

Non-Closure of Doubling Vesica Chains: A Constructive Proof of Irreducible Remainder in Self-Similar Geometric Expansions

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Abstract

We present a constructive geometric proof demonstrating that a chain of Vesica Piscis figures following a doubling rule ($R, 2R, 4R, \dots, 2^{n-1}R$) cannot be enclosed by a Vesica of radius $2^n R$ that closes at the chain's origin. The linear extent of n doubling iterations sums to $2^n - 1$, producing an irreducible remainder of exactly one base unit R . This remainder is invariant across all scales of the construction and cannot be eliminated by further iteration. We show that this non-closure property originates at the unit construction itself: the radius R , which generates the Vesica, cannot be consumed by the figure it produces. The result is established through direct lattice construction without appeal to limiting processes or infinite series. We discuss connections to self-similar fractal structures, aperiodic tilings, and renormalization group theory. The construction arises from Geosectometry, a geometric framework developed by the author for investigating natural-state properties of constructive geometry.

Keywords: Vesica Piscis, geometric series, non-closure, self-similarity, compass-and-straightedge, lattice construction, Geosectometry

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1. Introduction

The Vesica Piscis is among the oldest geometric constructions known to mathematics. Formed by two equal circles whose centers lie on each other's circumference, it produces an almond-shaped intersection whose width-to-height ratio is $1:\sqrt{3}$. Each arc of the Vesica subtends exactly one-sixth of its generating circle, giving rise to the designation V6 used throughout this paper.

In classical and sacred geometry, Vesica constructions are studied for their generative properties: from a single radius R , the V6 produces the equilateral triangle, the hexagonal lattice, and higher-order symmetric structures. These constructions require nothing beyond a compass and straightedge, placing them firmly within the Euclidean tradition [1].

This paper investigates a specific property of V6 chains constructed under a doubling rule, where each successive Vesica has twice the radius of its predecessor. We prove that such chains exhibit a non-closure property: the encompassing Vesica at the next doubling scale cannot close at the chain's point of origin. The deficit between the chain's linear extent and the span required by the encompassing Vesica is exactly one base unit R , regardless of the number of iterations.

We further show that this non-closure is not an artifact of the chain construction but is inherent in the unit construction itself—the generating radius R cannot be closed by the figure it produces. A V6 of radius R has major axis length R ; the generating element is identical to the figure’s extent. Closure at the origin would require $R = 0$, which contradicts the premise of construction.

The construction is carried out on a regular dot lattice (isometric grid) using compass-and-straightedge methods, and the proof is entirely constructive: no limits, no infinite processes, no axiom of choice. Every claim is verified by direct computation on the lattice.

This work emerges from Geosectometry, a geometric framework developed by the author for studying what is termed the “natural state” of geometric construction—the properties that emerge from compass-and-straightedge methods without preconception or imposed axioms. A full treatment of the natural-state interpretation is reserved for a companion paper; here we focus on the mathematical content of the non-closure theorem.

2. Definitions and Notation

We establish the following definitions and conventions used throughout.

Definition 2.1 (*Unit radius R*). The fundamental length used to set the compass. We normalize $R = 1$ as the base unit. All subsequent constructions are expressed in terms of R .

Definition 2.2 (*V6 — Vesica Piscis petal*). The figure formed by two circles of radius R whose centers are separated by distance R . Each arc subtends $1/6$ of its generating circle (60°). The major axis (length along the line of centers) of the V6 equals R . The minor axis (perpendicular width at the midpoint) equals $R\sqrt{3}$.

Definition 2.3 (*V6 Chain*). A linear sequence of V6 figures placed end-to-end along a common major axis, where the right vertex of one V6 coincides with the left vertex of the next.

Definition 2.4 (*Doubling Rule*). A prescription for constructing a V6 chain in which each successive V6 has radius $2^k \cdot R$ for $k = 0, 1, 2, \dots, n-1$. Thus the sequence of radii is $R, 2R, 4R, \dots, 2^{n-1}R$.

