
THE GEODESIC LINES

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Abstract. — Dans cet article, nous présentons les équations des lignes géodésiques d'une surface dans \mathbb{R}^3 puis on détermine le calcul des lignes géodésiques de l'ellipsoïde de révolution avec un exemple numérique.

Résumé. — In this article, we present the equations of the geodesic lines of a surface in \mathbb{R}^3 and then we determine the calculation of the geodesic lines of the ellipsoid of revolution with a numerical example.

To My Brother, My Sisters and My granddaughter Rayhane

Besides the main difficulty, the one that lies at the very heart of the matter, there are a host of secondary difficulties that further complicate the researcher's task. It would therefore be advantageous to first study a problem in which this main difficulty is encountered, but where all the secondary difficulties are absent. This problem is readily available: it is that of **the geodesic lines of a surface**; it is still a problem of dynamics, so the main difficulty remains; but it is the simplest of all dynamics problems.

(H. Poincaré [1]⁽¹⁾)

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⁽¹⁾**Henri Poincaré** (1854-1912): French mathematician, among the greatest of the 19th century.

1. Introduction and Notations

Let (S) be a surface defined by the parameters (u, v) with $(u, v) \in \mathcal{D}$ a domain $\subset \mathbb{R}^2$. A point $M \in (S)$ satisfies :

$$\mathbf{OM} = \mathbf{OM}(u, v) \begin{vmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{vmatrix}$$

We introduce the usual notations:

$$(1) \quad \begin{aligned} E &= \frac{\partial \mathbf{M}}{\partial u} \cdot \frac{\partial \mathbf{M}}{\partial u} = \left\| \frac{\partial \mathbf{M}}{\partial u} \right\|^2 \\ F &= \frac{\partial \mathbf{M}}{\partial u} \cdot \frac{\partial \mathbf{M}}{\partial v} \\ G &= \frac{\partial \mathbf{M}}{\partial v} \cdot \frac{\partial \mathbf{M}}{\partial v} = \left\| \frac{\partial \mathbf{M}}{\partial v} \right\|^2 \end{aligned}$$

From the relations (1), we obtain the equations:

$$(2) \quad \begin{aligned} \frac{\partial E}{\partial u} &= 2 \frac{\partial \mathbf{M}}{\partial u} \cdot \frac{\partial^2 \mathbf{M}}{\partial u^2} \\ \frac{\partial E}{\partial v} &= 2 \frac{\partial \mathbf{M}}{\partial u} \cdot \frac{\partial^2 \mathbf{M}}{\partial u \partial v} \\ \frac{\partial F}{\partial u} &= \frac{\partial^2 \mathbf{M}}{\partial u^2} \cdot \frac{\partial \mathbf{M}}{\partial v} + \frac{\partial \mathbf{M}}{\partial u} \cdot \frac{\partial^2 \mathbf{M}}{\partial u \partial v} \\ \frac{\partial F}{\partial v} &= \frac{\partial^2 \mathbf{M}}{\partial v^2} \cdot \frac{\partial \mathbf{M}}{\partial u} + \frac{\partial \mathbf{M}}{\partial v} \cdot \frac{\partial^2 \mathbf{M}}{\partial u \partial v} \\ \frac{\partial G}{\partial u} &= 2 \frac{\partial \mathbf{M}}{\partial v} \cdot \frac{\partial^2 \mathbf{M}}{\partial u \partial v} \\ \frac{\partial G}{\partial v} &= 2 \frac{\partial \mathbf{M}}{\partial v} \cdot \frac{\partial^2 \mathbf{M}}{\partial v^2} \end{aligned}$$

Let \mathbf{n} be the unit normal vector at $M(u, v)$ to the surface (S) , \mathbf{n} is given by:

$$\mathbf{n} = \frac{\frac{\partial \mathbf{M}}{\partial u} \wedge \frac{\partial \mathbf{M}}{\partial v}}{H}$$

where:

$$H = \left\| \frac{\partial \mathbf{M}}{\partial u} \wedge \frac{\partial \mathbf{M}}{\partial v} \right\|$$

Then :

$$(3) \quad ds^2 = E \cdot du^2 + 2 \cdot F \cdot du \cdot dv + G \cdot dv^2$$

The equation (3) represents the infinitesimal square of the arc length. It is ds^2 the first fundamental form.

