Residual Cancellation in Ordered Abelian Groups

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Abstract

This paper develops the ordered-structure core of the Theory of Residual Cancellation (TRC). The main idea is that, in a suitable ordered setting, the common part of two positive elements can be recovered from ordered difference and positive-part structure rather than assumed independently. Let G be an ordered abelian group equipped with a positive-part operation $u \mapsto u^+$, and define $u^- := (-u)^+$. For $x, y \in G$, define the TRC common-part candidate

$$m(x, y) := x - (x - y)^+.$$

Under a positive/negative-part decomposition axiom, this operation is symmetric and yields a two-sided residual/common decomposition. Under the additional assumption that the positive-part map is monotone, the matched-content operation becomes monotone, maximal among common lower bounds, and equal to the meet on positive pairs. This gives an axiom-separated theorem ladder: one axiom governs the algebraic decomposition layer, while the second upgrades the decomposition into genuine order-theoretic meet recovery. The result identifies the abstract core of TRC as a compatibility principle between difference, positive part, and common part.

1 Introduction

The elementary numerical identity

$$x = (x - y)^+ + \min(x, y), \qquad y = (y - x)^+ + \min(x, y)$$

suggests that a positive pair can be decomposed into a residual part and a common part. At the numerical level this is elementary. The deeper question is whether the same structure reflects a more general ordered principle.

The purpose of this paper is to isolate the ordered-structure core of that principle. We ask whether the common part of a positive pair can be recovered from ordered difference and positive-part structure alone, and if so, under what assumptions. This leads to a clean axiom-separated theory. One axiom yields a symmetric two-sided decomposition formula. A second axiom transports order through the positive-part operation and upgrades the decomposition into maximality and meet recovery.

The paper is deliberately focused. It does not address process theory, descriptor theory, or applications. Its sole purpose is to present the ordered-structure core in theorem-driven form.

2 Ambient Setting and Axioms

Let G be an ordered abelian group. We write G_+ for its positive cone.

Assume there is an operation

$$u \mapsto u^+$$

on G, and define

$$u^- := (-u)^+.$$

For $x, y \in G$, define the TRC common-part candidate by

$$m(x, y) := x - (x - y)^+.$$

We shall use the following two assumptions.

(A1) *Positive/negative-part decomposition.* For every $u \in G$,

$$u = u^+ - u^-, \qquad u^- = (-u)^+.$$

(A2) *Monotonicity of positive part.* For all $u, v \in G$,

$$u \le v \implies u^+ \le v^+.$$

Remark 2.1. The role of (A1) is algebraic: it yields a symmetric common-part formula and a two-sided decomposition. The role of (A2) is order-theoretic: it allows order information to pass through the positive-part map.

3 Consequences of $(A1)$ Alone

This section develops the algebraic layer of the theory.

Theorem 3.1 (Symmetry)**.** *Assume* (A1)*. Then for all $x, y \in G$,*

$$m(x, y) = y - (y - x)^+.$$

In particular,

$$m(x, y) = m(y, x).$$

Proof. By (A1),

$$x - y = (x - y)^+ - (x - y)^-.$$

Since

$$(x - y)^- = (-(x - y))^+ = (y - x)^+,$$

we obtain

$$x - y = (x - y)^+ - (y - x)^+.$$

Rearranging gives

$$x - (x - y)^+ = y - (y - x)^+.$$

The left-hand side is $m(x, y)$ and the right-hand side is $m(y, x)$. □

Corollary 3.2 (Two-sided TRC decomposition)**.** *Assume* (A1)*. Then for all $x, y \in G$,*

$$x = (x - y)^+ + m(x, y), \qquad y = (y - x)^+ + m(x, y).$$

Proof. The first identity is the definition of $m(x, y)$. The second follows from Theorem 3.1. □

Corollary 3.3 (Idempotence and zero law)**.** *Assume* (A1)*. If $x \in G_+$, then*

$$m(x, x) = x, \qquad m(x, 0) = 0.$$

Proof. Since $(x-x)^+ = 0$,

$$m(x,x) = x - (x-x)^+ = x.$$

Also,

$$m(x,0) = x - (x-0)^+ = x - x = 0.$$

□

Corollary 3.4 (Boundedness on positive pairs). *Assume* (A1). *If* $x, y \in G_+$, *then*

$$m(x,y) \le x, \qquad m(x,y) \le y.$$

Proof. Since $(x-y)^+ \in G_+$,

$$m(x,y) = x - (x-y)^+ \le x.$$

By symmetry,

$$m(x,y) = y - (y-x)^+ \le y.$$

□

Remark 3.5. Under (A1) alone, TRC already yields a symmetric common-part formula, a two-sided decomposition, idempotence, the zero law, and boundedness on positive pairs.

4 Consequences of $(A1)$ and $(A2)$

This section develops the order-theoretic strengthening.

