ON A CONVERGING INFINITE SERIES DEFINED BY SUPER-EXPONENTIALS

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Abstract. We introduce and study the entire function $\lambda(s) = \sum_{n=1}^{\infty} n^{s+1/2}/n^n$, defined by a Dirichlet-type series with super-exponential coefficients. We prove that $\lambda(s)$ converges absolutely for all $s \in \mathbb{C}$, uniformly on compact sets, and is therefore an entire function of order zero. We establish a closed-form evaluation of the special value $\lambda(-1/2) = \int_0^1 x^{-x}\,dx$, connecting λ to the classical Sophomore's Dream identity of Bernoulli. We further prove that $\lambda(s)$ is term-by-term differentiable, with $\lambda^{(k)}(s) = \sum_{n=1}^{\infty} (\ln n)^k n^{s+1/2}/n^n$ for all $k \geq 0$, justified by the Weierstrass M-test. Finally, we propose a conjecture connecting $\lambda(s)$ to the Riemann zeta function.

1. Definitions

Definition 1.1. *We define a function $\lambda(s)$ such that:*

$$\lambda(s) = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^{n-s}}$$

2. Theorems

Theorem 2.1. *The series defining $\lambda(s)$ converges absolutely for all $s \in \mathbb{C}$.*

Proof. Let $a_n(s) = \frac{\sqrt{n}}{n^{n-s}}$. We simplify the term as:

$$a_n(s) = \frac{n^{1/2} \cdot n^s}{n^n} = \frac{n^{s+1/2}}{n^n}.$$

We apply the Cauchy Root Test. Let $s = \sigma + it$. The modulus of the general term is:

$$|a_n(s)| = \frac{n^{\sigma+1/2}}{n^n}.$$

Consider the n-th root:

$$\sqrt[n]{|a_n(s)|} = \frac{n^{\frac{\sigma+1/2}{n}}}{n}.$$

We evaluate the limit of the numerator. Let $L_n = n^{\frac{\sigma+1/2}{n}}$. Then:

$$\lim_{n\to\infty} \ln(L_n) = \lim_{n\to\infty} \frac{\sigma + 1/2}{n} \ln(n) = 0 \Rightarrow \lim_{n\to\infty} L_n = 1.$$

Thus, the limit of the root is:

$$\lim_{n\to\infty} \sqrt[n]{|a_n(s)|} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Since $0 < 1$, the series converges absolutely for all $s \in \mathbb{C}$ by the Root Test. ☐

Theorem 2.2. *The series defining $\lambda(s)$ converges uniformly on every compact subset $K \subset \mathbb{C}$.*

Proof. Let $K \subset \mathbb{C}$ be compact. Since K is compact, the real part $\sigma = \mathrm{Re}(s)$ is bounded on K; let $B = \sup_{s \in K} \mathrm{Re}(s) < \infty$.

For all $s \in K$, the modulus of the general term satisfies:

$$|a_n(s)| = \frac{n^{\mathrm{Re}(s)+1/2}}{n^n} \leq \frac{n^{B+1/2}}{n^n} =: M_n.$$

The bound M_n is independent of $s \in K$. It remains to verify that $\sum_{n=1}^{\infty} M_n < \infty$. We apply the Root Test to the series $\sum M_n$. Writing $M_n = n^{B+1/2-n}$, we compute:

$$M_n^{1/n} = n^{\frac{B+1/2}{n}-1}.$$

Taking the limit:

$$\lim_{n\to\infty} M_n^{1/n} = \lim_{n\to\infty} n^{\frac{B+1/2}{n}-1} = \lim_{n\to\infty} \frac{n^{\frac{B+1/2}{n}}}{n}.$$

Since $\lim_{n\to\infty} n^{\frac{B+1/2}{n}} = 1$ (see Theorem 2.1), this limit equals:

$$\lim_{n\to\infty} \frac{1}{n} = 0 < 1.$$

Hence $\sum_{n=1}^{\infty} M_n < \infty$. By the Weierstrass M-test, the series $\sum_{n=1}^{\infty} a_n(s)$ converges uniformly on K. Since $K \subset \mathbb{C}$ was an arbitrary compact set, $\lambda(s)$ converges uniformly on every compact subset of $\mathbb{C}$. ☐

