Quadrature approximation to the Magnus exponent for numerical solution of non-autonomous coupled linear differential equations

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Version 09-Nov-2026
https://vixra.org/abs/2604.0065

Abstract

The Magnus-exponent method of solving non-autonomous (variable-coefficient) coupled linear differential equations is reviewed. A 6th-order, 3-commutator approximation to the Magnus exponent is derived using centered Legendre moments of the coefficient matrix, with the moment integrals evaluated via 3-point Gauss-Legendre quadrature.

Linear differential equations

A *first-order, linear differential equation* or system of coupled differential equations has the form

$$F'(t) = A(t) \cdot F(t) + B(t), \quad F(t_0) = F_0 \tag{1}$$

where $F(t)$, $A(t)$, and $B(t)$ are matrix functions of scalar argument t, F_0 is a specified initial value, and $F(t)$ is to be determined. The equation is *autonomous* if A and B are constant, and is *homogeneous* if $B(t)$ is identically zero.

A nonhomogeneous equation can be converted to a homogeneous equation by using an alternative formulation of Eq. 1,

$$\frac{d}{dt}\begin{pmatrix} F(t) \\ \mathbf{I} \end{pmatrix} = \begin{pmatrix} A(t) & B(t) \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} F(t) \\ \mathbf{I} \end{pmatrix} \tag{2}$$

where $\mathbf{I}$ and $\mathbf{0}$ represent identity and zero matrices, respectively. Therefore, we need only consider homogeneous equations, although Eq. (2) might be implemented to efficiently take advantage of the zeros in the coefficient matrix.

With $B(t) = \mathbf{0}$, we want to determine $F(t)$ at some point $t = t_1$ from its initial value at $t = t_0$, based on the differential equation $F'(t) = A(t) \cdot F(t)$. The t coordinate can be shifted and scaled so that the origin is at the integration interval's midpoint, and the interval width is $2h$:

$$t_1 = h = -t_0 \tag{3}$$

So we seek to determine $F(h)$ from a specified initial value $F(-h)$.

The solution $F(t)$ will be represented as,

$$F(t) = U(t) \cdot F(-t) \tag{4}$$

Differentiating this relation substituting $F'(\pm t) = A(\pm t) \cdot F(\pm t)$, we obtain

$$\begin{aligned} F'(t) &= U'(t) \cdot F(-t) - U(t) \cdot F'(-t) = U'(t) \cdot F(-t) - U(t) \cdot A(-t) \cdot F(-t) \\ &= A(t) \cdot F(t) = A(t) \cdot U(t) \cdot F(-t) \end{aligned} \tag{5}$$

Hence, $\big(U'(t) - U(t) \cdot A(-t) - A(t) \cdot U(t)\big) \cdot F(-t) = \mathbf{0}$. This condition holds for any initial value $F(-t)$, so U is defined by the differential equation

$$U'(t) = A(t) \cdot U(t) + U(t) \cdot A(-t), \quad U(0) = \mathbf{I} \tag{6}$$

The initial value $U(0) = \mathbf{I}$ is consistent with Eq. (4), which also implies that

$$U(-t) = U(t)^{-1} \tag{7}$$

(because $F(-t) = U(-t) \cdot F(t) = U(-t) \cdot U(t) \cdot F(-t)$).

The Magnus exponent

Eq. (6) has an exponential-matrix solution,

$$U(t) = \exp(\Omega(t)), \tag{8}$$

$\Omega(t)$ is the *Magnus exponent* defined by the conditions

$$\frac{d}{dt} \exp(\Omega(t)) = A(t) \cdot \exp(\Omega(t)) + \exp(\Omega(t)) \cdot A(-t), \quad \Omega(0) = \mathbf{0} \tag{9}$$

Eq. (7) implies that $\Omega(t)$ has odd symmetry,

$$\Omega(-t) = -\Omega(t), \tag{10}$$

Commutator algebra

The derivations below are adapted from Ref. 1 and make use of the following definitions and relations: The *commutator* […] is defined, for square matrix arguments A and B, as

$$[A, B] = A \cdot B - B \cdot A, \tag{11}$$

The commutator is antisymmetric and satisfies the Jacobi identity,

$$[B,A] = -[A,B], \tag{12}$$

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]] = \mathbf{0}, \tag{13}$$

The commutator expression $[\Omega, X]$ is a linear function of X, which will be represented as the matrix product $ad_\Omega \cdot X$, where X is interpreted as a block-partitioned column vector in this context and ad_Ω, the *adjoint mapping* of Ω, is a block-partitioned matrix.

4-D arrays will be used to represent partitioned matrices. For example, 4-D array A with elements $A_{i,j,m,n}$ represents matrix blocks labeled as $A_{i,j}$, each block comprising elements $(A_{i,j})_{m,n}$, where

$$\text{For 4-D array } A, \quad (A_{i,j})_{m,n} = A_{i,j,m,n}, \tag{14}$$

The matrix product of 4-D arrays A and B is defined by generalizing the standard product definition,

$$\text{For 4-D arrays } A, B, \text{ and } C = A \cdot B,$$

$$C_{i,k,m,p} = (C_{i,k})_{m,p} = \left(\sum_j A_{i,j} \cdot B_{j,k} \right)_{m,p} = \sum_j \left(A_{i,j} \cdot B_{j,k} \right)_{m,p}, \tag{15}$$

$$= \sum_j \sum_n (A_{i,j})_{m,n} \cdot (B_{j,k})_{n,p} = \sum_{j,n} A_{i,j,m,n} \cdot B_{j,k,n,p}$$

The product of 4-D array A and 2-D matrix B is implemented as a block matrix-vector multiply in which B is interpreted as a block column vector comprising blocks B_i, each block comprising elements $(B_i)_m$,

$$\text{For 2-D matrix } B, \quad (B_i)_m = B_{i,m}, \tag{16}$$

$$\text{For 4-D array } A \text{ and 2-D matrices } B \text{ and } C = A \cdot B,$$

$$C_{i,m} = (C_i)_m = \left(\sum_j A_{i,j} \cdot B_j \right)_m = \sum_j \left(A_{i,j} \cdot B_j \right)_m, \tag{17}$$

$$= \sum_j \sum_n (A_{i,j})_{m,n} \cdot (B_j)_n = \sum_{j,n} A_{i,j,m,n} \cdot B_{j,n}$$

The block matrix-vector product operation defined by Eq. (17) is left-associative but not right-associative,

$$\text{For 4-D arrays } A \text{ and } B, \text{ and 2-D matrix } C, \quad A \cdot (B \cdot C) = (A \cdot B) \cdot C, \tag{18}$$

