The Non-Existence of Non-Trivial Periodic Orbits in the Collatz Mapping

Xiaofeng Hu

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Abstract

The Collatz conjecture states that for any given positive integer N, if N is even, divide it by 2; if N is odd, multiply it by 3 and add 1. Repeating this process,N will eventually become 1. This paper proves that any positive odd integer O other than 1 cannot return to itself no matter how many times the iteration is performed. We derive the general formula satisfying this condition and rigorously prove by mathematical induction that this formula equals 1 uniquely in the set of positive odd integers. We thus conclude that there are no non-trivial periodic orbits in the Collatz mapping.

1 Introduction

Research on the Collatz conjecture falls into two categories: on the computational side, Barina (2025) extended the verification bound to 2^{71} ; on the theoretical side, Tao (2020) proved that almost all orbits are bounded in logarithmic density, which is the strongest deterministic conclusion to date. Existing results still have limitations: computational verification only covers a finite order of magnitude and cannot be generalized to infinity; the non-existence of non-trivial periodic orbits in the Collatz mapping has not yet been fully proven. This paper proves that any positive odd integers O other than 1 cannot return to itself after any number of iterations, leading to the conclusion that there are no non-trivial periodic orbits in the Collatz mapping. This paper focuses on the Collatz conjecture: for any positive integer N , define the iterative function$f(\mathrm{N}) = \begin{cases} \mathrm{N}/2, & 2 \mid \mathrm{N} \\ 3\mathrm{N}{+}1, & 2 \nmid \mathrm{N} \end{cases}$

For a positive integer$N$,if$N$is even compute$N/2$ if the result is still even,continue dividing by 2 until f(N)is odd Let O_0=f(N),If$N$is odd letO_0=N

Let $a_1 \in \mathbb{N}^+$. We define one iteration as the process of deriving the odd number O_1 from the odd number O_0 , given by:

$$O_1 = \frac{3O_0 + 1}{2^{a_1}} \tag{1.1}$$

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2 Establishment of the General Formula

Let O be a positive odd number, and $i, a_1, a_2 ... a_n, n, \in \mathbb{N}^+$

If the result of 1 iteration equals O itself, we have:

$$O = \frac{3O+1}{2^{a_1}}$$

Rearranging gives:

$$O = \frac{1}{2^{a_1} - 3}$$

If the result of 2 iteration equals O itself, we have:

$$\frac{3 \cdot \frac{3O+1}{2^{a_1}} + 1}{2^{a_2}} = O$$

Rearranging gives:

$$O = \frac{3 + 2^{a_1}}{2^{a_1+a_2} - 3^2}$$

If the result of 3 iteration equals O itself, we have:

$$\frac{3 \cdot \frac{3 \cdot \frac{3O+1}{2^{a_1}} + 1}{2^{a_2}} + 1}{2^{a_3}} = O$$

Rearranging gives:

$$O = \frac{3^2 + 3 \cdot 2^{a_1} + 2^{a_1+a_2}}{2^{a_1+a_2+a_3} - 3^3}$$

If the result of n iteration equals O itself, we have:

$$O = \frac{3^{n-1} + 3^{n-2} \cdot 2^{a_1} + \cdots + 3 \cdot 2^{a_1+a_2+\cdots+a_{n-2}} + 2^{a_1+a_2+\cdots+a_{n-1}}}{2^{a_1+a_2+\cdots+a_n} - 3^n}$$

Let the numerator be:

$$M_n = 3^{n-1} + 3^{n-2} \cdot 2^{a_1} + \cdots + 3 \cdot 2^{a_1+a_2+\cdots+a_{n-2}} + 2^{a_1+a_2+\cdots+a_{n-1}}$$

Let the denominator be:

$$D_n = 2^{a_1+a_2+\cdots+a_n} - 3^n$$

Then we obtain the general formula:

$$O = \frac{M_n}{D_n} \tag{2.1}$$

3 Establishment of Recurrence Relations

We derive the recurrence relations for M_n and D_n for different values of n from the general formula:
- For $n = 1$

$$M_1 = 1, D_1 = 2^{a_1} - 3$$

- For $n = 2$

$$M_2 = 3 + 2^{a_1} = 3 \cdot M_1 + 2^{a_1}$$
$$D_2 = 2^{a_1+a_2} - 3^2$$

- For $n = 3$

$$M_3 = 3^2 + 3 \cdot 2^{a_1} + 2^{a_1+a_2} = 3 \cdot M_2 + 2^{a_1+a_2}$$

$$D_2 = 2^{a_1+a_2} - 3^2$$

- For $n = n$

$$M_n = 3^{n-1} + 3^{n-2} \cdot 2^{a_1} + \cdots + 3 \cdot 2^{a_1+a_2+\cdots+a_{n-2}} + 2^{a_1+a_2+\cdots+a_{n-1}}$$

$$M_n = 3 \cdot M_{n-1} + 2^{a_1+a_2+\cdots+a_{n-1}}$$
$$D_n = 2^{a_1+a_2+\cdots+a_n} - 3^n$$

Let$S_n = a_1 + a_2 + \cdots + a_n$ We can rewrite the recurrence relations as:

$$\begin{aligned}
M_1 &= 1 \\
M_2 &= 3 \cdot M_1 + 2^{a_1} \\
M_3 &= 3 \cdot M_2 + 2^{a_1+a_2} \\
&\vdots \\
M_n &= 3 \cdot M_{n-1} + 2^{S_{n-1}} && (3.1) \\
M_{n+1} &= 3 \cdot M_n + 2^{S_n} && (3.2) \\
D_1 &= 2^{a_1} - 3 \\
D_2 &= 2^{a_1+a_2} - 3^2 \\
D_3 &= 2^{a_1+a_2+a_3} - 3^3 \\
&\vdots \\
D_n &= 2^{S_n} - 3^n && (3.3) \\
D_{n+1} &= 2^{S_{n+1}} - 3^{n+1} && (3.4)
\end{aligned}$$

