New Research on Divergence and Curl

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Abstract: Maxwell's equations are an integration of divergence, curl, and classical electromagnetism. This study reveals that divergence and curl can apply to static electromagnetic fields, but not to dynamic, time-varying electromagnetic fields.

Keywords: Maxwell's equations, Classical electromagnetism, Divergence, Curl, Vector.

Maxwell's equations

Maxwell's equations [1] [2] [3] [4] [5] were the greatest scientific achievement of physics in the 19th century. Maxwell's equations describe the relationship among charge, current, an electric field, and a magnetic field, predicting that light is an electromagnetic wave. The following are Maxwell's equations in integral form.

$$\oiint_s E \cdot ds = \frac{Q}{\varepsilon_0} \qquad (1\text{-}1A)$$

$$\oiint_s B \cdot ds = 0 \qquad (1\text{-}2A)$$

$$\oint_L E \cdot d\ell = -\frac{d\Phi_B}{dt} \qquad (1\text{-}3A)$$

$$\oint_L B \cdot d\ell = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \qquad (1\text{-}4A)$$

Equation (1-1A) is Gauss's law for the electric flux. It states that the electric flux passing through a certain closed surface is proportional to the amount of charge Q enclosed by the closed surface, and the coefficient is $1/\varepsilon_0$.

Equation (1-2A) is the law for a magnetic flux. It states that the magnetic flux passing through a certain closed surface must be equal to 0. Since there is no magnetic monopole in nature, the N pole and the S pole cannot be separated.

Equation (1-3A) is Faraday's law of electromagnetic induction. The law states that a magnetic field induces an electric field; that is, the induced electromotive force in a coil is proportional to the rate of change of the magnetic flux passing through the cross-section of the coil, and the coefficient is -1.

Equation (1-4A) is the Ampere-Maxwell law. According to Ampere's circuital law, the line integral of the magnetic induction intensity **B** along a closed curve L is equal to μ_0 multiplied by the conducting current passing through the closed curve L. Based on the polarization of dielectric, Maxwell introduced the "displacement current" hypothesis in 1865 and defined that the "displacement current" is proportional to the rate of change of the electric flux. Therefore, Maxwell believed that a magnetic field could be generated by a conducting current, or by a changing electric field, without the participation of charges.

According to the Ampere-Maxwell law in Equation (1-4A), a changing electric field induces a magnetic field. According to Faraday's law in Equation (1-3A), a changing magnetic field induces an electric field. The electric field and the magnetic field are closely linked and induce each other to form electromagnetic waves.

Maxwell introduced divergence and curl operators, which allow the integral form of the equations to be converted into differential form. The following are Maxwell's equations in integral form.

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \qquad (1\text{-}1B)$$

$$\nabla \cdot B = 0 \qquad (1\text{-}2B)$$

$$\nabla \times E = \frac{\partial B}{\partial t} \qquad (1\text{-}3B)$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \qquad (1\text{-}4B)$$

Definition of the divergence of a vector F(x, y, z)

Let a closed surface S enclose a volume ΔV. As ΔV → 0, the divergence of the vector **F**(x, y, z) is defined as the limit of the ratio of the flux of **F**(x, y, z) through S to the volume ΔV:

$$\text{div}\, \boldsymbol{F}(x, y, z) = \lim_{\Delta V \to 0} \frac{\oint \boldsymbol{F}(x, y, z) \bullet dS}{\Delta V} \qquad (2-1)$$

Definition of the curl of a vector F (x, y, z)

Let a closed curve L enclose an area ΔS. As ΔS → 0, the curl of the vector **F**(x, y, z) along the surface normal **n** is defined as the limit of the ratio of the circulation of **F**(x, y, z) around L to the area ΔS:

$$\text{rot}\, \boldsymbol{n}\, \boldsymbol{F}(x, y, z) = \lim_{\Delta S \to 0} \frac{\oint \boldsymbol{F}(x, y, z) \bullet \mathrm{d}l}{\Delta S} \qquad (2-2)$$

Definition of the operator ∇

In the rectangular coordinate system, the operator ∇ is defined as follows:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} \qquad (2-3)$$

where ***i, j,*** and ***k*** are the unit vectors in the rectangular coordinate system.