Definition 2.5 (*Encompassing Vesica*). A V6 of radius $2^n \cdot R$ centered to enclose the entire chain. Its major axis span is $2^n \cdot R$.

Definition 2.6 (*Lattice*). A regular isometric (triangular) dot grid with spacing equal to R . All compass centers and intersection points lie on lattice nodes.

Definition 2.7 (*Non-closure*). The property that the encompassing Vesica’s boundary does not pass through the origin point of the chain. Equivalently, the linear extent of the chain is strictly less than the major axis span of the encompassing Vesica.

3. Construction

We describe the step-by-step construction of the doubling V6 chain on an isometric lattice. The construction proceeds through five stages, each revealing additional structure.

Stage 1 — The Seed Vesicas.

On an isometric dot lattice with spacing $R = 1$, construct three V6 petals along a common horizontal axis:

- V6₁ with radius $R = 1$ (the base unit), occupying 1 lattice unit along the axis.
- V6₂ with radius $2R = 2$, occupying 2 lattice units along the axis.
- V6₃ with radius $4R = 4$, occupying 4 lattice units along the axis.

Each Vesica is placed so that its left vertex coincides with the right vertex of the preceding one. The chain therefore extends $1 + 2 + 4 = 7$ lattice units from the origin.

Stage 2 — Projection to Baseline.

Drop perpendicular markers from the chain axis to a common baseline, revealing the linear footprint of each Vesica and the total extent of the chain. This projection makes explicit that the footprints are contiguous and non-overlapping: each V6 occupies exactly its radius along the axis.

Stage 3 — Internal Self-Similar Tiling.

Tile the interior of V6₂ ($R = 2$) with two $R = 1$ diamond figures (rhombi formed by pairs of equilateral triangles). Tile the interior of V6₃ ($R = 4$) with four $R = 1$ diamonds. This reveals the self-similar structure: each larger Vesica contains exact copies of V6₁ at the base scale. The number of base-scale diamonds in a V6 of radius $2^k R$ is 2^k .

Stage 4 — The Encompassing Vesica (Attempted Closure).

Construct V6₄ with radius $8R = 8$, attempting to encompass the entire chain from point B (left origin) to point A (right terminus). The encompassing Vesica requires a span of 8 lattice units along the major axis. However, the chain occupies only 7 lattice units. The encompassing Vesica does *not* close at the chain's origin: there is a gap of exactly 1 lattice unit—one $R = 1$. The doubling sequence is annotated V6: 1–8, 1–4, 1–2, 1–1.

Stage 5 — Full Internal Structure.

Populate all scales with their internal $R = 1$ diamonds and vertical structural elements. The complete structure occupies $7/8$ of the encompassing Vesica's span. The non-closure ratio is confirmed: $V6 \sim 1 - 7/8$.

4. Theorem and Proof

Theorem 1 (Non-Closure of Doubling V6 Chains).

Let C_n be a V6 chain of n Vesica petals constructed under the doubling rule, with radii $R, 2R, 4R, \dots, 2^{n-1}R$. The total linear extent of C_n along the major axis is

$$L(C_n) = R \cdot \sum_{k=0}^{n-1} 2^k = R \cdot (2^n - 1). \tag{1}$$

An encompassing Vesica V of radius $2^n \cdot R$ requires a major axis span of $2^n \cdot R$. The deficit is

$$\Delta = 2^n \cdot R - (2^n - 1) \cdot R = R. \quad (2)$$

The deficit $\Delta = R$ is constant, equal to the base unit, and independent of n . Consequently, the encompassing Vesica never closes at the chain's origin.

Proof.

The proof proceeds by mathematical induction on the number of petals n .

Base case ($n = 1$). The chain consists of a single V6 with radius R . Its linear extent is $L(C_1) = R$. The encompassing Vesica has radius $2R$ and requires a span of $2R$. The deficit is $\Delta = 2R - R = R$.

