

# On the Iterative Convergence of Fermat Numbers through Continued Fractions

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## Abstract

This paper presents a novel iterative representation of Fermat numbers, defined by the sequence  $F_n = 2^{2^n} + 1$ . By leveraging the fundamental recurrence relation  $F_{n+1} - 2 = F_n(F_n - 2)$ , we define a functional equation  $x = \frac{A}{x} - 2$ , where  $A = F_{n+1} - 2$ . We demonstrate that this equation yields two integer solutions,  $x_1 = F_n - 2$  and  $x_2 = -F_n$ . Through an analysis of the derivative of the map  $f(x) = \frac{A}{x} - 2$ , we prove that  $x = -F_n$  is an attractive fixed point and  $x = F_n - 2$  is a repulsive fixed point, leading to a unique, convergent infinite continued fraction for the negative of any Fermat number. This provides a bridge between the rapid growth of Fermat sequences and the stability of iterative rational functions.

## 1 Introduction

The Fermat numbers are a sequence of natural numbers of the form  $F_n = 2^{2^n} + 1$ . These numbers are known for their rapid growth and their role in number theory, specifically in primality testing and the construction of regular polygons. This paper explores a specific continued fraction identity derived from the quadratic relationship between consecutive terms.

## 2 Derivation of the Fixed-Point Equation

We begin by considering the iterative functional equation:

$$x = \frac{F_{n+1} - 2}{x} - 2$$

To solve for  $x$ , we multiply the entire expression by  $x$  to form a quadratic equation:

$$x^2 + 2x - (F_{n+1} - 2) = 0$$

Applying the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-(F_{n+1} - 2))}}{2}$$

$$x = \frac{-2 \pm \sqrt{4 + 4(F_{n+1} - 2)}}{2}$$

$$x = -1 \pm \sqrt{1 + F_{n+1} - 2}$$

$$x = -1 \pm \sqrt{F_{n+1} - 1}$$

Using the definition  $F_{n+1} = 2^{2^{n+1}} + 1$ , we substitute  $F_{n+1} - 1 = 2^{2^{n+1}} = (2^{2^n})^2$ :

$$x = -1 \pm \sqrt{(2^{2^n})^2}$$

$$x = -1 \pm 2^{2^n}$$

Substituting  $2^{2^n} = F_n - 1$ :

$$1. x_1 = -1 + (F_n - 1) = F_n - 2$$

$$2. x_2 = -1 - (F_n - 1) = -F_n$$

### 3 Convergence and Stability Analysis

To determine which solution the continued fraction converges to, we define the map:

$$f(x) = \frac{F_{n+1} - 2}{x} - 2$$

The stability of a fixed point is determined by the derivative  $f'(x)$ . For the function  $f(x) = Ax^{-1} - 2$ , the derivative is:

$$f'(x) = -\frac{F_{n+1} - 2}{x^2}$$

Substituting the fundamental Fermat identity  $F_{n+1} - 2 = F_n(F_n - 2)$ , we evaluate the derivative at both fixed points:

- For the first root  $x_1 = F_n - 2$ :

$$|f'(F_n - 2)| = \left| -\frac{F_n(F_n - 2)}{(F_n - 2)^2} \right| = \left| \frac{F_n}{F_n - 2} \right| = 1 + \frac{2}{F_n - 2}$$

Since  $1 + \frac{2}{F_n - 2} > 1$  for all  $F_n > 2$ , this root is a repulsive fixed point.

- For the second root  $x_2 = -F_n$ :

$$|f'(-F_n)| = \left| -\frac{F_n(F_n - 2)}{(-F_n)^2} \right| = \left| \frac{F_n - 2}{F_n} \right| = \left| 1 - \frac{2}{F_n} \right|$$

Since  $F_n \geq 3$  for all  $n \geq 0$ , it follows that  $0 < |f'(-F_n)| < 1$ . Because the magnitude of the derivative is strictly less than 1, the fixed point  $x = -F_n$  is an attractive attractor.

Therefore, the iterative process converges exclusively to  $-F_n$ .

## 4 Final Expression

Based on the convergence established above, the negative of the  $n$ -th Fermat number is represented by the following infinite continued fraction:

$$-F_n = -2 + \frac{F_{n+1} - 2}{-2 + \frac{F_{n+1} - 2}{-2 + \frac{F_{n+1} - 2}{\ddots}}}$$