For 4-D array A and 2-D matrices B and C,
$A \cdot (B \cdot C) \neq (A \cdot B) \cdot C$ (generally) , (19)

The adjoint mapping ad_Ω is defined by

$$ad_\Omega \cdot X = [\Omega, X] \tag{20}$$

The commutator can be represented as follows,

$$\begin{aligned}[\Omega, X]_{i,j} &= \sum_m \Omega_{i,m} \cdot X_{m,j} - \sum_n X_{i,n} \cdot \Omega_{n,j} = \sum_{m,n} \Omega_{i,m} \cdot \delta_{j,n} \cdot X_{m,n} - \sum_{m,n} X_{m,n} \cdot \delta_{i,m} \cdot \Omega_{n,j} \\ &= \sum_{m,n} (\Omega_{i,m} \cdot \delta_{j,n} - \delta_{i,m} \cdot \Omega_{n,j}) \cdot X_{m,n}\end{aligned} \tag{21}$$

(δ is the Kronecker delta symbol.) Thus,

$$(ad_\Omega)_{i,m,j,n} = \Omega_{i,m} \cdot \delta_{j,n} - \delta_{i,m} \cdot \Omega_{n,j} \tag{22}$$

This operation can be represented in terms of Kronecker products: $ad_\Omega = \Omega \otimes I - I \otimes \Omega^{\mathsf{T}}$, where $(A \otimes B)_{i,m,j,n} = A_{i,m} \cdot B_{j,n}$ and Ω^{T} is the transpose of Ω. For example, a 2-by-2 matrix Ω has the adjoint mapping

$$\begin{aligned}ad_\Omega &= \begin{pmatrix} (ad_\Omega)_{1,1} & (ad_\Omega)_{1,2} \\ (ad_\Omega)_{2,1} & (ad_\Omega)_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} (ad_\Omega)_{1,1,1,1} & (ad_\Omega)_{1,1,1,2} \\ (ad_\Omega)_{1,1,2,1} & (ad_\Omega)_{1,1,2,2} \end{pmatrix} & \begin{pmatrix} (ad_\Omega)_{1,2,1,1} & (ad_\Omega)_{1,2,1,2} \\ (ad_\Omega)_{1,2,2,1} & (ad_\Omega)_{1,2,2,2} \end{pmatrix} \\ \begin{pmatrix} (ad_\Omega)_{2,1,1,1} & (ad_\Omega)_{2,1,1,2} \\ (ad_\Omega)_{2,1,2,1} & (ad_\Omega)_{2,1,2,2} \end{pmatrix} & \begin{pmatrix} (ad_\Omega)_{2,2,1,1} & (ad_\Omega)_{2,2,1,2} \\ (ad_\Omega)_{2,2,2,1} & (ad_\Omega)_{2,2,2,2} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} \Omega_{1,1} & 0 \\ 0 & \Omega_{1,1} \end{pmatrix} & \begin{pmatrix} \Omega_{1,2} & 0 \\ 0 & \Omega_{1,2} \end{pmatrix} \\ \begin{pmatrix} \Omega_{2,1} & 0 \\ 0 & \Omega_{2,1} \end{pmatrix} & \begin{pmatrix} \Omega_{2,2} & 0 \\ 0 & \Omega_{2,2} \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} \Omega_{1,1} & \Omega_{2,1} \\ \Omega_{1,2} & \Omega_{2,2} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \Omega_{1,1} & \Omega_{2,1} \\ \Omega_{1,2} & \Omega_{2,2} \end{pmatrix} \end{pmatrix}\end{aligned} \tag{23}$$

ad_Ω is a linear function of Ω,

$$ad_{A+B} = ad_A + ad_B \tag{24}$$

$$ad_{\sigma \cdot \Omega} = \sigma \cdot ad_\Omega \quad (\text{scalar } \sigma) \tag{25}$$

The centered Magnus theorem

The similarity transformation $\exp(\Omega)\cdot X\cdot\exp(-\Omega)$ has a series expansion of the form

$$\exp(\Omega)\cdot X\cdot\exp(-\Omega)=\sum_j\frac{1}{j!}\cdot ad_\Omega^j\cdot X \tag{26}$$

This relation is derived as follows: For scalar σ, the following relation is obtained by induction,

$$\frac{d^j}{d\sigma^j}\left(\exp(\sigma\cdot\Omega)\cdot X\cdot\exp(-\sigma\cdot\Omega)\right)=\exp(\sigma\cdot\Omega)\cdot(ad_\Omega^j\cdot X)\cdot\exp(-\sigma\cdot\Omega) \tag{27}$$

The induction step is detailed as follows,

$$\begin{aligned}
&\frac{d^{j+1}}{d\sigma^{j+1}}\left(\exp(\sigma\cdot\Omega)\cdot X\cdot\exp(-\sigma\cdot\Omega)\right)\\
&=\frac{d}{d\sigma}\left(\exp(\sigma\cdot\Omega)\cdot(ad_\Omega^j\cdot X)\cdot\exp(-\sigma\cdot\Omega)\right)\\
&=\exp(\sigma\cdot\Omega)\cdot[\Omega,(ad_\Omega^j\cdot X)]\cdot\exp(-\sigma\cdot\Omega)\\
&=\exp(\sigma\cdot\Omega)\cdot(ad_\Omega\cdot(ad_\Omega^j\cdot X))\cdot\exp(-\sigma\cdot\Omega)\\
&=\exp(\sigma\cdot\Omega)\cdot((ad_\Omega\cdot ad_\Omega^j)\cdot X)\cdot\exp(-\sigma\cdot\Omega)\\
&=\exp(\sigma\cdot\Omega)\cdot(ad_\Omega^{j+1}\cdot X)\cdot\exp(-\sigma\cdot\Omega)
\end{aligned} \tag{28}$$

Hence, the expression $\exp(\sigma\cdot\Omega)\cdot X\cdot\exp(-\sigma\cdot\Omega)$ has the Taylor series expansion

$$\exp(\sigma\cdot\Omega)\cdot X\cdot\exp(-\sigma\cdot\Omega)=\sum_j\frac{\sigma^j}{j!}\cdot ad_\Omega^j\cdot X \tag{29}$$

(For a general analytic function f, $f(\sigma)=\sum_j f^{(j)}(0)\cdot\sigma^j/j!$ where $f^{(j)}$ is the j-th derivative of f.) Eq. (26) follows from Eq. (29) with $\sigma=1$.