Inductive step. Assume that for $n = k$, the chain extent is $L(C_k) = R(2^k - 1)$ and the deficit is R . For $n = k + 1$, we append a V6 of radius $2^k \cdot R$ to obtain:

$$L(C_{k+1}) = L(C_k) + 2^k \cdot R = R(2^k - 1) + 2^k \cdot R = R(2^{k+1} - 1). \quad (3)$$

The encompassing Vesica at scale $k + 1$ has radius $2^{k+1} \cdot R$, requiring a span of $2^{k+1} \cdot R$. The deficit is:

$$\Delta = 2^{k+1} \cdot R - (2^{k+1} - 1) \cdot R = R. \quad (4)$$

Therefore, by induction, the deficit is R for all $n \geq 1$. The encompassing Vesica never closes at the chain origin. QED

Theorem 2 (Irreducibility of the Generative Element).

The non-closure property of Theorem 1 originates at the unit construction. A V6 petal of radius R has major axis length R . The generating radius R is thus identical to the figure's extent along the construction axis. The V6 does not "consume" R —it is R , deployed as structure.

The deficit R in the chain is not an artifact of series arithmetic; it is the base unit itself, persisting because it is the generative element of every Vesica in the chain. Closure at the origin would require $R = 0$, which contradicts the premise of construction: a Vesica of radius zero is undefined.

We present this as a constructive observation rather than a formal theorem in the classical sense. The content is established by direct inspection of the construction: the unit Vesica $V6_1$ has extent R along the axis and is generated by a compass opening of R . These are the same quantity. No deductive chain beyond this identification is needed; the irreducibility is definitional.

5. Properties of the Non-Closure

The following table summarizes the key quantitative properties of the doubling V6 chain for the first four values of n .

Table 1. Properties of the doubling V6 chain for $n = 1, 2, 3, 4$ with $R = 1$.

| Property | Formula | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
|--------------------------|-----------------|---------|---------|---------|---------|
| Chain extent | $R(2^n - 1)$ | 1 | 3 | 7 | 15 |
| Encompassing radius | $2^n \cdot R$ | 2 | 4 | 8 | 16 |
| Deficit (absolute) | R | 1 | 1 | 1 | 1 |
| Deficit (relative) | $1/2^n$ | 1/2 | 1/4 | 1/8 | 1/16 |
| Chain/encompassing ratio | $(2^n - 1)/2^n$ | 1/2 | 3/4 | 7/8 | 15/16 |

The absolute deficit is scale-invariant: it is always exactly R , regardless of n . The relative deficit decreases geometrically as $1/2^n$ but never reaches zero for any finite n . The chain-to-encompassing ratio $(2^n - 1)/2^n$ approaches 1 but never equals it.

In conventional analysis, the limit as $n \rightarrow \infty$ of the chain-to-encompassing ratio is 1, and the sum of the geometric series converges to 2^n . However, our construction is always performed at finite n , on a finite lattice. At every finite stage, the gap is present and always exactly R . The passage to the limit, while analytically valid, does not correspond to any constructive operation on the lattice.

6. Discussion

6.1 Relation to Geometric Series and Zeno's Paradox

The sum $1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$ is a well-known finite geometric series, recorded in essentially this form since antiquity [2]. Our contribution is not the arithmetic identity itself but its geometric realization on a lattice as a non-closure property of self-similar Vesica constructions. The conventional treatment passes to the infinite limit and declares the series equal to 2^n ; our construction remains finite and reveals the structural role of the remainder.

Zeno's dichotomy paradox (halving forward: $1/2 + 1/4 + 1/8 + \dots \rightarrow 1$) and our construction (doubling forward: $1 + 2 + 4 + \dots \rightarrow 2^n - 1$) are complementary views of the same structural boundary. In both cases, the process asymptotically approaches a bound it cannot cross. The difference is that Zeno's version is resolved by passing to the infinite limit; our version, being constructed on a finite lattice, makes the remainder explicit and irreducible.

6.2 Connection to Self-Similar and Fractal Structures

The V6 chain is self-similar by construction: each Vesica at scale 2^k contains exact copies of the base V6 (as demonstrated in Stage 3 of Section 3). This nested containment of identical structures at decreasing scales is the defining property of fractal geometry, as articulated by Mandelbrot [3].

