pm 1$ of timelike/spacelike velocities, when the velocity is timelike (subluminal particle) one has $(d\tau)^2 > 0$, so the acceleration is spacelike $a^2 = a_\mu a^\mu < 0$. Therefore $m^2 a^2 < 0$ since $m^2 > 0$. And viceversa, when the velocity is spacelike (superluminal particle) one has $(d\tau)^2 < 0$, so the acceleration is timelike $a^2 = a_\mu a^\mu > 0$. Therefore $m^2 a^2 < 0$ since $m^2 < 0$ (tachyonic particle). Consequently, the proper force squared $\mathcal{F}^2 \equiv (\frac{dE}{d\tau})^2 - (\frac{dP}{d\tau})^2 = \frac{(dE)^2 - (dP)^2}{(d\tau)^2} = m^2 a^2 < 0$ is always negative. Therefore, one may rewrite the negative definite $\mathcal{F}^2$ as $\mathcal{F}^2 \equiv m^2 a^2 = -F^2 < 0$, with $F^2 > 0$, so that the factorization can always be rewritten as $(d\tau)^2 (1 + \frac{\mathcal{F}^2}{b^2}) = (d\tau)^2 (1 - \frac{F^2}{b^2})$, with $F^2 > 0$. When $m = 0$, one has $(d\tau)^2 = (dE)^2 - (dP)^2 = 0$ so that $(d\omega)^2 = 0$. No factorization is needed.

Consequently, the *negative* sign appearing inside the parenthesis in eqs-(1.2) furnishes the analog of the Lorentz relativistic factor in special relativity and it involves the ratio of the square of two *proper* forces. These results can be generalized to the $8D$-dim phase space (and to higher dimensions)

The $U(1,1)$ group transformations involving the velocity and force boosts (along the X direction) acting on the phase-space coordinates X, t, P, E and which leave invariant the interval (1.2) are given by [3], [4]

$$t' = t \cosh \xi + (\xi_v \, x + \frac{\xi_a \, P}{b}) \frac{\sinh \xi}{\xi} \quad (1.3a)$$

$$E' = E \cosh \xi + (b \, \xi_a \, X + \xi_v P) \frac{\sinh \xi}{\xi} \quad (1.3b)$$

$$X' = X \cosh \xi + (\xi_v \, t + \frac{\xi_a \, E}{b}) \frac{\sinh \xi}{\xi} \quad (1.3c)$$

$$P' = P \cosh \xi + (\xi_v \, E + b \, \xi_a \, t) \frac{\sinh \xi}{\xi} \quad (1.3d)$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. The rapidity parameters ξ_a, ξ_v, ξ are defined, respectively, in terms of the spatial velocity $v = dx/dt$, and proper force $F = ma$, as follows

$$\tanh(\xi_v) = v; \quad \tanh(\xi_a) = \frac{F}{F_{max}}, \quad F_{max} = b, \quad \xi = \sqrt{(\xi_v)^2 + (\xi_a)^2} \quad (1.3e)$$

When $\xi_v \to \infty \Rightarrow v \to c = 1$. And $\xi_a \to \infty \Rightarrow F \to F_{max} = b$.

It is straight-forward to verify that the transformations of eqs-(1.3) leave invariant the phase space interval $(dt)^2 - (dX)^2 + ((dE)^2 - (dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dt^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination* is truly left invariant under force (acceleration) boosts

$$(d\omega)^2 = (d\tau)^2 \left(1 + \frac{\mathcal{F}^2}{F_{max}^2} \right) = (d\tau)^2 \left(1 - \frac{F^2}{F_{max}^2} \right), \quad F_{max}^2 = b^2 \quad (1.4)$$

where $\mathcal{F}^2 \equiv m^2 a^2 = -F^2 < 0$, with $F^2 > 0$. The transformations of eqs-(1.3a-1.3d) also leave invariant the symplectic 2-form (phase space areas) $\Omega = - \, dt \wedge dE + dX \wedge dP$, see [3], [4] for full details.