4 Proof that M_n are Coprime with 3

4.1 Proof of $\gcd(M_n, 3) = 1$

From the recurrence relation
$M_n = 3 \cdot M_{n-1} + 2^{S_{n-1}}(3.1)$
Taking modulo 3:
$M_n \equiv 3 \cdot M_{n-1} + 2^{S_{n-1}} \equiv 0 + 2^{S_{n-1}} = 2^{S_{n-1}} \pmod 3$
As before, $2^{S_{n-1}} \not\equiv 0 \pmod 3$
(since powers of 2 are never divisible by 3). Thus, $M_n \not\equiv 0 \pmod 3$ so

$$gcd(M_n, 3) = 1. \tag{4.1}$$

5 Proof That O Can Only Equal 1 Uniquely

5.1 Proof of $gcd(M_n, D_n) = 1$ (by Induction)

- Base case: $n = 1$,$M_1 = 1$,$D_1 = 2^{a_1} - 3$,Since:$\gcd(M_1, D_1) = 1$
- Inductive step: $n = k$,Assume $gcd(M_k, D_k) = 1$ for some positive integer k.We need to prove $gcd(M_{k+1}, D_{k+1}) = 1$.
- Inductive recursion: Assume $gcd(M_{k+1}, D_{k+1}) \neq 1$

since $gcd(M_{k+1}, D_{k+1}) \neq 1$,Let a positive integer p be able to divide bothM_{k+1}, D_{k+1} then:

$$M_{k+1} \equiv 0 \pmod p \tag{5.1}$$

$$D_{k+1} \equiv 0 \pmod p \tag{5.2}$$

since:$M_{k+1} \equiv 0 \pmod p$ and since:$gcd(M_{k+1}, 3) = 1(4.1)$
thus:

$$gcd(p, 3) = 1 \tag{5.3}$$

since:$M_{k+1} = 3 \cdot M_k + 2^{S_k}(3.2)$ and since:$M_{k+1} \equiv 0 \pmod p(5.1)$
thus:

$$3 \cdot M_k + 2^{S_k} \equiv 0 \pmod p$$

Sorted out as:

$$3 \cdot M_k \equiv -2^{S_k} \pmod p$$

and since:$gcd(p, 3) = 1(5.3)$. thus:

$$M_k \equiv -3^{-1}2^{S_k} \pmod p$$

since:

$$-3^{-1}2^{S_k} \equiv 0 \pmod 2$$

thus:

$$M_k \equiv 0 \pmod 2 \tag{5.4}$$

since:$D_{k+1} = 2^{S_{k+1}} - 3^{k+1}$ (3.4).and since:$D_{k+1} \equiv 0 \pmod p(5.2)$

thus:

$$2^{S_{k+1}} - 3^{k+1} \equiv 0 \pmod{p}$$

Sorted out as:

$$3 \cdot 3^k \equiv 2^{S_{k+1}} \pmod{p}$$

and since: $gcd(p, 3) = 1(5.3)$
thus:

$$3^k \equiv 3^{-1} 2^{S_{k+1}} \pmod{p}$$

and since: $D_k = 2^{S_k} - 3^k(3.3)$
thus:

$$D_k \equiv 2^{S_k} - 3^{-1} 2^{S_{k+1}} \pmod{p}$$

since: $2^{S_k} - 3^{-1} 2^{S_{k+1}} \equiv 0 \pmod{2}$
thus:

$$D_k \equiv 0 \pmod{2} \tag{5.5}$$

since:$M_k \equiv 0 \pmod{2}(5.4), D_k \equiv 0 \pmod{2}(5.5)$
thus:

$$gcd(M_k, D_k) \neq 1$$

This contradicts the assumption that $gcd(M_k, D_k) = 1$. Therefore, the assumption that $gcd(M_{k+1}, D_{k+1}) \neq 1$ cannot hold.

Therefore,$gcd(M_{k+1}, D_{k+1}) = 1$ holds.

To sum up:

$$\gcd(M_n, D_n) = 1 \tag{5.6}$$

5.2 Comparison of M_n and D_n

since:$O = \frac{M_n}{D_n}(2.1)$,

if:M_n<D_n,thus: $O < 1$(Non-positive odd numbers are discarded.)

ifM_n=D_n,thus: $O = 1$

ifM_n>D_n,thus: $O > 1$,At this point, since $\gcd(M_n, D_n) = 1(5.6)$ then: at this point,O has no positive odd values.

To sum up: O can only be uniquely equal to 1.

6 Conclusion

With the exception of 1, no positive odd integer can return to itself under repeated application of the Collatz map, no matter how many iterations are performed. For any positive even integer, it will always iterate to a positive odd integer according to the rules. If a positive even integer could return to itself, there would exist some odd integer that also returns to itself — a contradiction to the fact that no positive odd integer other than 1 can return to itself. Therefore, no positive integer other than 1, 2, and 4 can return to itself. This establishes the non-existence of non-trivial periodic orbits in the Collatz map.

Address: Shanguan Town, Jiangyin City, Wuxi City, Jiangsu Province China
Postal code:214437
Email:790815670@qq.com