Using the ∇ operator, the divergence and curl of the vector **F**(x, y, z) can be expressed as follows:

$$\text{div}\, \boldsymbol{F}(x,y,z) = \nabla \bullet \boldsymbol{F}(x,y,z) \qquad (2-4)$$

$$\text{rot}\, \boldsymbol{n}\, \boldsymbol{F}(x,y,z) = \nabla \times \boldsymbol{F}(x,y,z) \qquad (2-5)$$

According to the above definitions of divergence and curl, a three-dimensional vector:

$$\boldsymbol{F}(x,y,z) = \boldsymbol{F}(x)\,\boldsymbol{i} + \boldsymbol{F}(y)\,\boldsymbol{j} + \boldsymbol{F}(z)\,\boldsymbol{k}$$

Its divergence is as follows:

$$\nabla \bullet \boldsymbol{F}(x,y,z) = \frac{\partial \boldsymbol{F}(x)}{\partial x} + \frac{\partial \boldsymbol{F}(y)}{\partial y} + \frac{\partial \boldsymbol{F}(z)}{\partial z}$$

$$\nabla \bullet \boldsymbol{F}(x, y, z) = \lim_{\Delta V \to 0} \frac{\oint \boldsymbol{F}(x, y, z) \bullet dS}{\Delta V} \qquad (2-6)$$

Its curl is as follows:

$$\nabla \times \boldsymbol{F}(x,y,z) = (\frac{\partial \boldsymbol{F}(z)}{\partial y} - \frac{\partial \boldsymbol{F}(y)}{\partial z})\boldsymbol{i} + (\frac{\partial \boldsymbol{F}(x)}{\partial z} - \frac{\partial \boldsymbol{F}(z)}{\partial x} \text{-})\boldsymbol{j} + (\frac{\partial \boldsymbol{F}(y)}{\partial x} - \frac{\partial \boldsymbol{F}(x)}{\partial y})\boldsymbol{k}$$

$$\nabla \times \boldsymbol{F}\,(x,y,z) = \lim_{\Delta S \to 0} \frac{\oint \boldsymbol{F}\,(x,y,z) \bullet \mathrm{d}l}{\Delta S} \qquad (2-7)$$

In the above definition of divergence and curl, the vector **F**(x, y, z) is only related to the spatial position and is independent of time. Therefore, **F**(x, y, z) is a steady-state vector, that is, **F**(x, y, z) is a static, time-invariant vector.

According to high school mathematics, the theorems and formulas of the divergence and curl can apply to a static electric field **E**(x, y, z) and a static magnetic field **B**(x, y, z), which depend on three spatial variables. However, they may not be valid for time-varying electric fields **E**(x, y, z, t) and magnetic fields **B**(x, y, z, t), which depend on three spatial variables and one time variable.

For example, in two-dimensional plane geometry, two points uniquely determine a straight line, and this principle also applies in three-dimensional geometry. However, in plane geometry, the sum of the interior angles of any quadrilateral is always 360°. This rule does not hold in three-dimensional geometry. As shown in Figure 2.1, for a closed spatial quadrilateral formed by the sides AB, BC, CD, and DA, the sum of its interior angles is not equal to 360°

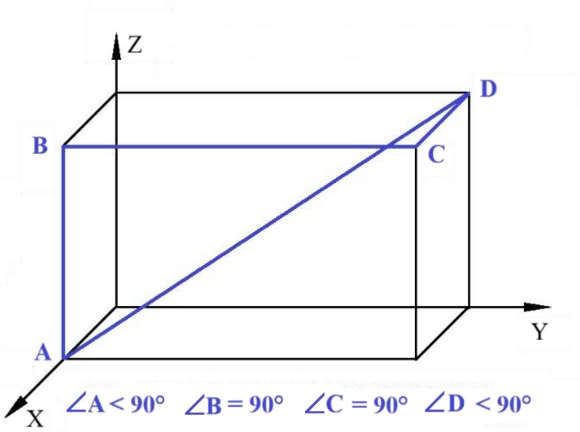

Figure 2.1 Sum of interior angles of a spatial quadrilateral

The following examples will illustrate that the divergence does not apply to a dynamic, time-varying electric field.

To simplify the discussion without losing generality, we analyze the divergence of a one-dimensional planar electric field **E**(x) in a vacuum, which can be generated by the two parallel plates of a capacitor.