One of the many consequences of Non-inertial Relativity is that it redefines the notion of mass [7]. There is a key difference between a truly $U(1,3)$-invariant mass $\mathcal{M}$ in $D = 4$ spacetime, and the $SO(1,3)$ (Lorentz, Poincare) invariant mass m. The relation between $\mathcal{M}$ and m is [7]

$$m = m(F) \;=\; \frac{\mathcal{M}}{\sqrt{1 - \frac{F^2}{b^2}}} \;=\; \frac{\mathcal{M}}{\sqrt{1 - \frac{mg^2}{b^2}}} \tag{1.5}$$

where F is the proper force. As a result, in general, it leads to the following modified mass-energy relation involving both the velocity and proper force [8]

$$E = \mathcal{M} \frac{dt}{d\omega} \;=\; \mathcal{M} \frac{dt}{d\tau \sqrt{1 - \frac{F^2}{b^2}}} \;=\; \mathcal{M} \; \frac{1}{\sqrt{1 - v^2}} \; \frac{1}{\sqrt{1 - \frac{F^2}{b^2}}} \tag{1.6}$$

which is a generalization of the relativistic expression $E = m(1 - v^2)^{-1/2}$ in special relativity with m being the invariant rest mass (proper mass) of the particle. In the $b \to \infty$ limit, eq-(1.6) yields $m = \mathcal{M}$ as expected and one recovers in eq-(1.6) the standard relation $E = m(1-v^2)^{-1/2}$ of special relativity. A similar relation to eq-(1.6) but involving only accelerations, and m instead of $\mathcal{M}$ can be found in [12].

When the proper force acting on a particle is not uniform and depends on the proper time $F(\tau)$, eq-(1.5) leads to a τ dependent $g(\tau)$, and in turn, it leads to a τ-dependent mass obeying the relation

$$m(\tau) \;=\; \frac{\mathcal{M}}{\sqrt{1 - \frac{m^2(\tau) g^2(\tau)}{b^2}}} \tag{1.7}$$

A τ dependent mass $m(\tau)$ is just the point-mass case of strings and p-branes with a *dynamical* (variable) tension $T = T(\sigma_a), a = 1, 2, \ldots, p+1$ along the string's world-sheet (brane's world-volume) parametrized by the coordinates σ^a that have been extensively studied by Guendelman [15] over the years. In the modified measure formulation of strings/ branes, the tension appear as an additional dynamical degree of freedom. There are many important physical consequences of these variable tension models of strings and branes. Recently, Guendelman has reviewed how the model avoids the Swampland constraints making treatments for Dark energy and inflation more realistic and how strings with a different tension appear as Dark Matter to us. We refer to [15] and the many references therein for specific details.

One could also have a spacetime-dependent force like $F = F(x^\mu)$, and which in turn, as a result of eq-(1.5) means that the mass $m(x^\mu)$ becomes x^μ-dependent. How is this possible ? A careful thought reveals that this could occur as a result of the *back reaction* of spacetime on matter. As the particle probes the spacetime points along its (non-uniformly) accelerated trajectory, space-time back reacts on the particle affecting its mass. Special relativity led to the unification of space and time. Non-inertial relativity seems to indicate a space-time-matter "unification". To be more precise, matter curves spacetime, and in turn, spacetime back reacts on matter curving momentum space. In [7] we formulated the generalized field equations in phase space and how curvature in spacetime and momentum space resulted from the presence of matter.

The concept of a position dependent mass has appeared in the literature before but based on very different physical principles. Most recently, a non-commutative Hamilton-Jacobi equation based on Moyal-type noncommutative spacetimes was studied and it was found that all noncommutative effects could be absorbed into an effective, position-dependent mass function $M(x)$, appearing in an otherwise standard relativistic dispersion relation. See [13], [14] and references therein.