The derivative of $\exp(\Omega(t))$ has the following relationship to the derivative $\Omega'(t)$,

$$\frac{d}{dt}\exp(\Omega(t))=\left(\sum_j\frac{1}{(j+1)!}\cdot ad_{\Omega(t)}^j\cdot\Omega'(t)\right)\cdot\exp(\Omega(t)) \tag{30}$$

This relation is derived by first applying Eq's. (26) and (25) to simplify the following expression:

$$\begin{aligned}
&\frac{\partial}{\partial\sigma}\left(\left(\frac{\partial}{\partial t}\exp(\sigma\cdot\Omega(t))\right)\cdot\exp(-\sigma\cdot\Omega(t))\right)\\
&=\left(\frac{\partial}{\partial t}\left(\exp(\sigma\cdot\Omega(t))\cdot\Omega(t)\right)\right)\cdot\exp(-\sigma\cdot\Omega(t))\\
&\quad-\left(\frac{\partial}{\partial t}\exp(\sigma\cdot\Omega(t))\right)\cdot\Omega(t)\cdot\exp(-\sigma\cdot\Omega(t))\\
&=\exp(\sigma\cdot\Omega(t))\cdot\Omega'(t)\cdot\exp(-\sigma\cdot\Omega(t))\\
&=\sum_j\frac{1}{j!}\cdot ad^j_{\sigma\cdot\Omega(t)}\cdot\Omega'(t)=\sum_j\frac{\sigma^j}{j!}\cdot ad^j_{\Omega(t)}\cdot\Omega'(t)
\end{aligned}\tag{31}$$

Eq. (31) is integrated over σ to obtain Eq. (30),

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}\exp(\Omega(t))\right)\cdot\exp(-\Omega(t))\\
&=\int_0^1 d\sigma\cdot\frac{\partial}{\partial\sigma}\left(\left(\frac{\partial}{\partial t}\exp(\sigma\cdot\Omega(t))\right)\cdot\exp(-\sigma\cdot\Omega(t))\right)\\
&=\sum_j\frac{1}{(j+1)!}\cdot ad^j_{\Omega(t)}\cdot\Omega'(t)
\end{aligned}\tag{32}$$

Eq. (30) is substituted into Eq. (9), and Eq. (26) is applied with $X = A(-t)$,

$$A(t)+\sum_j\frac{1}{j!}\cdot ad^j_{\Omega(t)}\cdot A(-t)=\sum_j\frac{1}{(j+1)!}\cdot ad^j_{\Omega(t)}\cdot\Omega'(t)\tag{33}$$

The series factors can be simplified to the following form,

$$A(t)+\exp(ad_{\Omega(t)})\cdot A(-t)=\frac{\exp(ad_{\Omega(t)})-\mathbf{I}}{ad_{\Omega(t)}}\cdot\Omega'(t)\tag{34}$$

This relation is solved for $\Omega'(t)$ and simplified using the relation $-ad_{\Omega(t)}=ad_{-\Omega(t)}=ad_{\Omega(-t)}$ (Eq's. (25) and (10)),

$$\begin{aligned}
\Omega'(t)&=\frac{ad_{\Omega(t)}}{\exp(ad_{\Omega(t)})-\mathbf{I}}\cdot\left(A(t)+\exp(ad_{\Omega(t)})\cdot A(-t)\right)\\
&=\frac{ad_{\Omega(t)}}{\exp(ad_{\Omega(t)})-\mathbf{I}}\cdot A(t)+\frac{ad_{\Omega(t)}}{\mathbf{I}-\exp(-ad_{\Omega(t)})}\cdot A(-t)\\
&=\frac{ad_{\Omega(t)}}{\exp(ad_{\Omega(t)})-\mathbf{I}}\cdot A(t)+\frac{ad_{\Omega(-t)}}{\exp(ad_{\Omega(-t)})-\mathbf{I}}\cdot A(-t)
\end{aligned}\tag{35}$$

Eq. (35) is consistent with the relation $\Omega'(-t)=\Omega'(t)$, which is implied by Eq. (10).

The leading factor in Eq. (35) is reformulated to separate its even and odd parts,

$$\frac{ad_{\Omega(t)}}{\exp(ad_{\Omega(t)})-\mathbf{I}} = \tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\left(\coth(\tfrac{1}{2}\cdot ad_{\Omega(t)})-\mathbf{I}\right) \tag{36}$$

This further simplifies Eq. (35) to

$$\begin{aligned}\Omega'(t) &= \tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\left(\coth(\tfrac{1}{2}\cdot ad_{\Omega(t)})-\mathbf{I}\right)\cdot A(t) + \tfrac{1}{2}\cdot ad_{\Omega(-t)}\cdot\left(\coth(\tfrac{1}{2}\cdot ad_{\Omega(-t)})-\mathbf{I}\right)\cdot A(-t)\\ &= \tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\left(\coth(\tfrac{1}{2}\cdot ad_{\Omega(t)})-\mathbf{I}\right)\cdot A(t) - \tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\left(-\coth(\tfrac{1}{2}\cdot ad_{\Omega(t)})-\mathbf{I}\right)\cdot A(-t)\\ &= \tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\coth(\tfrac{1}{2}\cdot ad_{\Omega(t)})\cdot\left(A(t)+A(-t)\right) - \tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\left(A(t)-A(-t)\right)\end{aligned} \tag{37}$$

A Taylor series can be used for the leading terms in Eq. (37),

$$\tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\coth(\tfrac{1}{2}\cdot ad_{\Omega(t)}) = \sum_{j=0}^{\infty} c_j\cdot ad_{\Omega(t)}^{2\cdot j} \tag{38}$$

The series coefficients are defined by the following recursion relation, which can be solved for c_k after $c_0,\ldots,c_{k-1}$ have been determined:

$$\sum_{j=0}^{k}\frac{c_j}{(2k-2j+1)!} = \frac{1}{2(2k)!}\quad\left(\{c_0,c_1,\ldots\}=\{1,\tfrac{1}{12},-\tfrac{1}{720},\ldots\}\right) \tag{39}$$

(The coefficients are related to the Bernoulli numbers: $c_j = B_{2j}/(2j)!$).

The convergence condition for Eq. (38) is

$$\rho(ad_{\Omega(t)}) < 2\cdot\pi \tag{40}$$

where $\rho(\ldots)$ is the spectral radius (maximum absolute eigenvalue) of $ad_{\Omega(t)}$. It is presumed that t is sufficiently small to satisfy Eq. (40).