Theorem 4.1 (Monotonicity in the first variable). *Assume* (A1) *and* (A2). *If* $x, x', y \in G_+$ *and* $x \le x'$, *then*

$$m(x,y) \le m(x',y).$$

Proof. By Theorem 3.1,

$$m(x,y) = y - (y-x)^+, \qquad m(x',y) = y - (y-x')^+.$$

Since $x \le x'$, we have

$$y - x' \le y - x.$$

By (A2),

$$(y-x')^+ \le (y-x)^+.$$

Subtracting from y yields

$$m(x,y) \le m(x',y).$$

□

Theorem 4.2 (Monotonicity in the second variable). *Assume* (A1) *and* (A2). *If* $x, y, y' \in G_+$ *and* $y \le y'$, *then*

$$m(x,y) \le m(x,y').$$

Proof. We have

$$m(x, y) = x - (x - y)^+, \qquad m(x, y') = x - (x - y')^+.$$

Since $y \le y'$, we obtain

$$x - y' \le x - y.$$

By (A2),

$$(x - y')^+ \le (x - y)^+.$$

Subtracting from x yields

$$m(x, y) \le m(x, y').$$

□

Corollary 4.3 (Full monotonicity)**.** *Assume* (A1) *and* (A2)*. If $x, x', y, y' \in G_+$ and*

$$x \le x', \qquad y \le y',$$

then

$$m(x, y) \le m(x', y').$$

Proof. Apply Theorem 4.1 and then Theorem 4.2. □

Theorem 4.4 (Maximality on positive pairs)**.** *Assume* (A1) *and* (A2)*. If $x, y, z \in G_+$ and*

$$z \le x, \qquad z \le y,$$

then

$$z \le m(x, y).$$

Proof. By Corollary 3.3,

$$m(z, z) = z.$$

By Corollary 4.3 and the assumptions $z \le x$, $z \le y$,

$$m(z, z) \le m(x, y).$$

Hence

$$z \le m(x, y).$$

□

Theorem 4.5 (Meet recovery on positive pairs)**.** *Assume* (A1) *and* (A2)*. Then for all $x, y \in G_+$, the element $m(x, y)$ is the greatest lower bound of x and y in G_+. In particular,*

$$m(x, y) = x \wedge y.$$

Proof. By Corollary 3.4, $m(x, y)$ is a lower bound of x and y. By Theorem 4.4, every positive lower bound z of x and y satisfies $z \le m(x, y)$. Thus $m(x, y)$ is the greatest lower bound of x and y in G_+. □

Corollary 4.6 (Ordered TRC decomposition on positive pairs)**.** *Assume* (A1) *and* (A2)*. Then for all $x, y \in G_+$,*

$$x = (x - y)^+ + (x \wedge y), \qquad y = (y - x)^+ + (x \wedge y).$$

Proof. Combine Corollary 3.2 with Theorem 4.5. □

5 Axiom Separation Theorem

Theorem 5.1 (Axiom Separation)**.** *Let G be an ordered abelian group with a positive-part operation $u \mapsto u^+$, define $u^- := (-u)^+$, and define*

$$m(x, y) := x - (x - y)^+.$$

Then:

(i) *Under* (A1) *alone, TRC yields:*

(a) *symmetry of m,*
(b) *two-sided decomposition,*
(c) *boundedness on positive pairs,*
(d) *idempotence,*
(e) *the zero law.*

(ii) *Under* (A1) *and* (A2)*, TRC additionally yields:*

(a) *monotonicity of m,*
(b) *maximality of m on positive pairs,*
(c) *meet recovery on positive pairs:*

$$m(x, y) = x \wedge y.$$

Proof. Part (i) is given by Theorem 3.1, Corollary 3.2, Corollary 3.4, and Corollary 3.3. Part (ii) is given by Corollary 4.3, Theorem 4.4, and Theorem 4.5. □

6 Examples

Remark 6.1 (Numbers)**.** If $G = \mathbb{R}$ with its usual order, then

$$u^+ = \max(u, 0), \qquad m(x, y) = x - (x - y)^+ = \min(x, y).$$

So the ordered TRC decomposition reduces to the familiar numerical identity

$$x = (x - y)^+ + \min(x, y), \qquad y = (y - x)^+ + \min(x, y).$$

Remark 6.2 (Functions)**.** If G is a lattice of real-valued functions under pointwise order, then

$$(f - g)^+(t) = \max(f(t) - g(t), 0),$$

and the recovered common part is pointwise meet:

$$m(f, g)(t) = \min(f(t), g(t)).$$

Remark 6.3 (Classical lattice identity)**.** In a lattice-ordered abelian group, the formula

$$m(x, y) = x - (x - y)^+$$

is exactly the meet:

$$m(x, y) = x \wedge y.$$

Thus the TRC construction recovers a standard lattice operation from positive-part structure.

7 Conclusion

The ordered-structure branch of TRC may be summarized as follows.

- The operation
  $$m(x, y) := x - (x - y)^+$$
  is the natural TRC candidate for the common part.
- Under (A1), this candidate is symmetric and yields a two-sided decomposition formula.
- Under (A1) and (A2), the same candidate becomes monotone, maximal among common lower bounds, and equal to the meet on positive pairs.

Thus TRC begins as a decomposition principle and becomes a meet-recovery principle once positive part transports order.