As shown in Figure 2.2, consider an infinitesimal cylindrical element at position x in the one-dimensional planar electric field **E**(x) in a vacuum. The cylinder has height Δx and radius r. Its volume is ΔV, and the area of its closed cylindrical surface is ΔS.

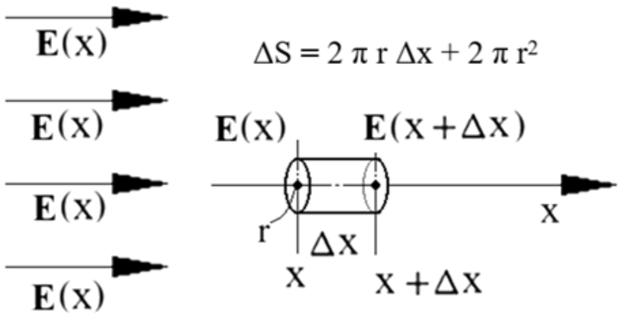

Figure 2.2 One-dimensional planar electric fields in a vacuum

According to the definition of the divergence for a three-dimensional vector in Equation (2-6), the divergence of a one-dimensional electric field **E**(x) is given as follows:

$$\nabla \bullet \boldsymbol{E}(x) = \lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(x) \bullet dS}{\Delta V} \qquad (2-8)$$

Assume a constant voltage is applied to the two plates of a parallel capacitor, generating a one-dimensional static electric field, **E**(x) = **E**(x + Δx) = **E**C, where EC is a constant.

The expression on the left side of Equation (2-8) is as follows:

$$\nabla \bullet \boldsymbol{E}(x) = \frac{\mathrm{d}\boldsymbol{E}(x)}{\mathrm{d}x} = \frac{\mathrm{d}\boldsymbol{E}_{\mathrm{C}}}{\mathrm{d}x} = 0$$

In the expression on the right side of Equation (2-8), the flux of **E**(x) through the closed cylindrical surface ΔS of the infinitesimal cylinder ΔV is as follows:

$$\oint \boldsymbol{E}(x) \bullet \mathrm{dS} = (\boldsymbol{E}(x+\Delta x) - \boldsymbol{E}(x))\pi r^2 = (\boldsymbol{E}_{\mathrm{C}} - \boldsymbol{E}_{\mathrm{C}})\pi r^2 = 0$$

Then, the expression on the right side of Equation (2-8) is as follows

$$\lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(x) \bullet dS}{\Delta V} = \lim_{\Delta V \to 0} \frac{0}{\Delta V} = 0$$

From the above analysis, both expressions on the left and right-hand sides of Equation (2-8) are always equal to 0, so Equation (2-8) holds. This shows that the divergence can apply to an electrostatic field in a vacuum.

Assume a sinusoidal time-varying voltage is applied to the two plates of a parallel capacitor, generating a one-dimensional dynamic time-varying electric field. Without loss of generality, at position x, let the electric field intensity be **E**(x) = **E**C Sinωt, where **E**C is a constant.

The expression on the left side of Equation (2-8) is as follows:

$$\nabla \bullet \boldsymbol{E}(x) = \frac{\mathrm{d}\boldsymbol{E}(x)}{\mathrm{d}x} = \mathrm{Sin}\omega t \frac{\mathrm{d}\boldsymbol{E}_{\mathrm{C}}}{\mathrm{d}x} = 0$$

For the time-varying electric field **E**(x) = **E**C Sinωt, **E**(x+Δx) = **E**C Sin(ωt + Δφ), where Δφ = ω Δx/c。
The expression on the right side of Equation (2-8) is as follows:

$$\lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(\mathrm{x}) \bullet dS}{\Delta V} = \lim_{\Delta V \to 0} \frac{(\boldsymbol{E}(\mathrm{x}+\Delta \mathrm{x}) - \boldsymbol{E}(\mathrm{x}))\pi \mathrm{r}^2}{\Delta \mathrm{x}\, \pi \mathrm{r}^2} = \lim_{\Delta V \to 0} \frac{\mathrm{Sin}(\omega \mathrm{t} + \Delta\varphi) - \mathrm{Sin}\, \omega \mathrm{t}}{\Delta \mathrm{x}} \boldsymbol{E}_{\mathrm{C}}$$