To sum up, one ends up with the following $m, \mathcal{M}$ relation

$$m \ = \ \frac{\mathcal{M}}{\sqrt{1 - \frac{m^2 g^2}{b^2}}} \ \Leftrightarrow \ \frac{m^2}{\mathcal{M}^2} \ = \ \frac{1 \pm \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \tag{1.8}$$

The minus sign choice in front of the square root in the right hand side of eq-(1.8) is the adequate one on physical grounds. A plus sign yields $m = \infty$ when $b = \infty$ and must be disregarded. Therefore, the minus sign choice gives

$$\frac{m^2}{\mathcal{M}^2} \ = \ \frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \ \Rightarrow \ m \ = \ \mathcal{M} \ \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \right)^{1/2} \tag{1.9}$$

Another more transparent way to rewrite eq-(1.9), after defining $F = mg$, and $\mathcal{F} = \mathcal{M} g$, is

$$F \ = \ \mathcal{F} \ \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{F}^2}{b^2}}}{\frac{2\mathcal{F}^2}{b^2}} \right)^{1/2} \quad F = mg, \quad \mathcal{F} = \mathcal{M} g \tag{1.10}$$

in other words, the ratio $F/\mathcal{F}$ is a function of the ratio $\mathcal{F}/b$ involving the maximal proper force $b : \ F/\mathcal{F} = f(\mathcal{F}/b)$, such $b \rightarrow \infty \Rightarrow \mathcal{F} \rightarrow F$ since $f(0) = 1$.

After defining

$$\Omega \ = \ \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \right)^{1/2} \tag{1.11}$$

the Galilean limit $v << 1$ of eq-(1.10) yields the following deviation (modification) of the Newtonian dynamical law of motion

$$\vec{f} \ = \ \mathcal{M} \vec{a} \ \Omega(\frac{M|\vec{a}|}{b}), \quad \frac{M|\vec{a}|}{b} < 1 \tag{1.12}$$

with

$$\Omega = \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2|\vec{a}|^2}{b^2}}}{\frac{2\mathcal{M}^2|\vec{a}|^2}{b^2}} \right)^{1/2} \tag{1.13}$$

and which follows directly from eq-(1.10) simply by replacing $F \rightarrow |\vec{f}|$ and $\mathcal{M}g \rightarrow \mathcal{M}|\vec{a}|$ in the Galilean limit.

Eq-(1.12) can be rewritten as $\vec{f} = \mathcal{M}[a]\vec{a}$, with $\mathcal{M}[a] \equiv \mathcal{M}\Omega(\frac{M|\vec{a}|}{b})$, instead of $\vec{f} = \mathcal{M}\vec{a}$. Writing $\vec{f}$ as $\vec{f} = \mathcal{M}\vec{a}'$, one finds $|\vec{a}'| = |\vec{a}|\Omega(\frac{M|\vec{a}|}{b})$ providing the relation between $|\vec{a}'|$ and $|\vec{a}|$, and where Ω is the "interpolating" function between $|\vec{a}'|$ and $|\vec{a}|$.

The above deviations from Newtonian dynamics differ from Milgromian's MOND (modified Newtonian dynamics) [16], [17], mainly because the interpolating functions between the Milgromian acceleration a_M and the Newtonian acceleration a_N differ considerably from the Ω function described above. There are various theoretical attempts to effectively embed the modifications of Newtonian dynamics within a relativistic theory of gravity, see for example [17], [18]. Non-inertial relativity theory provides a very different approach.

The reader may also ask : is there any role of Milgrom's acceleration constant a_o in all of this ? To answer this question one may notice that one can rewrite the maximal proper force b in terms of the Planck mass M_P, the Planck scale L_P; as well as the observable mass of the Universe M_U, and the Hubble Radius R_H as follows $b = m_P(c^2/L_P) = M_U(c^2/R_H)$, and such that c^2/R_H is closer to Milgrom's acceleration constant $a_o \sim 1.2 \times 10^{-10} m/s^2$.