Eq. (38) is substituted in Eq. (37) to obtain the following result (with the initial condition from Eq. (9)),

The centered Magnus theorem:

$$\Omega'(t) = \left(\sum_{j=0}^{\infty} c_j\cdot ad_{\Omega(t)}^{2\cdot j}\cdot\left(A(t)+A(-t)\right)\right) - \tfrac{1}{2}\cdot ad_{\Omega(t)}\cdot\left(A(t)-A(-t)\right),\quad \Omega(0)=\mathbf{0} \tag{41}$$

This implies that $\Omega(t) = Ot$ (to first order, $\Omega(t) = 2\cdot A(0)\cdot t + Ot^3$), hence $ad_{\Omega(t)} = Ot$. The first few terms of Eq. (41), up to order 5, are

$$\begin{aligned}\Omega'(t) = {} & A(t) + A(-t) + \tfrac{1}{12}\cdot[\Omega(t),[\Omega(t), A(t) + A(-t)]] \\ & - \tfrac{1}{720}\cdot[\Omega(t),[\Omega(t),[\Omega(t),[\Omega(t), A(t) + A(-t)]]]] + Ot^6 \\ & - \tfrac{1}{2}\cdot[\Omega(t), A(t) - A(-t)]\end{aligned} \tag{42}$$

Series expansion of the Magnus exponent

$\Omega(t)$ and $A(t)$ can be represented by Taylor series expansions,

$$\Omega(t) = \sum_k \frac{\Omega^{(k)}(0)}{k!}\cdot t^k \tag{43}$$

$$A(t) = \sum_k \frac{A^{(k)}(0)}{k!}\cdot t^k \tag{44}$$

With $\Omega(0) = \mathbf{0}$, Eq. (41) determines $\Omega^{(k)}(0)$ from $\Omega'(0)$, $\Omega''(0)$, … $\Omega^{(k-1)}(0)$. The process can be illustrated using Eq. (42) with substitution from Eq. (44),

$$\begin{aligned}\Omega'(t) = {} & 2\cdot\left(A(0) + \tfrac{1}{2}\cdot t^2\cdot A''(0) + \tfrac{1}{24}\cdot t^4\cdot A^{(4)}(0)\right) \\ & + \tfrac{1}{6}\cdot[\Omega(t),[\Omega(t), A(0) + \tfrac{1}{2}\cdot t^2\cdot A''(0)]] \\ & - \tfrac{1}{360}\cdot[\Omega(t),[\Omega(t),[\Omega(t),[\Omega(t), A(0)]]]] + Ot^6 \\ & - [\Omega(t), t\cdot A'(0) + \tfrac{1}{6}\cdot t^3\cdot A^{(3)}(0)] \\ & + Ot^6\end{aligned} \tag{45}$$

With $t = 0$, this yields

$$\Omega'(0) = 2\cdot A(0) \tag{46}$$

The substitution $\Omega(t) = t\cdot\Omega'(0) + \tfrac{1}{6}\cdot t^3\cdot\Omega^{(3)}(0) + Ot^5$ (Eq. (43)) is made on the right side of Eq. (45) (omitting even-order terms due to symmetry, Eq. (10)), and $\Omega'(t) = \Omega'(0) + \tfrac{1}{2}\cdot t^2\cdot\Omega^{(3)}(0) + Ot^4$ is substituted on the left, resulting in an equation that reduces to

$$\Omega^{(3)}(0) = 2\cdot A''(0) - 4\cdot[A(0), A'(0)] \tag{47}$$

The process is repeated, with the higher-order substitutions $\Omega(t) = t\cdot\Omega'(0) + \tfrac{1}{6}\cdot t^3\cdot\Omega^{(3)}(0) + \tfrac{1}{120}\cdot t^5\cdot\Omega^{(5)}(0) + Ot^7$ and $\Omega'(t) = \Omega'(0) + \tfrac{1}{2}\cdot t^2\cdot\Omega^{(3)}(0) + \tfrac{1}{24}\cdot t^4\cdot\Omega^{(5)}(0) + Ot^6$, to obtain $\Omega^{(5)}(0)$,

$$\begin{aligned}\Omega^{(5)}(0) = & \\ & 2\cdot A^{(4)}(0) \\ & +\tfrac{16}{3}\cdot[A(0),[A(0),A''(0)+[A(0),A'(0)]]] \\ & -8\cdot[A(0),A^{(3)}(0)] \\ & -8\cdot[A''(0)-2\cdot[A(0),A'(0)],A'(0)]\end{aligned} \tag{48}$$

Eq's. (46)-(48) can be used to define the order-6 approximation $\Omega(t) = t\cdot\Omega'(0) + \frac{1}{6}\cdot t^3\cdot\Omega^{(3)}(0) + \frac{1}{120}\cdot t^5\cdot\Omega^{(5)}(0) + Ot^7$, which contains five A-dependent terms ($A(0)$, …, $A^{(4)}(0)$) and five distinct commutator terms. Five-point Savitzky-Golay filters (Ref. 2) can be used to estimate the derivatives within the error order Ot^7.

However, a simplified approximation of the same accuracy order using fewer A-dependent terms and fewer commutators can be obtained by parameterizing the function $A(t)$ in terms of its Legendre moments over the integration interval, rather than its derivatives.

Legendre-moment expansion of the Magnus exponent

$\Omega(t)$ can be estimated by integrating Eq. (41),

$$\Omega(t) = \int_0^t d\tau\cdot\left(\left(\sum_{j=0}^{\infty} c_j\cdot ad_{\Omega(\tau)}^{2\cdot j}\cdot\left(A(\tau)+A(-\tau)\right)\right) - \tfrac{1}{2}\cdot ad_{\Omega(\tau)}\cdot\left(A(\tau)-A(-\tau)\right)\right) \tag{49}$$

Eq. (49) implies that

$$\Omega(0) = \mathbf{0},\quad \Omega'(0) = 2\cdot A(0) \tag{50}$$

Hence,

$$\Omega(t) = Ot \tag{51}$$

A sequence of successively more accurate approximations to $\Omega(t)$, denoted as $\Omega_0(t)$, $\Omega_1(t)$, $\Omega_2(t)$, …, are obtained by starting with $\Omega_0(t) = Ot$ and defining $\Omega_{k+1}(t)$ by substituting $\Omega_k(t)$ on the right side of Eq. (49),

$$\begin{aligned}&\Omega_0(t) = Ot, \\ &\Omega_{k+1}(t) = \int_0^t d\tau\cdot\left(\left(\sum_{j=0}^{\infty} c_j\cdot ad_{\Omega_k(\tau)}^{2\cdot j}\cdot\left(A(\tau)+A(-\tau)\right)\right) - \tfrac{1}{2}\cdot ad_{\Omega_k(\tau)}\cdot\left(A(\tau)-A(-\tau)\right)\right)\end{aligned} \tag{52}$$