When both Δx and Δφ approach infinitesimal values, the following is derived:

$$\mathrm{Sin}(\omega t + \Delta\varphi) - \mathrm{Sin}(\omega t) = 2\,\mathrm{Cos}(\omega t + \Delta\varphi/2)\,\mathrm{Sin}(\Delta\varphi/2)$$

$$\approx \Delta\varphi\,\mathrm{Cos}(\omega t + \Delta\varphi/2)$$

$$= \Delta\varphi\,[\mathrm{Cos}(\omega t)\,\mathrm{Cos}(\Delta\varphi/2) - \mathrm{Sin}(\omega t)\,\mathrm{Sin}(\Delta\varphi/2)]$$

$$\approx \Delta\varphi\,[\mathrm{Cos}(\omega t) - (\Delta\varphi/2)\,\mathrm{Sin}(\omega t)]$$

$$= \Delta\varphi\,\mathrm{Cos}(\omega t) - (\Delta\varphi^2/2)\,\mathrm{Sin}(\omega t)$$

$$\approx \Delta\varphi\,\mathrm{Cos}(\omega t)$$

The expression on the right side of Equation (2-8) is as follows**:**

$$\lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(\mathrm{x}) \bullet dS}{\Delta V} = \lim_{\Delta V \to 0} \frac{\Delta\varphi\,\mathrm{Cos}\,\omega \mathrm{t}}{\Delta \mathrm{x}} \boldsymbol{E}_{\mathrm{C}} = \frac{\omega}{\mathrm{c}} \boldsymbol{E}_{\mathrm{C}}\,\mathrm{Cos}\,\omega \mathrm{t}$$

where c is the speed of the electric field. According to the above analysis, when **E**(x) is a dynamic, time-varying electric field, the expression on the left side of Equation (2-8) is always zero. However, the expression on the right side is a cosine function. Therefore, Equation (2-8) no longer holds, and the divergence does not apply to a time-varying electric field in a vacuum.

Assume a periodic step voltage is applied to the two plates of a parallel capacitor. The electric field intensity **E**(x) at position x is a periodic step function, as shown in Figure 2.3.

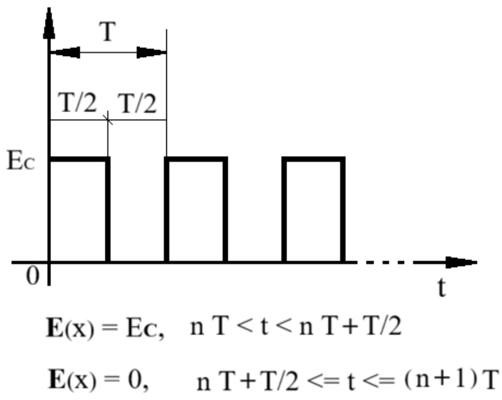

Figure 2.3 Electric field signal of periodic step function

At the moment t = n T/2, the electric field intensity **E**(x) at point x suddenly changes from 0 to $\mathbf{E}_{\mathrm{C}}$, or from $\mathbf{E}_{\mathrm{C}}$ to 0. In this case, **E**(x) is no longer continuous, and both expressions on the left and right-hand sides of Equation (2-8) become meaningless.

The above examples illustrate that the divergence does not apply to the time-varying electric fields in a vacuum. Below, we will further demonstrate that the divergence does not apply to the time-varying electric field in circuit wires.

As shown in Figure 2.4, a section of metal wire is taken in the circuit. The one-dimensional electric field intensity at position x of the wire is **E**(x). Consider an infinitesimal cylindrical wire element at position x, and the cylinder has height Δx and radius r. Its volume is ΔV, and the area of its closed cylindrical surface is ΔS.

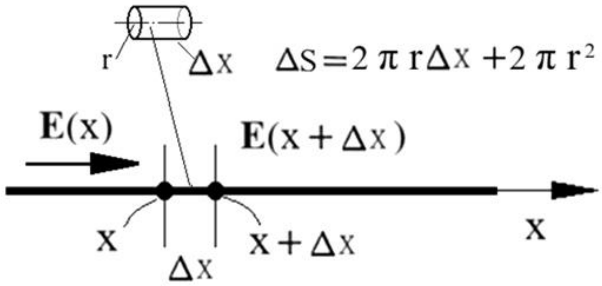

Figure 2.4 One-dimensional electric fields in circuit wires

According to the definition of the divergence for a three-dimensional vector in Equation (2-6), the divergence of a one-dimensional electric field **E**(x) is given as follows:

$$\nabla \bullet \boldsymbol{E}(x) = \lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(x) \bullet dS}{\Delta V} \qquad (2-9)$$

Assume the circuit is a DC circuit, the electric field in the wire is a one-dimensional static electric field, **E**(x) = **E**(x + Δx) = $\mathbf{E}_C$, where $\mathbf{E}_C$ is a constant.