Theorem 2.3. *The function $\lambda(s)$ is entire, i.e., holomorphic on all of $\mathbb{C}$.*

Proof. Each term of the series is entire as a function of s. Indeed, writing:

$$a_n(s) = \frac{n^{s+1/2}}{n^n} = \frac{e^{(s+1/2)\ln n}}{n^n},$$

we see that $a_n(s)$ is an exponential function of s scaled by the constant n^{-n}, and is therefore holomorphic on all of $\mathbb{C}$.

By the Weierstrass theorem on uniform limits of holomorphic functions, and since the series $\sum_{n=1}^{\infty} a_n(s)$ converges uniformly on every compact subset $K \subset \mathbb{C}$ (see Theorem 2.2), the limit:

$$\lambda(s) = \sum_{n=1}^{\infty} a_n(s)$$

is holomorphic on every such K. Since every point of $\mathbb{C}$ belongs to some compact set, $\lambda(s)$ is holomorphic on all of $\mathbb{C}$, and is therefore an entire function. ☐

Theorem 2.4. *The function $\lambda(s)$ satisfies:*

$$\lambda\left(-\tfrac{1}{2}\right) = \int_0^1 x^{-x}\, dx.$$

Proof. From Definition 1.1, evaluating at $s = -1/2$:

$$\lambda\left(-\tfrac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

We show $\int_0^1 x^{-x}\,dx$ equals the same sum. Apply the substitution $x = e^{-t}$, so $dx = -e^{-t}\,dt$, giving:

$$\int_0^1 x^{-x}\,dx = \int_0^{\infty} e^{te^{-t}} e^{-t}\,dt.$$

Expanding $e^{te^{-t}} = \sum_{k=0}^{\infty} \frac{t^k e^{-kt}}{k!}$, we obtain:

$$\int_0^{\infty} e^{te^{-t}} e^{-t}\,dt = \int_0^{\infty} \sum_{k=0}^{\infty} \frac{t^k e^{-(k+1)t}}{k!}\,dt.$$

Each term $\frac{t^k e^{-(k+1)t}}{k!}$ is non-negative on $[0, \infty)$. By Tonelli's theorem, we may interchange the sum and integral freely:

$$\int_0^{\infty} \sum_{k=0}^{\infty} \frac{t^k e^{-(k+1)t}}{k!}\,dt = \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^{\infty} t^k e^{-(k+1)t}\,dt.$$

Substituting $u = (k+1)t$ in each integral:

$$\int_0^{\infty} t^k e^{-(k+1)t}\,dt = \frac{\Gamma(k+1)}{(k+1)^{k+1}} = \frac{k!}{(k+1)^{k+1}}.$$

Therefore:

$$\int_0^1 x^{-x}\,dx = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{k!}{(k+1)^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}} = \sum_{n=1}^{\infty} \frac{1}{n^n} = \lambda\left(-\tfrac{1}{2}\right). \quad \square$$

Theorem 2.5. *The function $\lambda(s)$ is term-by-term differentiable, and its derivative is:*

$$\lambda'(s) = \sum_{n=1}^{\infty} \frac{n^{s+1/2} \ln n}{n^n}.$$

More generally, for any $k \geq 0$:

$$\lambda^{(k)}(s) = \sum_{n=1}^{\infty} \frac{(\ln n)^k\, n^{s+1/2}}{n^n}.$$

Proof. Each term $a_n(s) = \frac{n^{s+1/2}}{n^n} = \frac{e^{(s+1/2)\ln n}}{n^n}$ is entire, and differentiating with respect to s:

$$a_n'(s) = \frac{n^{s+1/2} \ln n}{n^n}.$$

We show the differentiated series $\sum_{n=1}^{\infty} a_n'(s)$ converges uniformly on every compact set $K \subset \mathbb{C}$. Let $B = \sup_{s \in K} \operatorname{Re}(s) < \infty$. Then for all $s \in K$:

$$|a_n'(s)| = \frac{n^{\operatorname{Re}(s)+1/2} \ln n}{n^n} \leq \frac{n^{B+1/2} \ln n}{n^n} =: M_n.$$