By induction, each $\Omega_k(t)$ estimate is of order t and has accuracy order $2\cdot k$ (i.e., the error residual is $Ot^{2\cdot k+1}$):

$$\Omega_k(t) = \Omega'(0)\cdot t + \ldots + Ot^{2\cdot k+1} = Ot \tag{53}$$

Eq. (53) implies that $ad_{\Omega_k(t)}$ is correspondingly of order t and has error $Ot^{2\cdot k+1}$. The right-hand integrand term in Eq. (52) has error $O\tau^{2\cdot k+2}$ (because $ad_{\Omega_k(\tau)} = O\tau^{2\cdot k+1}$ and $A(\tau) - A(-\tau) = O\tau$); and for $j > 0$, the summand factor $ad^{2\cdot j}_{\Omega_k(\tau)}$ has error of order $\tau^{2\cdot k+2\cdot j}$. Thus, the integrand error order is $O\tau^{2\cdot k+2}$ and the integral error is $O\tau^{2\cdot k+3}$, establishing that $\Omega_{k+1}(t)$ has accuracy order $2\cdot k+2$.

The $ad^{2\cdot j}_{\Omega_k(\tau)}$ summand factor is of order $\tau^{2\cdot j}$ and is subsumed by the integrand error $O\tau^{2\cdot k+2}$ if $2\cdot j \geq 2\cdot k+2$. The infinite sum can thus be truncated to $j \leq k$,

$$\Omega_{k+1}(t) = \int_0^t d\tau \cdot \left(\left(\sum_{j=0}^{k} c_j \cdot ad^{2\cdot j}_{\Omega_k(\tau)} \cdot \big(A(\tau) + A(-\tau)\big) \right) - \tfrac{1}{2} \cdot ad_{\Omega_k(\tau)} \cdot \big(A(\tau) - A(-\tau)\big) \right) \tag{54}$$

The integral can be evaluated by projecting $A(\tau)$ onto Legendre basis functions $P_i(\tau / t)$,

$$A(\tau) = \sum_{i=0}^{\infty} P_i(\tau / t) \cdot \alpha_i(t) \tag{55}$$

The Legendre functions (Ref. 3) are polynomials defined by

$$P_0(u) = 1, \quad P_1(u) = u, \quad (i+1)\cdot P_{i+1}(u) = (2\cdot i+1)\cdot u \cdot P_i(u) - i \cdot P_{i-1}(u) \tag{56}$$

The polynomials satisfy the orthogonality relations,

$$\int_{-1}^{1} P_i(u) \cdot P_j(u) \cdot du = \frac{2}{2\cdot i+1} \cdot \delta_{i,j} \tag{57}$$

($\delta_{i,j}$ is the Kronecker delta symbol.) They also satisfy the parity conditions,

$$P_i(-u) = (-1)^i \cdot P_i(u) \tag{58}$$

The coefficients $\alpha_i(t)$ in Eq. (55) are matrices determined from the orthogonality relations, Eq. (57),

$$\alpha_i(t) = \frac{i+\frac{1}{2}}{t} \cdot g_i(t) \tag{59}$$

where $g_i(t)$ is the i-th Legendre moment of $A(\tau)$,

$$g_i(t) = \int_{-t}^{t} P_i(\tau / t) \cdot A(\tau) \cdot d\tau \quad \left(\text{Equivalently, } g_i(t) = t \cdot \int_{-1}^{1} P_i(u) \cdot A(t \cdot u) \cdot du \right) \tag{60}$$

$P_i(u)$ is a polynomial of degree i with highest-degree term $c \cdot u^i$ ($c = (2 \cdot i - 1)!!/\, i!$). It follows that u^i is equal to $P_i(u) / c$ minus a polynomial of degree $i-1$, and by induction, u^i is a linear combination of $P_0(u)$, $P_1(u)$, …, $P_i(u)$. Hence, the orthogonality condition (Eq. (57)) implies that

$$\int_{-1}^{1} P_i(u) \cdot u^j \cdot du = 0 \quad \text{for } j < i \tag{61}$$

(Replace u^j with an equivalent linear combination of $P_0(u)$, $P_1(u)$, …, $P_j(u)$ and apply Eq. (57).) If a Taylor series is substituted for $A(t \cdot u)$ in Eq. (60), the terms of degree less than i have zero contribution to the integral as a result of Eq. (61); hence

$$g_i(t) = t \cdot \int_{-1}^{1} P_i(u) \cdot \left(\frac{A^{(i)}(0)}{i!} \cdot (t \cdot u)^i + \ldots \right) \cdot du = t^{i+1} \cdot \frac{A^{(i)}(0)}{i!} \cdot \left(\int_{-1}^{1} P_i(u) \cdot u^i \cdot du \right) + \ldots \tag{62}$$

("…" represents higher-order terms, and the integral is nonzero because $u^i \cdot P_i(u)$ is an even function.) $g_i(t)$ is therefore of order t^{i+1} (and $\alpha_i(t)$ is of order t^i),

$$g_i(t) = O t^{i+1}, \quad \alpha_i(t) = O t^i \tag{63}$$

Eq. (55) is substituted in Eq. (54), and the parity conditions, Eq. (58), are applied to obtain

$$\begin{aligned} \Omega_{k+1}(t) = {} & 2 \cdot \sum_{i=0}^{\infty} \left(\sum_{j=0}^{k} c_j \cdot \int_0^t d\tau \cdot P_{2 \cdot i}(\tau / t) \cdot ad_{\Omega_k(\tau)}^{2 \cdot j} \right) \cdot \alpha_{2 \cdot i}(t) \\ & - \sum_{i=0}^{\infty} \left(\int_0^t d\tau \cdot P_{2 \cdot i+1}(\tau / t) \cdot ad_{\Omega_k(\tau)} \right) \cdot \alpha_{2 \cdot i+1}(t) \end{aligned} \tag{64}$$