If the maximal proper force b acting on a fundamental particle is set to be M_P^2, where M_P is the Planck mass in $D = 4$ spacetime dimensions, in $\hbar = c = 1$ units, it is clear that one cannot set $\mathcal{M}$ to be a huge planetary mass, unless the magnitude of the acceleration $|\vec{a}|$ is *very* small such that the ratio $\frac{M|\vec{a}|}{b} < 1$ remains small. Hence, for very small accelerations one could still use eq-(2.28). In the case of fundamental particles whose masses are very small compared to the Planck mass M_P, due to the fact that they can acquire very large accelerations, the ratio $\frac{M|\vec{a}|}{b} < 1$ could still be small. Therefore, eq-(2.28) has a wide range of validity.

To finalize this introduction, we should point out that a maximal proper force b does not necessarily entail a minimal length like the Planck scale L_P, for example. b can be written as $m_1c^2/L_1 = m_2c^2/L_2 = \ldots = m_nc^2/L_n = \ldots$ in terms of an infinite number of mass-length ratios m/L. If b is chosen to be given by m_Pc^2/L_P, where m_P, L_P are the Planck mass, length, respectively, one can always find a sequence of masses and lengths obeying

$$\ldots = m_1c^2/L_1 = m_Pc^2/L_P = m_2c^2/L_2 = \ldots \tag{1.14}$$

such that $L_1 < L_P < L_2$. As $L_1 \rightarrow 0 \Rightarrow m_1 \rightarrow 0$. And as $L_2 \rightarrow \infty \Rightarrow m_2 \rightarrow \infty$.

2 Finite Modified Newtonian Potentials and Asymptotic Freedom

Note : Before we begin this section we shall set aside the subtleties between the $U(1,3)$ invariant mass $\mathcal{M}$ and the $SO(1,3)$ invariant mass m. For notational *convenience* we shall set $\mathcal{M} = m$ in all of our equations, however we remind the reader that one must *not* conflate $\mathcal{M}$ with the mass m, which in general can vary with position and time.

Let us propose the following ansatz for a *modification* of the Newtonian gravitational force with the main requirement that the magnitude of the force attains its maximum $b \neq \infty$ at the origin $r = 0$

$$F(r) \;=\; -\;\frac{Gm_1m_2\left[1-\exp(-\kappa r^2)\right]}{r^2} \tag{2.1}$$

Performing a Taylor expansion of the exponential around $r = 0$, and equating the magnitude of the force at $r = 0$ to b, gives

$$|F(r=0)| \;=\; Gm_1m_2\kappa \;=\; b \;\Rightarrow\; \kappa = \frac{b}{Gm_1m_2} \;\Rightarrow$$

$$F(r) \;=\; -\;\frac{Gm_1m_2\left[1-\exp(-br^2/Gm_1m_2)\right]}{r^2} \tag{2.2}$$

Given $F(r)$ one learns that : (i) $F(r=0) = -b$; (ii) $F(r=\infty) = 0$ as expected; (iii) when $\kappa r^2 >> 1$, then $F \to -Gm_1m_2/r^2$ recovering the Newtonian form; (iv) whereas, when $\kappa r^2 << 1$ clear deviations from the Newtonian force are manifest.

To be more precise, given $\kappa r^2 << 1 \Rightarrow r^2 << \frac{1}{\kappa} = \frac{Gm_1m_2}{b}$, for very *large* masses m_1, m_2 (compared to $\sqrt{b}$) the *threshold* in the values of r obeying $\kappa r^2 << 1$, where the non-Newtonian regime becomes manifest, becomes larger and larger as m_1, m_2 become larger and larger. Whereas for very *small* masses (compared to $\sqrt{b}$) the *threshold* in the values of r obeying $\kappa r^2 << 1$, where the non-Newtonian regime becomes manifest, becomes smaller and smaller as m_1, m_2 become smaller and smaller. This is one of the most relevant findings in this work. Clearly, in the $b = \infty$ limit one recovers the Newtonian gravitational force for all values of $r > 0$.