Let (Γ) be a curve drawn on (S) and \mathbf{N} is the unit vector of the principal normal along (Γ) .

Definition 1.1. — A curve (Γ) is said to be a geodesic line of the surface (S) if and only if the vectors \mathbf{n} and \mathbf{N} are collinear [2].

It is demonstrated by the calculus of variations [3] that the geodesic line between two points of a surface (S) when it exists is the curve of minimum length joining the two points.

2. Determination of the Differential Equations of Geodesic Lines

We calculate the expression for \mathbf{N} , we obtain:

$$\mathbf{N} = R \frac{d\mathbf{T}}{ds}$$

but:

$$\mathbf{T} = \frac{d\mathbf{M}}{ds} = \frac{\partial \mathbf{M}}{\partial u} \frac{du}{ds} + \frac{\partial \mathbf{M}}{\partial v} \frac{dv}{ds}$$

It follows:

$$\frac{d\mathbf{T}}{ds} = \frac{\partial^2 \mathbf{M}}{\partial u^2} \left(\frac{du}{ds} \right)^2 + 2 \frac{\partial^2 \mathbf{M}}{\partial u \partial v} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial \mathbf{M}}{\partial u} \frac{d^2 u}{ds^2} + \frac{\partial \mathbf{M}}{\partial v} \frac{d^2 v}{ds^2} + \frac{\partial^2 \mathbf{M}}{\partial v^2} \left(\frac{dv}{ds} \right)^2$$

The condition $\mathbf{n} // \mathbf{N}$ can be written:

$$\mathbf{N} \wedge \mathbf{n} = 0$$

then :

$$R \frac{d\mathbf{T}}{ds} \wedge \left(\frac{\frac{\partial \mathbf{M}}{\partial u} \wedge \frac{\partial \mathbf{M}}{\partial u}}{H} \right) = 0$$

Using the wedge product formula:

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

we obtain:

$$\left(\frac{d\mathbf{T}}{ds} \cdot \frac{\partial \mathbf{M}}{\partial v} \right) \frac{\partial \mathbf{M}}{\partial u} - \left(\frac{d\mathbf{T}}{ds} \cdot \frac{\partial \mathbf{M}}{\partial u} \right) \frac{\partial \mathbf{M}}{\partial v} = 0$$

But $\frac{\partial \mathbf{M}}{\partial u}$ and $\frac{\partial \mathbf{M}}{\partial v}$ form a basis of the tangent plane at M , hence the two conditions:

$$\frac{d\mathbf{T}}{ds} \cdot \frac{\partial \mathbf{M}}{\partial v} = 0 \quad \text{and} \quad \frac{d\mathbf{T}}{ds} \cdot \frac{\partial \mathbf{M}}{\partial u} = 0$$

This gives us two second-order differential equations:

$$(4) \quad \frac{\partial^2 \mathbf{M}}{\partial u^2} \cdot \frac{\partial \mathbf{M}}{\partial v} \left(\frac{du}{ds} \right)^2 + F \frac{d^2 u}{ds^2} + 2 \frac{\partial^2 \mathbf{M}}{\partial u \partial v} \cdot \frac{\partial \mathbf{M}}{\partial v} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial^2 \mathbf{M}}{\partial v^2} \cdot \frac{\partial \mathbf{M}}{\partial v} \left(\frac{dv}{ds} \right)^2 + G \frac{d^2 v}{ds^2} = 0$$

and :

$$(5) \quad \frac{\partial^2 \mathbf{M}}{\partial v^2} \cdot \frac{\partial \mathbf{M}}{\partial u} \left(\frac{dv}{ds} \right)^2 + F \frac{d^2 v}{ds^2} + 2 \frac{\partial^2 \mathbf{M}}{\partial u \partial v} \cdot \frac{\partial \mathbf{M}}{\partial u} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial^2 \mathbf{M}}{\partial u^2} \cdot \frac{\partial \mathbf{M}}{\partial u} \left(\frac{du}{ds} \right)^2 + E \frac{d^2 u}{ds^2} = 0$$