However, our construction adds a boundary condition not typically present in classical fractal analysis: when the self-similar structure attempts to close—that is, when we seek an encompassing figure at the next scale that precisely contains the chain—it fails by exactly one base unit. This connects to open questions in fractal boundary value problems, where classical differential equations break down

on self-similar boundaries. Jonsson and Wallin [7] have studied Brownian motion and boundary value problems on fractal domains, finding that the irregular boundary geometry fundamentally alters the behavior of solutions.

6.3 Connection to Aperiodic Tilings

The non-closure of the V6 chain is structurally analogous to the aperiodicity of Penrose tilings [5], where the irrational ratio of tile types (governed by the golden ratio $\phi = (1 + \sqrt{5})/2$) prevents periodic closure. In our case, the mechanism is different—the deficit is rational (exactly R)—but the consequence is the same: the structure cannot close on itself. The chain always falls short of the encompassing boundary by a fixed quantum.

The recent discovery of aperiodic monotiles by Smith, Myers, Kaplan, and Goodman-Strauss [4] renewed interest in finite geometric structures with non-periodic properties. Their “hat” tile achieves aperiodicity with a single prototile—a remarkable economy that parallels the economy of the V6 construction, which produces non-closure from a single generating element R .

6.4 Connection to Renormalization and Scale Invariance

In lattice quantum field theory, renormalization group methods coarse-grain a lattice by doubling the lattice spacing at each step [6]. Information is lost at each scale transition: short-distance degrees of freedom are integrated out, leaving an effective description at the larger scale. The V6 chain provides a geometric model of this process.

At each doubling, the encompassing structure captures $2^n - 1$ of the 2^n available units. The “missing” unit R is the information that does not survive the scale transition. The invariance of the deficit—always R , regardless of the number of doubling steps—mirrors the universality of renormalization group fixed points, where certain quantities remain unchanged across scale transformations. Wilson’s foundational work on the renormalization group [6] established that such scale-invariant quantities carry fundamental physical significance.

6.5 Geosectometry and the Natural State Framework

This construction arises from Geosectometry, a geometric framework developed by the author for investigating what is termed the “natural state” of geometric construction—the properties that emerge from compass-and-straightedge methods without preconception or imposed axioms.

Within this framework, the non-closure of the V6 chain is interpreted as a fundamental property of constructive geometry: the generative element (R) is irreducible and cannot be eliminated by the structures it produces. Just as a seed is not consumed by the tree it grows, the radius is not consumed by the Vesica it generates. This irreducibility is not a defect to be overcome by passage to the limit; it is a structural feature of finite, constructive mathematics.

A full treatment of the natural-state interpretation, including connections to biological growth patterns, philosophical systems, and additional geometric constructions within Geosectometry, is reserved for a companion paper.

7. Conclusion

We have established the following results:

- A V6 doubling chain of n petals with radii $R, 2R, \dots, 2^{n-1}R$ has linear extent $R(2^n - 1)$, producing an irreducible remainder of exactly R when compared to the encompassing Vesica of radius $2^n R$.
- This non-closure is not an artifact of series arithmetic but a structural property of the unit construction itself: the generating radius R is identical to the V6's extent and cannot be consumed by the figure it produces.
- The result is established entirely by finite, constructive methods on an isometric lattice, without appeal to limits, infinite series, or the axiom of choice.
- The construction connects to active research areas in fractal geometry [3, 7], aperiodic tilings [4, 5], and lattice field theory [6], providing a simple geometric model of non-closure under scale transformation.
- The Geosectometry framework from which this result arises offers a systematic approach to identifying such natural-state properties of constructive geometry.

The non-closure theorem reveals that the relationship between a geometric structure and its generating element contains an irreducible gap—a unit of construction that persists at every scale and cannot be absorbed into the structure it creates. Whether this property has deeper implications for the foundations of constructive mathematics, or for the physical theories that rely on lattice regularization, remains an open question.

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