The salient feature of the functional form of $F(r)$ above, besides a repulsive core provided by the exponential term, is that it *is* the values of the masses themselves which encode the domain of the regions in r where the *modifications* of the Newtonian gravitational force will emerge, and the domain of the regions where the standard Newtonian gravitational force is recovered. These regions are defined by the inequalities

$$r_1^2 << \frac{1}{\kappa} = \frac{Gm_1m_2}{b} << r_2^2 \tag{2.3}$$

r_1 signals the regime where the modification of the Newtonian force becomes manifest. And the scale r_2 signals the regime where the Newtonian force becomes manifest. The region $r_1 < r < r_2$ represents the transition bridge between the two regimes. In other words, the values of $m_1 m_2$, can be seen as the "dial" which can be moved to the right/left, shifting in turn, the locations of $r_1 < r_2$.

In passing, one may remark that given the Newtonian gravitational force $\tilde{F}_N(r) \equiv -(G m_1 m_2/r^2)$, the modified Newtonian gravitational force $F(r)$ can be rewritten in the form

$$F \;=\; \tilde{F}_N \; \Big[1 - \exp(-b/\tilde{F}_N) \Big] \;\equiv\; \tilde{F}_N \; \Phi(b/\tilde{F}_N) \qquad (2.4)$$

expressing the explicit map from $\tilde{F}_N$ to F. Had one be able to invert the relation (2.4), the inverse map would have rendered the relation for $\tilde{F}_N$ in terms of F. This won't be necessary.

These results can be generalized to higher spacetime dimensions $D = d + 1$ than $D = 4$. The ansatz for the modified force in D-dim is

$$F_D(r) \;=\; -\frac{G_D m_1 m_2}{r^{D-2}} \; \left[1 - \exp\left(-b \; r^{D-2}/G_D m_1 m_2 \right) \right] \qquad (2.5)$$

where G_D is the gravitational coupling in $D = (d+1)$-dimensions and which has units of $(length)^{D-2}$. The magnitude of the force at the origin $F_D(r = 0) = -b$ attains its maximum value.

Mathematically speaking there are an *infinite* number of functional expressions for the force that are regular (finite) at $r = 0$. Besides the exponential term in eq-(2.1), there many other possibilities. Any *convergent* power series at $r = \infty$ of the form $\Phi(b/\tilde{F}_N) = \sum_{n=1}^{\infty} a_n r^{2n}$ in eq-(2.1), with $a_1 = \kappa = (b/G m_1 m_2)$, yields a finite force at $r = 0 : F(r = 0) = -b$, and vanishing at $r = \infty$. When we discuss below the asymptotic-freedom-like properties of the force we shall see the importance of involving the exponential function compared to other expressions.

Given the modified Newtonian gravitational force $F(r)$ we could equate it to $m_1 a_1 = m_2 a_2$ in order to determine the accelerations of the masses m_1, m_2, respectively. In this case the Newtonian dynamics (law of motion) is *not* modified, only the Newtonian gravitational force *is*. The familiar results are obtained $a_1 = F/m_1; a_2 = F/m_2$, with the key difference that violations of the weak equivalence principle [20] will occur since in this case a_1 will acquire a dependence on both m_1 and m_2. The same argument applies to a_2. For example, taking m_1, m_2 to be the mass of the earth M and a test object m, respectively, the acceleration of the test object would be dependent on both M and m

$$a(r) \;=\; -\; \frac{GM \left[1 - \exp(-br^2/GMm) \right]}{r^2} \qquad (2.6)$$

In the Newtonian domain region, $a(r) \rightarrow -(GM/r)$, the acceleration of the test particle is independent of its mass, and only depends on the mass of the earth consistent with the weak equivalence principle [20] that all test objects fall with the same acceleration under the influence of the earth's gravitational field irrespective of the values of their masses. One could rescue the weak equivalence principle by setting $\kappa = b$ to start with. However in this case the values of the magnitude of the force at $r = 0$ would be bGm_1m_2 instead of b and the maximal force postulate would be violated when $Gm_1m_2 > 1$.