Let :

$$\begin{aligned} E'_u &= \frac{\partial E}{\partial u}; & E'_v &= \frac{\partial E}{\partial v}; & F'_u &= \frac{\partial F}{\partial u} \\ F'_v &= \frac{\partial F}{\partial v}; & G'_u &= \frac{\partial G}{\partial u}; & G'_v &= \frac{\partial G}{\partial v} \end{aligned}$$

and we use equations (2), (4) and (5), these last 2 equations can be written as:

$$(6) \quad \boxed{\left(F'_u - \frac{E'_v}{2}\right) \left(\frac{du}{ds}\right)^2 + F \frac{d^2u}{ds^2} + G'_u \frac{du}{ds} \frac{dv}{ds} + \frac{G'_v}{2} \left(\frac{dv}{ds}\right)^2 + G \frac{d^2v}{ds^2} = 0}$$

$$(7) \quad \boxed{\left(F'_v - \frac{G'_u}{2}\right) \left(\frac{dv}{ds}\right)^2 + F \frac{d^2v}{ds^2} + E'_v \frac{dv}{ds} \frac{du}{ds} + \frac{E'_u}{2} \left(\frac{du}{ds}\right)^2 + E \frac{d^2u}{ds^2} = 0}$$

3. Determination of the Geodesic Lines of the Ellipsoid of Revolution

We now consider as a surface the ellipsoid of revolution which we parameterize as follows:

$$\begin{cases} X = N \cos\varphi \cos\lambda \\ Y = N \cos\varphi \sin\lambda \\ Z = N(1 - e^2) \sin\varphi \end{cases}$$

with :

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2\varphi}} = aW^{-1/2}$$

is the radius of curvature of the major normal with:

$$W = 1 - e^2 \sin^2\varphi$$

We note:

$$r = N \cos\varphi$$

the radius of the parallel of latitude φ and ρ the radius of curvature of the meridian given by:

$$\rho = \frac{a(1 - e^2)}{(1 - e^2 \sin^2\varphi) \sqrt{1 - e^2 \sin^2\varphi}} = a(1 - e^2)W^{-3/2}$$

So the first fundamental form is written:

$$ds^2 = \rho^2 d\varphi^2 + r^2 d\lambda^2$$

Taking $u = \varphi$ and $v = \lambda$ as variables, we obtain:

$$(8) \quad E = E(\varphi) = \rho^2, \quad F = 0, \quad G = r^2$$

$$(9) \quad E'_\varphi = 2\rho\rho', \quad E'_\lambda = 0, \quad F'_\varphi = F'_\lambda = 0, \quad G'_\varphi = 2rr' = -2r\rho \sin\varphi, \quad G'_\lambda = 0$$

Then equations (6) and (7) become:

$$(10) \quad -2r\rho \sin\varphi \frac{d\varphi}{ds} \frac{d\lambda}{ds} + r^2 \frac{d^2\lambda}{ds^2} = 0$$

$$(11) \quad r\rho \sin\varphi \left(\frac{d\lambda}{ds}\right)^2 + \rho\rho' \left(\frac{d\varphi}{ds}\right)^2 + \rho^2 \frac{d^2\varphi}{ds^2} = 0$$

The first equation can be written as:

$$(12) \quad \frac{d}{ds} \left(r^2 \frac{d\lambda}{ds} \right) = 0$$

and its integration gives:

$$(13) \quad r^2 \frac{d\lambda}{ds} = C = \text{constante}$$

We then find Clairaut's relationship⁽²⁾ [2]:

$$(14) \quad \boxed{r \cdot \sin Az = \text{constante} = C = a \sin Aze}$$

where Az is the azimuth of the geodesic line at point M and Aze its initial azimuth at point M_0 at the equator.