The integrand in the first term is of order $\tau^{2 \cdot j}$ (because $ad_{\Omega_k(\tau)} = O\tau$ and $P_{2 \cdot i}(\tau / t) = O1$), so the integral is of order $t^{2 \cdot j+1}$ and the integral's product with $\alpha_{2 \cdot i}(t)$ is of order $t^{2 \cdot i+2 \cdot j+1}$ (because $\alpha_{2 \cdot i}(t) = O t^{2 \cdot i}$). The product will be subsumed by the $\Omega_{k+1}(t)$ error, $O t^{2 \cdot k+3}$, if $2 \cdot i + 2 \cdot j + 1 \geq 2 \cdot k + 3$, so the double sum can be truncated to $i + j \leq k$. Also, the $j = 0$ term is zero when $i > 0$ because the integral vanishes in that case. The integrand in the last term is of order τ^2 / t (because $ad_{\Omega_k(\tau)} = O\tau$ and $P_{2 \cdot i+1}(\tau / t) = O\tau / t$); hence the integral is of order t^2 and the last summand is of order $t^{2 \cdot i+3}$ (because $\alpha_{2 \cdot i+1}(t) = O t^{2 \cdot i+1}$). The summand will be subsumed

by the $\Omega_{k+1}(t)$ error, $Ot^{2\cdot k+3}$, if $2\cdot i+3\geq 2\cdot k+3$, so the second infinite sum can be truncated to $i\leq k-1$. With these modifications, Eq. (64) simplifies to

$$\begin{aligned}\Omega_{k+1}(t) &= 2\cdot\sum_{i=0}^{k}\left(\sum_{j=\min(i,1)}^{k-i} c_j\cdot\int_0^t d\tau\cdot P_{2\cdot i}(\tau/t)\cdot ad^{2\cdot j}_{\Omega_k(\tau)}\right)\cdot\alpha_{2\cdot i}(t)\\ &-\sum_{i=0}^{k-1}\left(\int_0^t d\tau\cdot P_{2\cdot i+1}(\tau/t)\cdot ad_{\Omega_k(\tau)}\right)\cdot\alpha_{2\cdot i+1}(t)\end{aligned} \tag{65}$$

The first iteration of Eq. (65) ($k=0$, $\Omega_k(t)=Ot$) yields

$$\Omega_1(t) = g_0(t)+Ot^3 \tag{66}$$

The second iteration takes the following form,

$$\begin{aligned}\Omega_2(t) &= g_0(t)+\tfrac{1}{12}\cdot\int_0^t d\tau\cdot ad^2_{g_0(\tau)}\cdot g_0(t)/t-\tfrac{3}{2}\cdot\int_0^t d\tau\cdot\tau\cdot ad_{g_0(\tau)}\cdot g_1(t)/t^2+Ot^5\\ &= g_0(t)+\tfrac{1}{12}\cdot\int_0^t d\tau\cdot[g_0(\tau),[g_0(\tau),g_0(t)]]/t-\tfrac{3}{2}\cdot\int_0^t d\tau\cdot\tau\cdot[g_0(\tau),g_1(t)]/t^2+Ot^5\end{aligned} \tag{67}$$

The integrand factor $g_0(\tau)$ is approximately proportional to τ ($g_0(\tau)=O\tau$), implying that $g_0(\tau)$ approximates to $(\tau/t)\cdot g_0(t)$. The approximation can be formally derived by projecting $g_0(\tau)$ onto the $g_i(t)$ bases via Eq's. (60), (55), and (59),

$$\begin{aligned}g_0(\tau) &= \int_{-\tau}^{\tau} A(\tau')\cdot d\tau'\\ &= \int_{-\tau}^{\tau}\left(\sum_{i=0}^{\infty}(i+\tfrac{1}{2})\cdot P_i(\tau'/t)\cdot g_i(t)/t\right)\cdot d\tau'\\ &= \sum_{i=0}^{\infty}(i+\tfrac{1}{2})\cdot\left(\int_{-\tau}^{\tau}P_i(\tau'/t)\cdot d\tau'\right)\cdot g_i(t)/t\\ &= (\tau/t)\cdot\sum_{i=0}^{\infty}(i+\tfrac{1}{2})\cdot\left(\int_{-1}^{1}P_i(u\cdot\tau/t)\cdot du\right)\cdot g_i(t)\\ &= (\tau/t)\cdot\sum_{i=0}^{1}(i+\tfrac{1}{2})\cdot\left(\int_{-1}^{1}P_i(u\cdot\tau/t)\cdot du\right)\cdot g_i(t)+O\tau\cdot t^2\\ &= (\tau/t)\cdot g_0(t)+O\tau\cdot t^2\end{aligned} \tag{68}$$

This substitution simplifies Eq. (67) to

$$\Omega_2(t) = g_0(t)-\tfrac{1}{2}\cdot[g_0(t),g_1(t)]+Ot^5 \tag{69}$$

The substitution $ad_{\Omega_2(\tau)} = ad_{g_0(\tau)}-\frac{1}{2}\cdot ad_{[g_0(\tau),g_1(\tau)]}+O\tau^5$ (from Eq's. (69), (24), and (25)) is made in Eq. (65) with $k=2$, and terms that are subsumed by the Ot^7 error are omitted,