Before continuing with the evaluation of the gravitational potential energy and other topics it is worth mentioning that one could modify both the Newtonian gravitational force *and* the Newtonian dynamics (law of motion) by equating $F(r)$ to $m_1\tilde{a}_1\Omega(m_1\tilde{a}_1/b) = m_2\tilde{a}_2\Omega(m_2\tilde{a}_2/b)$, with the provision that $m_1\tilde{a}_1 = m_2\tilde{a}_2$. After a close inspection it reveals that in order to be able to achieve this in a consistent manner, the maximal value of the magnitude of the force F at $r = 0$ must be $\frac{b}{\sqrt{2}}$ instead of b. Hence, the value of the κ in eq-(2.1) must be *reduced* by a factor of $\frac{1}{\sqrt{2}}$. Consequently, the new value of κ is $\kappa = (b/\sqrt{2}Gm_1m_2) \Rightarrow |F(r = 0)| = \frac{b}{\sqrt{2}}$. In doing so, one now can posit the relation

$$F(r) \ = \ m_1\tilde{a}_1(r)\ \Omega\left(m_1\tilde{a}_1(r)/b\right) \ = \ m_2\tilde{a}_2(r)\ \Omega\left(m_2\tilde{a}_2(r)/b\right) \tag{2.7}$$

Given the definition of Ω

$$\Omega(r) \ = \ \left(\frac{1 - \sqrt{1 - \frac{4m^2\tilde{a}(r)^2}{b^2}}}{\frac{2m^2\tilde{a}(r)^2}{b^2}} \right)^{1/2} \tag{2.8}$$

one can invert the relations (2.7) and arrive at

$$m_1\tilde{a}_1(r) \ = \ m_2\tilde{a}_2(r) \ = \ F(r)\ \sqrt{1 - \frac{F(r)^2}{b^2}} \tag{2.9}$$

And now one has

$$|F(r=0)| \ = \ \frac{b}{\sqrt{2}} \Rightarrow m_1|\tilde{a}_1(r=0)| \ = \ m_2|\tilde{a}_2(r=0)| \ = \ \frac{b}{\sqrt{2}}\ \frac{1}{\sqrt{2}} \ = \ \frac{b}{2} < \frac{b}{\sqrt{2}} < b \tag{2.10}$$

As a consistency check, one can verify that

$$m|\tilde{a}(r=0)| \ = \ \frac{b}{2} \Rightarrow \Omega(r=0) = \sqrt{2} \Rightarrow |F(r=0)| \ = \ m|\tilde{a}(r=0)|\ \Omega(r=0) \ = \ \frac{b}{2}\ \sqrt{2} \ = \ \frac{b}{\sqrt{2}} \tag{2.11}$$

The "dictionary" (map) between $a(r)$ and $\tilde{a}(r)$ is obtained from eq-(2.9)

$$\tilde{a}(r) \ = = \ \frac{F(r)}{m}\ \sqrt{1 - \frac{F(r)^2}{b^2}} \ = \ a(r)\ \sqrt{1 - \frac{ma(r)^2}{b^2}} \tag{2.12}$$

and whose inverse map is

$$a(r) \ = \ \tilde{a}(r)\ \Omega\left(m\tilde{a}(r)/b\right) \tag{2.13}$$

We should emphasize that this "dictionary" (map) between $a(r)$ and $\tilde{a}(r)$ involves very *different* functions than the interpolating functions relating the Milgromian a_M and Newtonian a_N accelerations [16], [17].

To sum up, one finds

$$|ma(r=0)| \ = \ \frac{b}{\sqrt{2}} \Leftrightarrow m|\tilde{a}(r=0)| \ = \ \frac{b}{2} \tag{2.14}$$

At $r = \infty \Rightarrow a = \tilde{a} = 0$, as expected, since $\Omega(r=\infty) = 1$ as a result of using L'Hopital's rule.