The equation (11) is written as:

$$(15) \quad \rho \left(r \sin \varphi \left(\frac{d\lambda}{ds} \right)^2 + \rho' \left(\frac{d\varphi}{ds} \right)^2 + \rho \frac{d^2\varphi}{ds^2} \right) = 0$$

It gives:

- $\rho = 0$ point M is on the equator: $\varphi = 0$ and $r = a$ the semi-major axis of the ellipsoid and the equation (10) becomes:

$$(16) \quad \frac{d^2\lambda}{ds^2} = 0$$

and its integration gives:

$$(17) \quad \lambda - \lambda_0 = l(s - s_0)$$

the point M describes the equator and the geodesic line is the great circle of radius a .

- $\rho \neq 0$, the point M is not on the equator, the equation (11) is written as follows:

$$(18) \quad \rho \frac{d^2\varphi}{ds^2} + \rho' \left(\frac{d\varphi}{ds} \right)^2 + r \sin \varphi \left(\frac{d\lambda}{ds} \right)^2 = 0$$

To integrate (18), we use a new function, namely:

$$(19) \quad Z = \frac{d\lambda}{d\varphi}$$

From (13), we obtain:

$$\frac{d\varphi}{ds} = \frac{d\varphi}{d\lambda} \frac{d\lambda}{ds} = \frac{C}{r^2} \frac{d\varphi}{d\lambda} = \frac{C}{r^2 Z}$$

let:

$$(20) \quad \frac{d\varphi}{ds} = \frac{C}{r^2 Z}$$

We now express the second derivative $d^2\varphi/ds^2$:

$$(21) \quad \frac{d^2\varphi}{ds^2} = \frac{d}{ds} \left(\frac{d\varphi}{ds} \right) = \frac{d}{d\varphi} \left(\frac{d\varphi}{ds} \right) \frac{d\varphi}{ds} = \frac{1}{2} \frac{d}{d\varphi} \left(\frac{d\varphi}{ds} \right)^2$$

Equation (18) can be written using (13) and (21):

$$(22) \quad \frac{\rho}{2} \frac{d}{d\varphi} \left[\left(\frac{d\varphi}{ds} \right)^2 \right] + \rho' \left(\frac{d\varphi}{ds} \right)^2 + \sin \varphi \left(\frac{C^2}{r^3} \right) = 0$$

Let :

$$(23) \quad U = \left(\frac{d\varphi}{ds} \right)^2$$

⁽²⁾ **Alexis Claude de Clairaut** (1713-1765): French mathematician, astronomer and geophysicist.

The equation (22) becomes:

$$(24) \quad \frac{\rho}{2} \frac{dU}{d\varphi} + \rho' U = -\frac{C^2 \sin\varphi}{r^3}$$

Equation (24) is a first-order linear differential equation with a non-homogeneous term. Solving it without a non-homogeneous term gives:

$$(25) \quad U = \frac{k}{\rho^2}$$

Using the second member of (24), we consider that k is a function of φ , we then have:

$$(26) \quad U = \frac{1}{\rho^2} \left(k_0 - \frac{C^2}{r^2} \right) = \frac{k_0 r^2 - C^2}{\rho^2 r^2}$$

with k_0 being the constant of integration. Since U is a positive function, we must have:

$$(27) \quad k_0 r^2 - C^2 > 0$$

Returning to equation (23), we obtain:

$$(28) \quad U = \left(\frac{d\varphi}{ds} \right)^2 = \frac{k_0 r^2 - C^2}{\rho^2 r^2}$$