The bound M_n is independent of $s \in K$. We verify $\sum_{n=1}^{\infty} M_n < \infty$ by the Root Test. Since $\ln n$ grows slower than any positive power of n, we have $(\ln n)^{1/n} \to 1$ as $n \to \infty$, and therefore:

$$M_n^{1/n} = n^{\frac{B+1/2}{n} - 1} \cdot (\ln n)^{1/n} \to 0 \cdot 1 = 0 < 1.$$

Hence $\sum_{n=1}^{\infty} M_n < \infty$, and by the Weierstrass M-test the differentiated series converges uniformly on K. Since each $a_n'(s)$ is holomorphic and the convergence is uniform on every compact set, the standard theorem on differentiation of uniformly convergent series of holomorphic functions gives:

$$\lambda'(s) = \sum_{n=1}^{\infty} a_n'(s) = \sum_{n=1}^{\infty} \frac{n^{s+1/2} \ln n}{n^n}.$$

The general case follows by induction. Differentiating $a_n(s)$ a total of k times:

$$a_n^{(k)}(s) = \frac{(\ln n)^k \, n^{s+1/2}}{n^n}.$$

The same M-test argument applies at each stage, since $(\ln n)^k$ for fixed k is absorbed by the super-exponential decay of n^n, giving $M_n^{1/n} \to 0$ uniformly. Therefore:

$$\lambda^{(k)}(s) = \sum_{n=1}^{\infty} \frac{(\ln n)^k \, n^{s+1/2}}{n^n}. \quad \square$$

3. Propositions

Conjecture 3.1. *For all $s \in \mathbb{C}$, the following identity holds:*

$$\lambda(s) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \, \zeta^{(\alpha)}\left(-s - \alpha - \frac{1}{2}\right),$$

where $\zeta^{(\alpha)}$ denotes the α-th derivative of the Riemann zeta function with respect to its argument, understood via analytic continuation, and the series on the right hand side converges absolutely for all $s \in \mathbb{C}$.

Remark 3.1. *The derivation of Conjecture 3.1 proceeds by substituting the Taylor expansion $e^{-n \ln n} = \sum_{\alpha=0}^{\infty} \frac{(-n \ln n)^\alpha}{\alpha!}$ into the definition of $\lambda(s)$, and identifying the resulting inner sum with $\zeta^{(\alpha)}(-s - \alpha - 1/2)$ via the analytic continuation of the Dirichlet series $\zeta^{(\alpha)}(w) = (-1)^\alpha \sum_{n=1}^{\infty} \frac{(\ln n)^\alpha}{n^w}$. The obstruction to a proof is the interchange of the sum over n and the sum over α, which cannot be justified by Fubini or Tonelli since the absolute double sum $\sum_{n=1}^{\infty} \sum_{\alpha=0}^{\infty} \frac{n^{\sigma+\alpha+1/2} (\ln n)^\alpha}{\alpha!}$ diverges. Absolute convergence of the right hand side follows from the functional equation of ζ and Stirling's approximation, which together yield the bound:*

$$\frac{|\zeta^{(\alpha)}(-s - \alpha - \frac{1}{2})|}{\alpha!} \lesssim C(s) \cdot \frac{\alpha^{\sigma+1/2}}{\pi^\alpha},$$

and hence $\sum_{\alpha=0}^{\infty} \frac{|\zeta^{(\alpha)}(-s-\alpha-1/2)|}{\alpha!} < \infty$ *since* $\pi > 1$*. A proof of the equality likely requires the theory of Ramanujan summation as developed in* [3]*, or a Mellin-kernel argument using the dominated convergence theorem applied to the representation* $\zeta^{(\alpha)}(w)\Gamma(w) = \int_0^\infty \frac{t^{w-1}(\ln t)^\alpha}{e^t-1}\,dt$*.*

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