The expression on the left side of Equation (2-9) is as follows:

$$\nabla \bullet \boldsymbol{E}(x) = \frac{\mathrm{d}\boldsymbol{E}(x)}{\mathrm{d}x} = \frac{\mathrm{d}\boldsymbol{E}_C}{\mathrm{d}x} = 0$$

In the expression on the right side of Equation (2-9), the flux of **E**(x) through the closed cylindrical surface ΔS of the infinitesimal cylinder ΔV is as follows:

$$\oint \boldsymbol{E}(x) \bullet \mathrm{dS} = (\boldsymbol{E}(x+\Delta x) - \boldsymbol{E}(x))\pi r^2 = (\boldsymbol{E}_C - \boldsymbol{E}_C)\pi r^2 = 0$$

Then, the expression on the right side of Equation (2-9) is as follows:

$$\lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(x) \bullet dS}{\Delta V} = \lim_{\Delta V \to 0} \frac{0}{\Delta V} = 0$$

From the above analysis, both expressions on the left and right-hand sides of Equation (2-9) are always equal to 0, so Equation (2-9) holds. This shows that the divergence can apply to an electrostatic field in circuit wires.

Assume the circuit is an AC circuit, the electric field in the wire is a one-dimensional time-varying electric field. Without loss of generality, at position x, let the electric field intensity be **E**(x) = $\mathbf{E}_C$ Sinωt, where $\mathbf{E}_C$ is a constant.

The expression on the left side of Equation (2-9) is as follows:

$$\nabla \bullet \boldsymbol{E}(x) = \frac{\mathrm{d}\boldsymbol{E}(x)}{\mathrm{d}x} = \mathrm{Sin}\omega t \frac{\mathrm{d}\boldsymbol{E}_C}{\mathrm{d}x} = 0$$

For the time-varying electric field $\mathbf{E}(x) = \mathbf{E}_C \, Sin\omega t$, $\mathbf{E}(x+\Delta x) = \mathbf{E}_C \, Sin(\omega t + \Delta\varphi)$, where $\Delta\varphi = \omega \, \Delta x/c$。 The expression on the right side of Equation (2-9) is as follows:

$$\lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(x) \bullet dS}{\Delta V} = \lim_{\Delta V \to 0} \frac{(\boldsymbol{E}(x+\Delta x) - \boldsymbol{E}(x))\pi r^2}{\Delta x \, \pi r^2} = \lim_{\Delta V \to 0} \frac{Sin(\omega t + \Delta\varphi) - Sin\, \omega t}{\Delta x} \boldsymbol{E}_C$$

When both Δx and Δφ approach infinitesimal values, the following is derived:

$$Sin(\omega t + \Delta\varphi) - Sin(\omega t) = \Delta\varphi \, Cos(\omega t)$$

The expression on the right side of Equation (2-9) is as follows**:**

$$\lim_{\Delta V \to 0} \frac{\oint \boldsymbol{E}(x) \bullet dS}{\Delta V} = \lim_{\Delta V \to 0} \frac{\Delta\varphi \, Cos\, \omega t}{\Delta x} \boldsymbol{E}_C = \frac{\omega}{c} \boldsymbol{E}_C \, Cos\, \omega t$$

where c is the speed of the electric field. According to the above analysis, when **E**(x) is a dynamic, time-varying electric field, the expression on the left side of Equation (2-9) is always zero. However, the expression on the right side is a cosine function. Therefore, Equation (2-9) no longer holds, and the divergence does not apply to a time-varying electric field in circuit wires.

The divergence does not apply to time-varying electromagnetic fields, and the curl also does not apply to time-varying electromagnetic fields. Both divergence and curl fail for dynamic, time-varying electromagnetic fields. Therefore, Maxwell's equations are facing serious challenges.

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