Using equations (20) and (28), we obtain:

$$(29) \quad \left(\frac{d\varphi}{ds} \right)^2 = \frac{k_0 r^2 - C^2}{\rho^2 r^2} = \left(\frac{C}{r^2 Z} \right)^2 = \frac{C^2}{r^4 Z^2} = \frac{C^2}{r^4} \left(\frac{d\varphi}{d\lambda} \right)^2$$

it gives :

$$(30) \quad \left(\frac{d\lambda}{d\varphi} \right)^2 = \frac{\rho^2}{r^2} \frac{C^2}{k_0 r^2 - C^2}$$

To determine the value of k_0 , we express $\frac{d\lambda}{ds}$ using equations (13) and (30). We write ds^2 :

$$ds^2 = \rho^2 d\varphi^2 + r^2 d\lambda^2 = \frac{r^2 (k_0 r^2 - C^2)}{C^2} d\lambda^2 + r^2 d\lambda^2$$

let:

$$(31) \quad ds^2 = \frac{r^4 k_0}{C^2} d\lambda^2 \Rightarrow \left(\frac{d\lambda}{ds} \right)^2 = \frac{C^2}{k_0 r^4}$$

According to (13):

$$\left(\frac{d\lambda}{ds} \right)^2 = \frac{C^2}{r^4}$$

Therefore, $k_0 = 1$ and consequently:

$$(32) \quad \left(\frac{d\lambda}{d\varphi} \right)^2 = \frac{\rho^2}{r^2} \frac{C^2}{r^2 - C^2}$$

To be able to integrate the previous equation, we express $r^2 - C^2$, hence:

$$(33) \quad \frac{r^2 - C^2 = N^2 \cos^2 \varphi - C^2 = \frac{a^2 \cos^2 \varphi}{1 - e^2 \sin^2 \varphi} - C^2 = (a^2 - C^2) \left(1 - \frac{a^2 - C^2 e^2}{a^2 - C^2} \sin^2 \varphi \right)}{W}$$

Let :

$$(34) \quad k^2 = \frac{a^2 - C^2 e^2}{a^2 - C^2}$$

Hence :

$$(35) \quad r^2 - C^2 = (a^2 - C^2)(1 - k^2 \sin^2 \varphi) / W$$

We note that the coefficient k is greater than 1, therefore the geodetic latitude φ remains less than the latitude φ_1 defined by $\sin \varphi_1 = 1/k$.

Then the equation (32) can be written as:

$$(36) \quad \left(\frac{d\lambda}{d\varphi} \right)^2 = \frac{(1 - e^2)^2 C^2}{(a^2 - C^2) \cos^2 \varphi (1 - e^2 \sin^2 \varphi) (1 - k^2 \sin^2 \varphi)}$$

Therefore, by replacing C with $a \cdot \sin(Aze)$ and since $tg(Aze)$ has the same sign as $(d\lambda/d\varphi)$, we can then write:

$$(37) \quad \frac{d\lambda}{d\varphi} = \frac{(1 - e^2) tg(Aze)}{\cos \varphi \sqrt{(1 - e^2 \sin^2 \varphi) (1 - k^2 \sin^2 \varphi)}}$$

Either by integrating between 0 and φ :

$$\begin{aligned} \lambda - \lambda_e &= \int_0^\varphi \frac{(1 - e^2) tg(Aze)}{\cos t \sqrt{(1 - e^2 \sin^2 t) (1 - k^2 \sin^2 t)}} dt = \\ &(1 - e^2) tg(Aze) \int_0^\varphi \frac{dt}{\cos t \sqrt{(1 - e^2 \sin^2 t) (1 - k^2 \sin^2 t)}} \end{aligned}$$

or again:

$$(38) \quad \lambda - \lambda_e = (1 - e^2) tg(Aze) \int_0^\varphi \frac{dt}{\cos t \sqrt{(1 - e^2 \sin^2 t) (1 - k^2 \sin^2 t)}}$$

Taking $w = \sin t$ as the variable, the integral (38) becomes:

$$(39) \quad \boxed{\lambda - \lambda_e = (1 - e^2) tg(Aze) \int_0^{\sin \varphi} \frac{dw}{(1 - w^2) \sqrt{(1 - e^2 w^2) (1 - k^2 w^2)}}$$