$$
\begin{aligned}
&\Omega_3(t) = g_0(t) \\
&\quad + \begin{pmatrix} \frac{1}{12}\cdot\int_0^t d\tau\cdot\left(ad_{g_0(\tau)}^2 - \frac{1}{2}\cdot ad_{g_0(\tau)}\cdot ad_{[g_0(\tau),g_1(\tau)]} - \frac{1}{2}\cdot ad_{[g_0(\tau),g_1(\tau)]}\cdot ad_{g_0(\tau)}\right) \\ -\frac{1}{720}\cdot\int_0^t d\tau\cdot ad_{g_0(\tau)}^4 \end{pmatrix}\cdot g_0(t)/t \\
&\quad + \frac{5}{12}\cdot\left(\int_0^t d\tau\cdot P_2(\tau/t)\cdot ad_{g_0(\tau)}^2\right)\cdot g_2(t)/t \\
&\quad - \frac{3}{2}\cdot\left(\int_0^t d\tau\cdot P_1(\tau/t)\cdot\left(ad_{g_0(\tau)} - \frac{1}{2}\cdot ad_{[g_0(\tau),g_1(\tau)]}\right)\right)\cdot g_1(t)/t \\
&\quad - \frac{7}{2}\cdot\left(\int_0^t d\tau\cdot P_3(\tau/t)\cdot ad_{g_0(\tau)}\right)\cdot g_3(t)/t \\
&\quad + Ot^7 \\
&= g_0(t) \\
&\quad + \begin{pmatrix} \frac{1}{12}\cdot\int_0^t d\tau\cdot\begin{pmatrix} [g_0(\tau),[g_0(\tau),g_0(t)]] \\ -\frac{1}{2}\cdot[g_0(\tau),[[g_0(\tau),g_1(\tau)],g_0(t)]] \\ -\frac{1}{2}\cdot[[g_0(\tau),g_1(\tau)],[g_0(\tau),g_0(t)]] \end{pmatrix} \\ -\frac{1}{720}\cdot\int_0^t d\tau\cdot[g_0(\tau),[g_0(\tau),[g_0(\tau),[g_0(\tau),g_0(t)]]]] \end{pmatrix}/t \\
&\quad + \frac{5}{12}\cdot\left(\int_0^t d\tau\cdot P_2(\tau/t)\cdot[g_0(\tau),[g_0(\tau),g_2(t)]]\right)/t \\
&\quad - \frac{3}{2}\cdot\left(\int_0^t d\tau\cdot P_1(\tau/t)\cdot\begin{pmatrix}[g_0(\tau),g_1(t)] \\ -\frac{1}{2}\cdot[[g_0(\tau),g_1(\tau)],g_1(t)]\end{pmatrix}\right)/t \\
&\quad - \frac{7}{2}\cdot\left(\int_0^t d\tau\cdot P_3(\tau/t)\cdot[g_0(\tau),g_3(t)]\right)/t \\
&\quad + Ot^7
\end{aligned}
\tag{70}
$$

The $g_0(\tau)$ integrand order is approximated by extending Eq. (68) to one higher order,

$$
\begin{aligned}
&g_0(\tau) = (\tau/t)\cdot\sum_{i=0}^{3}(i+\tfrac{1}{2})\cdot\left(\int_{-1}^{1} P_i(u\cdot\tau/t)\cdot du\right)\cdot g_i(t) + O\ \tau\cdot t^4 \\
&= (\tau/t)\cdot\left(g_0(t) - \tfrac{5}{2}\cdot(1-(\tau/t)^2)\cdot g_2(t)\right) + O\tau\cdot t^4
\end{aligned}
\tag{71}
$$

$g_1(\tau)$ approximates to $(\tau/t)^2\cdot g_1(t)$ because $g_1(\tau) = O\tau^2$. This is the leading term of the projection of $g_1(\tau)$ onto the $g_i(t)$ bases (via Eq's. (60), (55), and (59)), which extends the approximation to higher order (although higher-order terms will not actually be needed for Ω_3):

$$
\begin{aligned}
g_1(\tau) &= \int_{-\tau}^{\tau} P_1(\tau'/\tau)\cdot A(\tau')\cdot d\tau' \\
&= \int_{-\tau}^{\tau} P_1(\tau'/\tau)\cdot\left(\sum_{i=0}^{\infty}(i+\tfrac{1}{2})\cdot P_i(\tau'/t)\cdot g_i(t)/t\right)\cdot d\tau' \\
&= \sum_{i=0}^{\infty}(i+\tfrac{1}{2})\cdot\left(\int_{-\tau}^{\tau} P_1(\tau'/\tau)\cdot P_i(\tau'/t)\cdot d\tau'\right)\cdot g_i(t)/t \\
&= (\tau/t)\cdot\sum_{i=0}^{\infty}(i+\tfrac{1}{2})\cdot\left(\int_{-1}^{1} P_1(u)\cdot P_i(u\cdot\tau/t)\cdot du\right)\cdot g_i(t) \\
&= (\tau/t)\cdot\sum_{i=1}^{4}(i+\tfrac{1}{2})\cdot\left(\int_{-1}^{1} P_1(u)\cdot P_i(u\cdot\tau/t)\cdot du\right)\cdot g_i(t) + O\tau\cdot t^5 \\
&= (\tau/t)^2\cdot\left(g_1(t)-\tfrac{7}{2}\cdot(1-(\tau/t)^2)\cdot g_3(t)\right)+O\tau\cdot t^5
\end{aligned}
\tag{72}
$$

Eq's. (71) and (72) simplify Eq. (70) to

$$
\begin{aligned}
\Omega_3(t) &= g_0(t) \\
&\quad+\left(\tfrac{1}{12}\cdot\int_0^t d\tau\cdot\begin{pmatrix}-\tfrac{5}{2}\cdot(\tau/t)^2\cdot(1-(\tau/t)^2)\cdot[g_0(t),[g_2(t),g_0(t)]] \\ -\tfrac{1}{2}\cdot(\tau/t)^4\cdot[g_0(t),[[g_0(t),g_1(t)],g_0(t)]]\end{pmatrix}\right)/t \\
&\quad+\tfrac{5}{12}\cdot\left(\int_0^t d\tau\cdot P_2(\tau/t)\cdot(\tau/t)^2\cdot[g_0(t),[g_0(t),g_2(t)]]\right)/t \\
&\quad-\tfrac{3}{2}\cdot\left(\int_0^t d\tau\cdot P_1(\tau/t)\cdot\begin{pmatrix}(\tau/t)\cdot[g_0(t),g_1(t)] \\ -\tfrac{5}{2}\cdot(\tau/t)\cdot(1-(\tau/t)^2)\cdot[g_2(t),g_1(t)] \\ -\tfrac{1}{2}\cdot(\tau/t)^3\cdot[[g_0(t),g_1(t)],g_1(t)]\end{pmatrix}\right)/t \\
&\quad-\tfrac{7}{2}\cdot\left(\int_0^t d\tau\cdot P_3(\tau/t)\cdot(\tau/t)\cdot[g_0(t),g_3(t)]\right)/t \\
&\quad+Ot^7 \\
&= g_0(t) \\
&\quad-\tfrac{1}{2}\cdot[g_0(t),g_1(t)] \\
&\quad+\tfrac{1}{2}\cdot[g_2(t),g_1(t)] \\
&\quad+\tfrac{3}{20}\cdot[[g_0(t),g_1(t)],g_1(t)] \\
&\quad+\tfrac{1}{12}\cdot[g_0(t),[g_0(t),g_2(t)]] \\
&\quad+\tfrac{1}{120}\cdot[g_0(t),[g_0(t),[g_0(t),g_1(t)]]] \\
&\quad+Ot^7
\end{aligned}
\tag{73}
$$

