Euler Product-Power Series Representation of the Dirichlet Eta Function

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Abstract:

In this paper, I demonstrate how the Dirichlet Eta function may be represented as a sum of its fundamental frequencies through the use of a power series whose coefficients are the Euler Product representation of the Riemann Zeta function.

Introduction

Let the Zeta Function be defined as :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \tag{1}$$

Let the Dirichlet Eta Function be defined as the alternating version of (1):

$$\eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} \tag{2}$$

Equation (1) can be shown to be split into a sum of odd and even integers through factoring akin to Euler's method for deriving the Euler Product.

$$\zeta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) + \frac{1}{2^s}\,\zeta(s) \tag{3}$$

A simple sign change produces the Dirichlet Eta function:

$$\eta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{1}{2^s}\,\zeta(s) \tag{4}$$

Power Series Representation

When s = ½ + bi, this difference can be expressed as a complex conjugate division:

$$\eta(0.5+b\,i) = -\left(\frac{1-\frac{1}{2^{0.5+b\,i}}}{\frac{1}{2^{0.5+b\,i}}}\right)^{*}\zeta(s) \tag{5}$$

Which can be rearranged into a more appealing form:

$$\eta(0.5+b\,i) = \left(\frac{\frac{1-\frac{1}{2^{0.5+b\,i}}}{1-2^{0.5-b\,i}}}{1-\frac{1-\frac{1}{2^{0.5+b\,i}}}{1-2^{0.5-b\,i}}}\right)^{*}\zeta(0.5+b\,i) \tag{6}$$

Whose appeal is made obvious by a simple variable substitution, where the magnitude of x is always less than 1.

$$\eta(0.5+b\,i) = \left(\frac{x}{1-x}\right)^{*}\zeta(0.5+b\,i) \tag{7}$$

Which can be expressed as a Power Series:

$$\eta(0.5+b\,i) = \sum_{n=1}^{\infty} a\,(x^{*})^{n} \tag{8}$$

Where the coefficient is the Riemann Zeta Function, and more specifically, the Euler Product representation of the Riemann Zeta function.

$$\eta(0.5 + b\,i) = \sum_{n=1}^{\infty} \left(\prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{(p_n)^{0.5+b\,i}}} \right) (x^*)^n \tag{9}$$

Where,

$$x = \frac{1 - \frac{1}{2^{0.5+b\,i}}}{1 - 2^{0.5-b\,i}} \tag{10}$$

Thus, the Dirichlet Eta Function can be thought of as an infinite sum of its fundamental frequencies. This representation explicitly relies on the real part of input s being equal to1/2, and makes the meaning of the zeros of the Riemann Zeta Function apparent. In other words, an Euler Product can only equal zero if one of its factor is zero, but a sum of Euler Products can equal zero through elementary addition and substraction.

References

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