We now seek to express the curvilinear abscissa s as a function of φ . The expression for ds^2 is equal to:

$$ds^2 = \rho^2 d\varphi^2 + r^2 d\lambda^2 = \rho^2 d\varphi^2 + \frac{C^2}{r^2} ds^2$$

let:

$$(40) \quad ds^2 = \frac{r^2 \rho^2 d\varphi^2}{r^2 - C^2} = \frac{a^2 (1 - e^2)^2 \cos^2 \varphi d\varphi^2}{\cos^2(Aze) (1 - e^2 \sin^2 \varphi)^3 (1 - k^2 \sin^2 \varphi)}$$

Hence :

$$(41) \quad ds = \frac{a(1 - e^2) \cos \varphi d\varphi}{\cos(Aze) (1 - e^2 \sin^2 \varphi) \sqrt{(1 - k^2 \sin^2 \varphi) (1 - e^2 \sin^2 \varphi)}}$$

Taking $t = \sin\varphi$ as the new variable, the integral of (41) gives, taking as the origin of the curvilinear abscissa s a point on the equator:

$$(42) \quad s = \frac{a(1-e^2)}{\cos Aze} \int_0^{\sin\varphi} \frac{dt}{(1-e^2t^2)\sqrt{(1-k^2t^2)(1-e^2t^2)}}$$

The integrals (39) and (42) are called elliptic integrals of the third kind.

4. Applications to the Direct and Inverse Problems of Geodesic Line Calculation

In this second part, we will numerically treat the application of the previous formulas in the resolution of the so-called direct and inverse problems of the calculation of geodesic lines.

4.1. The Direct Problem. — We give :

- (φ_1, λ_1) of one point M_1 ,
- the length s of the geodesic line from M_1 to M_2 ,
- the geodetic azimuth Az_1 of the geodesic line from M_1 to M_2 .

Calculate:

- the geodetic coordinates (φ_2, λ_2) of M_2 ,
- the geodetic azimuth Az_2 at M_2 .

Solution:

1. Calculation of the constant C , $C = N(\varphi_1) \cdot \cos\varphi_1 \cdot \sin Az_1 = a \cdot \sin(Aze)$ hence Aze and k .
2. Determination of φ_2 from:

$$\Delta s = \frac{a(1-e^2)}{\cos Aze} \frac{\cos\varphi_1 \Delta\varphi}{(1-e^2 \sin^2\varphi_1) \sqrt{(1-k^2 \sin^2\varphi_1)(1-e^2 \sin^2\varphi_1)}}$$

with $\Delta\varphi = \varphi_2 - \varphi_1$.

3. Having φ_2 , we calculate λ_2 using :

$$\lambda_2 - \lambda_1 = (1-e^2) \operatorname{tg}(Aze) \int_{\sin\varphi_1}^{\sin\varphi_2} \frac{dw}{(1-w^2) \sqrt{(1-e^2w^2)(1-k^2w^2)}}$$

4. The calculation of Az_2 by $\sin(Az_2) = C/r(\varphi_2)$.

4.2. The Inverse Problem. — Given the coordinates (φ_1, λ_1) and (φ_2, λ_2) of two points M_1 and M_2 , calculate:

- the length s of the geodesic line from M_1 to M_2 ,
- the azimuth Az_1 at M_1 ,
- the geodetic azimuth Az_2 at M_2 .