The gravitational potential energy is given by the integral

$$U(r) \ = \ - \int_{\infty}^{r} dr \ F(r) \ = \ Gm_1m_2 \ \int_{\infty}^{r} dr \ \frac{1 - \exp(-\kappa r^2)}{r^2} \ =$$

$$- \frac{Gm_1m_2}{r} \ \left[\ 1 \ - \ \sqrt{\pi\kappa} \ r \ erf(\sqrt{\kappa} r) \ - \ \exp(-\kappa r^2) \right] \ - \ Gm_1m_2\sqrt{\pi\kappa} \tag{2.15}$$

where erf(z) is the error function defined by the integral

$$erf(z) \equiv \frac{2}{\sqrt{\pi}} \ \int_{0}^{z} \ \exp(-t^2) \ dt \tag{2.16}$$

with the properties erf (z = 0) = 0; erf ($z = \pm\infty$) = ± 1; erf (- z) = - erf (z). By a simple inspection of eq-(2.15) we can see that the above gravitational potential energy $U(r)$ clearly *differs* from the the short-ranged Yukawa potential (energy) [19]

$$U_Y(r) \ = \ - \frac{Gm_1m_2}{r} \ [1 - \exp(-r/L_P)] \tag{2.17}$$

where $L_P \simeq 10^{-33}$ cm is the Planck length in $D = 4$ spacetime. The Yukawa potential (energy) also has a repulsive core provided by the exponential term, such that $U_Y(r=0) = -\frac{Gm_1m_2}{L_P}$ at the origin.

Given the important properties of the error function, after performing a Taylor expansion of the exponential around $r = 0$, one can verify that $U(r=0) = -Gm_1m_2\sqrt{\pi\kappa} = -\sqrt{\pi b G m_1 m_2} \neq \infty$. Therefore U is *finite* and negative at the origin, for $b \neq \infty$. It is the *finite* value of the maximal force b which regularizes the gravitational potential energy at $r = 0$. The asymptotic behavior is Newtonian $U(r) \rightarrow (-Gm_1m_2/r)$ for very large values of r, so that $U(r=\infty) = 0$ as expected. The presence of the error function, resulting from the exponential integral, highlights the importance of the presence of the exponential term in the force (2.1) compared to other possible expressions. The fact that U is finite at the origin is also reflection of a "repulsive" core comprised by the exponential term.

Writing the gravitational potential energy in the form $U(r) = -\frac{G(r)m_1m_2}{r}$, allows to view the running coupling $G(r)$ in terms of a renormalization-group-like flow of the Newtonian coupling constant G_N. In this case, the expression for the running coupling $G(r)$ reads

$$G(r) \ = \ G_N \ \left[\ 1 \ - \ \sqrt{\pi\kappa} \ r \ erf(\sqrt{\kappa}r) \ - \ \exp(-\kappa r^2) \ + \ \sqrt{\pi\kappa} \ r \ \right] \tag{2.18}$$

From eq-(2.18) one learns that $G(r = \infty) = G_N$; and $G(r = 0) = 0$, therefore $G(r)$ exhibits asymptotic-freedom-like properties which are compatible with the finiteness of the force and gravitational potential energy at $r = 0$. The salient feature of these results is that Quantum corrections were *not* needed to decrease the strength of gravity at short distances. All that was required was to invoke the Galilean limit of non-inertial relativity theory and introduce a modification of the Newtonian gravitational force via the maximal force parameter $b \neq \infty$.

From eqs-(2.2, 2.18) one learns that in the extreme case $b = \infty \Rightarrow \kappa = \infty$ the gravitational coupling no longer runs with scale : $G(r) \rightarrow G_N$. The key properties of the error function erf $(z = 0) = 0$; erf $(z = \infty) = 1$ were essential in all these findings. Because the error function originated from the key exponential term in the expression for the force (2.1), this validates once more the use of the exponential.