The three-commutator, order-6 Magnus expansion

The $\Omega(t)$ approximation from Eq. (73) is succinctly restated with the functions' t dependence implicit,

$$\begin{aligned}\Omega &= g_0 \\ &\quad -\tfrac{1}{2}\cdot[g_0, g_1] \\ &\quad +\tfrac{1}{2}\cdot[g_2, g_1] \\ &\quad +\tfrac{3}{20}\cdot[[g_0, g_1], g_1] \\ &\quad +\tfrac{1}{12}\cdot[g_0,[g_0, g_2]] \\ &\quad +\tfrac{1}{120}\cdot[g_0,[g_0,[g_0, g_1]]] \\ &\quad +Ot^7\end{aligned} \tag{74}$$

Eq. (74) is consolidated by merging the first three commutators ($[\ldots, g_1]$) and the last two commutators ($[g_0(t),[g_0(t),\ldots]]$),

$$\begin{aligned}\Omega &= g_0 \\ &\quad -\tfrac{1}{2}\cdot[g_0 - g_2 - \tfrac{3}{10}\cdot[g_0, g_1], g_1] \\ &\quad +\tfrac{1}{12}\cdot[g_0,[g_0, g_2 + \tfrac{1}{10}\cdot[g_0, g_1]]] \\ &\quad +Ot^7\end{aligned} \tag{75}$$

The common subexpression $[g_0, g_1]$ is separated out to reduce this to a four-commutator formula,

$$\begin{aligned}C &= [g_0, g_1], \\ \Omega &= g_0 \\ &\quad -\tfrac{1}{2}\cdot[g_0 - g_2 - \tfrac{3}{10}\cdot C, g_1] \\ &\quad +\tfrac{1}{12}\cdot[g_0,[g_0, g_2 + \tfrac{1}{10}\cdot C]] \\ &\quad +Ot^7\end{aligned} \tag{76}$$

Eq. (76) can be transformed into a three-commutator formula by redistributing the commutators so that high-order terms are consolidated in a commutator term of order Ot^7, which can be neglected. The left operands of the two commutators in Eq. (76) differ by a quantity $-g_2 - \tfrac{3}{10}\cdot C$, which is of order Ot^3. The right operand of the second commutator, $[g_0, g_2 + \tfrac{1}{10}\cdot C]$, is of order Ot^4. The commutator formed from these two operands is therefore of order Ot^7,

$$[-g_2 - \tfrac{3}{10}\cdot C,[g_0, g_2 + \tfrac{1}{10}\cdot C]] = Ot^7 \tag{77}$$

A scalar multiple of this expression can therefore be added to the second commutator in Eq. (76), so that two commutators' first operands match, without impacting the approximation's accuracy order,

$$\begin{aligned}&\tfrac{1}{12}\cdot[g_0,[g_0,g_2+\tfrac{1}{10}\cdot C]]=\tfrac{1}{2}\cdot[g_0,\tfrac{1}{6}\cdot[g_0,g_2+\tfrac{1}{10}\cdot C]]\\&=\tfrac{1}{2}\cdot[g_0-g_2-\tfrac{3}{10}\cdot C,\tfrac{1}{6}\cdot[g_0,g_2+\tfrac{1}{10}\cdot C]]+Ot^7\end{aligned} \tag{78}$$

With this substitution, the two $[g_0+\ldots,\ldots]$ commutators in Ω can be merged,

$$\begin{aligned}\Omega&=g_0\\&\quad-\tfrac{1}{2}\cdot[g_0-g_2-\tfrac{3}{10}\cdot C,g_1]\\&\quad+\tfrac{1}{2}\cdot[g_0-g_2-\tfrac{3}{10}\cdot C,\tfrac{1}{6}\cdot[g_0,g_2+\tfrac{1}{10}\cdot C]]\\&\quad+Ot^7\\&=g_0\\&\quad-\tfrac{1}{2}\cdot[g_0-g_2-\tfrac{3}{10}\cdot C,g_1-\tfrac{1}{6}\cdot[g_0,g_2+\tfrac{1}{10}\cdot C]]\\&\quad+Ot^7\end{aligned} \tag{79}$$

In summary, the order-6, 3-commutator algorithm approximating the Magnus exponent Ω, over a symmetric integration interval $-t<\tau<t$, has the following form (from Eq's. (60), (56), (76), and (79)),

$$\begin{aligned}&g_0=\int_{-t}^{t}A(\tau)\cdot d\tau\\&g_1=\int_{-t}^{t}(\tau/t)\cdot A(\tau)\cdot d\tau\\&g_2=\int_{-t}^{t}(\tfrac{3}{2}\cdot(\tau/t)^2-\tfrac{1}{2})\cdot A(\tau)\cdot d\tau\\&C=[g_0,g_1],\\&\Omega=g_0-\tfrac{1}{2}\cdot[g_0-g_2-\tfrac{3}{10}\cdot C,g_1-\tfrac{1}{6}\cdot[g_0,g_2+\tfrac{1}{10}\cdot C]]+Ot^7\end{aligned} \tag{80}$$

The integrals in Eq's. (80) can be evaluated via an order-six quadrature approximation such as a three-point, Gauss-Legendre quadrature (Ref. 4),

$$\begin{aligned}&\text{With}\quad w_0=4/9,\quad w_1=5/18,\quad q=\sqrt{\tfrac{3}{5}}:\\&\int_{-t}^{t}f(\tau)\cdot d\tau=2\cdot t\cdot\left(w_0\cdot f(0)+w_1\cdot(f(-q\cdot t)+f(q\cdot t))\right)+Ot^7\end{aligned} \tag{81}$$

The Ot^7 error is inapplicable to g_1 and g_2 because the integrands depend on t. The t denominator in the g_1 integrand reduces the error to Ot^6, and the t^2 denominator in the g_2 integrand reduces the error to Ot^5. But in Eq's. (80) g_1 and g_2 are always paired with other factors that scale their errors to Ot^7.

For a general non-centered integration interval $t_0 < \tau < t_1$, Eq's. (80) are applied with t replaced with the interval half-width, $h = \frac{1}{2} \cdot (t_1 - t_0)$, and the integrand factor $A(\tau)$ is replaced with $A(t_{\text{ctr}} + \tau)$, $t_{\text{ctr}} = \frac{1}{2} \cdot (t_0 + t_1)$.

Additional resources

Software and documentation on GitHub:
https://github.com/kjinnovation/explde

Software on MathWorks File Exchange:
https://www.mathworks.com/matlabcentral/fileexchange/183695-explde

A more complete exposition of the method, including 8th- and 10th-order methods, is detailed in Ref's. [5] and [6].

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