Solution:

1. We need to calculate the constant C . From equation (32), we can write that:

$$\left(\frac{\Delta\lambda}{\Delta\varphi}\right)^2 = \frac{\rho^2(\varphi_1)}{r^2(\varphi_1)} \frac{C^2}{(r^2(\varphi_1) - C^2)} = \frac{(\lambda_2 - \lambda_1)^2}{(\varphi_2 - \varphi_1)^2}$$

It gives C :

$$C = \frac{\frac{r^2}{\rho} \frac{\Delta\lambda}{\Delta\varphi}}{\sqrt{1 + \frac{r^2}{\rho^2} \left(\frac{\Delta\lambda}{\Delta\varphi}\right)^2}}$$

Considering the azimuth to be between 0 and π , therefore Az is positive, and C is positive. Calculating it for φ_1 and φ_2 , we obtain C by taking the average value:

$$C = \frac{C_1(\varphi_1) + C_2(\varphi_2)}{2}$$

2. Therefore, we obtain the value of k by (34):

$$k = \frac{a^2 - C^2 e^2}{a^2 - C^2}$$

3. Given C , we have by (14), Az_1 and Az_2 :

$$\sin Az_1 = \frac{C}{r(\varphi_1)} \quad \text{and} \quad \sin Az_2 = \frac{C}{r(\varphi_2)}$$

4. then, we obtain also Az_e :

$$\sin Az_e = \frac{C}{a}$$

5. Finally, equation (42) determines s .

The process is iterated.

4.3. Calculation of the equation (42). — This section calculates in detail:

$$s = \frac{a(1 - e^2)}{\cos Az_e} \int_0^{\sin\varphi} \frac{dt}{(1 - e^2 t^2) \sqrt{(1 - k^2 t^2)(1 - e^2 t^2)}}$$

For $|x| < 1$, we have the following Taylor series expansions:

$$(43) \quad \frac{1}{(1+x)^{3/2}} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + \frac{315}{128}x^4 + \dots$$

$$(44) \quad \frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128} + \dots$$

By taking $x = -e^2 t^2$ and $x = k^2 t^2$, we obtain:

$$(45) \quad \frac{1}{(1 - e^2 t^2)^{3/2}} = 1 + \frac{3}{2}e^2 t^2 + \frac{15}{8}e^4 t^4 + \frac{35}{16}e^6 t^6 + \frac{315}{128}e^8 t^8 + \dots$$

$$\frac{1}{\sqrt{1 - k^2 t^2}} = 1 + \frac{k^2 t^2}{2} + \frac{3k^4 t^4}{8} + \frac{5k^6 t^6}{16} + \frac{35k^8 t^8}{128} + \dots$$

Hence:

$$(46) \quad \frac{1}{(1 - e^2 t^2) \sqrt{(1 - k^2 t^2)(1 - e^2 t^2)}} = 1 + \frac{k^2 + 3e^2}{2} t^2 + \frac{3k^4 + 6e^2 k^2 + 15e^4}{8} t^4 +$$

$$\frac{5k^6 + 9k^4 e^2 + 15k^2 e^4 + 35e^6}{16} t^6 +$$

$$\frac{35k^8 + 60k^6 e^2 + 90k^4 e^4 + 140k^2 e^6 + 315e^8}{128} t^8 + \dots$$

or even to order 4:

$$(47) \quad \frac{1}{(1-e^2t^2)\sqrt{(1-k^2t^2)(1-e^2t^2)}} = 1 + mt^2 + nt^4 + \dots$$

with:

$$m = \frac{k^2 + 3e^2}{2}; \quad n = \frac{3k^4 + 6e^2k^2 + 15e^4}{8}$$

4.4. Calculation of the expression (39). — We have:

$$\lambda - \lambda_e = (1 - e^2)tg(Aze) \int_0^{\sin\varphi} \frac{dw}{(1-w^2)\sqrt{(1-e^2w^2)(1-k^2w^2)}}$$

In our case:

$$\lambda_2 - \lambda_1 = (1 - e^2)tg(Aze) \int_{\sin\varphi_1}^{\sin\varphi_2} \frac{dt}{(1-t^2)\sqrt{(1-e^2t^2)(1-k^2t^2)}}$$

But according to (44):

$$\frac{1}{\sqrt{1-e^2t^2}} = 1 + \frac{1}{2}e^2t^2 + \frac{3}{8}e^4t^4 + \frac{5}{16}e^6t^6 + \frac{35}{128}e^8t^8 + \dots$$

and :

$$\frac{1}{\sqrt{1-k^2t^2}} = 1 + \frac{k^2t^2}{2} + \frac{3k^4t^4}{8} + \frac{5k^6t^6}{16} + \frac{35k^8t^8}{128} + \dots$$

and for $(1-t^2)^{-1}$, we obtain:

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + t^8 + \dots$$

Then:

$$\begin{aligned} \frac{1}{(1-t^2)\sqrt{(1-e^2t^2)(1-k^2t^2)}} &= 1 + \frac{2+k^2+e^2}{2}t^2 + \\ &\quad \frac{8+4k^2+4e^2+3k^4+2e^2k^2+3e^4}{8}t^4 + \\ &\quad \frac{16+8k^2+8e^2+6k^4+4e^2k^2+6e^4+5k^6+3k^4e^2+3k^2e^4+5e^6}{16}t^6 + \dots \end{aligned}$$

Which is written in the form:

$$(48) \quad \frac{1}{(1-t^2)\sqrt{(1-e^2t^2)(1-k^2t^2)}} = 1 + \alpha t^2 + \beta t^4 + \gamma t^6 + \dots$$

with :

$$(49) \quad \begin{cases} \alpha = \frac{2+k^2+e^2}{2} \\ \beta = \frac{8+4k^2+4e^2+3k^4+2e^2k^2+3e^4}{8} \\ \gamma = \frac{16+8k^2+8e^2+6k^4+4e^2k^2+6e^4+5k^6+3k^4e^2+3k^2e^4+5e^6}{16} \end{cases}$$

5. Processing an numerical example

5.1. The Direct Problem. — We consider the point M_1 with :

- $\varphi_1 = 10.45498299 \text{ gr}$,
- $\lambda_1 = 9.59542429 \text{ gr}$,
- $Az_1 = 249.310168 \text{ gr}$,
- $s = 16255.206 \text{ m}$.

Solution:

$$- C = N(\varphi_1) \cdot \cos\varphi_1 \cdot \sin Az_1 = -4401454.883 \text{ m},$$

$$- Az_e = 248.48428278 \text{ gr},$$

$$- k = \sqrt{\frac{a^2 - C^2 e^2}{a^2 - C^2}} = 1.98008417$$

-To calculate φ_2 , let $\Delta\varphi = \varphi_2 - \varphi_1$, and $s = \Delta s$. Using (47), we then have the equation

:

$$\frac{\Delta s \cdot \cos Az_e}{a(1-e^2)} = \int_{\sin\varphi_1}^{\sin\varphi_2} \frac{dt}{(1-e^2t^2)\sqrt{(1-k^2t^2)(1-e^2t^2)}} = \int_{\sin\varphi_1}^{\sin\varphi_2} (1+mt+nt^2)dt$$

To the order 1, we obtain : $\frac{\Delta s \cdot \cos Az_e}{a(1-e^2)} = \sin\varphi_2 - \sin\varphi_1 = -0.001860569 \implies \varphi_2 = 10.20545272 \text{ gr}$.

$$- \lambda_2 - \lambda_1 = \frac{(1-e^2)tgAz_e(\sin\varphi_2 - \sin\varphi_1)}{C} = -0.00176195 \text{ gr} \implies \lambda_2 = 9.59366234 \text{ gr}.$$

$$- \sin Az_2 = \frac{C}{r(\varphi_2)} = \frac{C}{N(\varphi_2)\cos\varphi_2} \implies Az_2 = 249.27200916 \text{ gr}.$$

5.2. The Inverse Problem. — I invite the reader to do the calculation of the inverse problem. We consider the point M_1 with :

- $\varphi_1 = 10.45498299 \text{ gr}$,
 - $\lambda_1 = 9.59542429 \text{ gr}$,
- and the point M_2 with :

- $\varphi_2 = 10.20545272 \text{ gr}$,
- $\lambda_2 = 9.59366234 \text{ gr}$.

To determine the distance $s = M_1M_2$ and the geodetic azimuth of the geodesic line M_1M_2 .

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