Because we were able to rewrite the the gravitational potential energy in terms of a running Newtonian coupling $G(r)$, one could ask if one could rewrite all of our expressions in terms of variable masses $m_1(r), m_2(r)$. As a reminder to the reader, for notational *convenience* we had set $\mathcal{M} = m$ in all of our above equations. To be rigorous we should replace m_1 for $\mathcal{M}_1$; m_2 for $\mathcal{M}_2$, ... in all of our equations above. This raises the possibility of attempting to write the modify Newtonian force in the form

$$F(r) \ = \ - \ \frac{G\mathcal{M}_1\mathcal{M}_2 \left[1 - \exp(-br^2/G\mathcal{M}_1\mathcal{M}_2)\right]}{r^2} \ = \ \frac{Gm_1(r)m_2(r)}{r^2} \tag{2.19}$$

after inserting the proper notation for the $U(1,3)$ invariant masses $\mathcal{M}_1; \mathcal{M}_2$ in the left hand side, and introducing the variable masses $m_1(r), m_2(r)$ in the right hand side. A detailed analysis reveals that after expressing $m_1(r), m_2(r)$ in the form

$$m_1(r) \ = \ \mathcal{M}_1 \ \Omega\left(\mathcal{M}_1\tilde{a}_1(r)/b\right); \quad m_2(r) \ = \ \mathcal{M}_2 \ \Omega\left(\mathcal{M}_2\tilde{a}_2(r)/b\right) \tag{2.20}$$

and factoring out all the common terms in eq-(2.19), it furnishes an algebraic equation leading to the trivial solutions $\tilde{a}_1(r)/b = \tilde{a}_2(r)/b = 0 \Rightarrow b = \infty$, and one ends up once again with the original Newtonian force and with $\mathcal{M}_1 = m_1; \mathcal{M}_2 = m_2$, since non-inertial relativity reduces to ordinary relativity when the maximal proper force is ∞.

To sum up, the modified Newtonian force cannot be rewritten in the form described by eq-(2.19) in terms of variable masses. However, it can be described in the form

$$F(r) \ = \ -\frac{\partial U(r)}{\partial r} \ = \ \frac{\partial}{\partial r} \ \left(\frac{G(r)m_1m_2}{r}\right) \ =$$

$$m_1 m_2 \frac{G'(r) r - G(r)}{r^2} = -\frac{G_N m_1 m_2}{r^2} \left[1 - \exp\left(-b r^2 / G_N m_1 m_2 \right) \right] \qquad (2.21)$$

in terms of a running Newtonian coupling $G(r)$ displayed in eq-(2.18). Our results can be generalized to higher spacetime dimensions $D = d+1 > 4$. Given the modified force $F_D(r)$ in eq-( ), the corresponding modified gravitational potential energy is given by a more complicated integral $U_D(r) = -\int_\infty^r F_D(r) dr$.

Concluding, after reviewing the basics of Non-inertial relativity theory based on the existence of a maximal proper force b, it allowed to postulate a modified Newtonian attractive gravitational force (and potential) which is $finite$ at the origin : $|F(r=0)| = b$, and which vanishes at $r = \infty$. Secondly, from the modified gravitational potential energy we were able to glean the expression for a running gravitational coupling $G(r)$ which exhibits asymptotic-freedom-like properties : $G(r=0) = 0$, and $G(r=\infty) = G_N$. No quantum corrections were necessary to decrease the strength of gravity at short distances. Thirdly, we found that for very *large* masses m_1, m_2 (compared to $\sqrt{b}$) the *threshold* in the values of r obeying $\kappa r^2 << 1$, where the non-Newtonian regime becomes manifest, becomes larger and larger as m_1, m_2 become larger and larger. Whereas for very *small* masses (compared to $\sqrt{b}$) the *threshold* in the values of r obeying $\kappa r^2 << 1$, where the non-Newtonian regime becomes manifest, becomes smaller and smaller as m_1, m_2 become smaller and smaller. In the $b = \infty$ limit one recovers the Newtonian gravitational force for all values of $r > 0$. These results were all possible by abandoning the weak equivalence principle at short distances.

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