Duality in Physics

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Abstract

We propose an ontological interpretation of quantum mechanics based on the principle of **subjective incompleteness**—a fundamental limitation arising from the fact that the observer is part of the world being observed. By formalizing consciousness as a set of states Subj and the world as a self-mapping W:Subj→Subj, we construct an ontological configuration space whose structure naturally gives rise to the Hilbert space of quantum states.

Within the proposed model, the hidden parameters in the representation of an observable operator are the eigenvalues of its canonical conjugate. In particular, the coordinate and momentum representations complement each other to form a complete ontological description, with the corresponding variables appearing as mutually hidden. From this perspective, the **phenomenon of duality** in physics, exemplified by pairs such as coordinate–momentum or time–energy, reflects the underlying subject–object structure of reality.

The model offers a new justification of Bohr's principle of complementarity and provides a geometrical account of noncommutativity in terms of subjective incompleteness. Furthermore, the entropic uncertainty relations of Hirschman–Everett are reinterpreted as quantitative measures of subjective incompleteness. This approach links the growth of thermodynamic entropy with the "motion" of the observer's consciousness along the gradient of ontological states, thereby providing a natural explanation of the thermodynamic arrow of time.

Thus, the key features of quantum mechanics emerge from the fundamental principle of subjective incompleteness. This article continues a series of works devoted to the role of the observer and consciousness in physics.

Keywords:
quantum foundations; complementarity; hidden variables; subjective incompleteness; consciousness; ontological models; entropic uncertainty

1. Quantum Mechanics on the Field of States of Consciousness

In [3] we showed how quantum mechanics can be derived on the basis of a formalized conception of consciousness. Here we demonstrate how the abstraction of a Hilbert space of quantum states emerges, and how it is related to the space of physical states. Let us begin by defining the key notions.

We introduce the concept of the Abstract Observer, denoted by *Subj* (the Subject). It is represented as the set of states of consciousness

$$Subj = \{\xi_1, \xi_2, \cdots, \xi_N\} \quad (1.1)$$

In the spirit of George Berkeley's philosophy, we identify this set with the set of elements of physical reality — everything that can be measured and perceived. Within this approach, the interpretation of quantum mechanics according to which quantum observables have no a priori (i.e. observer-independent) existence is justified in the most radical way, since the observables are nothing but the states of the observer himself.

We define the World W as the mapping of the abstract observer into itself:

$$W: Subj \rightarrow Subj \qquad (1.2)$$

That is, the World is the set of pairs of states of consciousness $\{\xi_i, \xi_j\}$. Here one may note an allusion to John Archibald Wheeler's "self-observing Universe" and to Johann Gottlieb Fichte's philosophy of self-positing. For us, however, this is simply taken as an axiom.

Ontological pairs are ordered. We take the first element of the pair to correspond to the current state of consciousness. This ordering follows from a fundamental property of consciousness known as ***intentionality*** (Franz Brentano), which denotes the directedness of consciousness toward an object. From the point of view of our formal construction, however, it is simply a necessary condition allowing us to build structures upon the set of states of consciousness Subj.

From the definitions of the observer Subj and the World W (see above) it follows that the cardinality of the set of ontological states exceeds that of the states of consciousness, $|W| > |Subj|$ (for N>1). We call this fact ***incompleteness***. From incompleteness it follows that the subject cannot distinguish between intentions sharing the same current state of consciousness:

$$\{\xi_i, \xi_j\} \sim \{\xi_i, \xi_k\};\ i \neq j \neq k \qquad (1.3)$$

Such states form classes of ***subjectively indistinguishable states***, denoted $\{Subj, \sim\}$. This fact plays a key role in the structure of the formalism of quantum mechanics.

Physical reality Subj is ***supervenient*** upon the ontological layer of reality W; that is, it is determined by the latter. Formally, this means that the wave function obeys a deterministic ontological dynamic, while the dynamics of projective measurements carried out by the subject is indeterministic. Understanding the fact that quantum indeterminacy is subjective is one of the most important consequences of our model. Thus, the combination of supervenience with incompleteness generates that remarkable hybrid of determinism and chance which is characteristic of quantum mechanics.

As the simplest example, consider a world consisting of an observer and a single observed particle. The configuration space of states of such a system is formed by all possible ontological pairs $\{x, x^{\mathrm{h}}\}$, where $x \in Subj$ corresponds to the observed coordinate (for instance, the coordinate at which a particle detector is located), while $x^{\mathrm{h}} \in Obj$ is the hidden coordinate of the object (the particle).

To each ontological state $\{x, x^{\mathrm{h}}\}$ we associate an element of the Galois field $GF(N^2)$, where N is prime. This field contains N^2 elements. For example, let N=5. This corresponds to a world in which the observer possesses only five states of consciousness. In Fig. 1 the elements of the Galois field $GF(5^2)$ are represented in the form of a rectangular table of logarithms of the field elements.

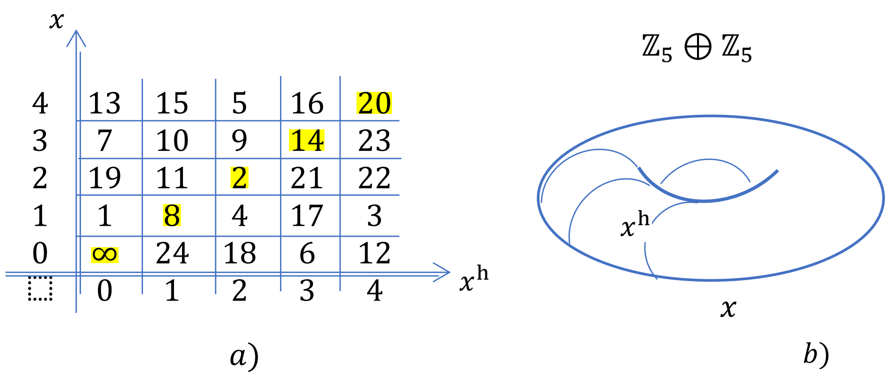

Fig.1 Ontological space Ω

The zero element (whose logarithm is = ∞) is located in the lower left corner of the table. It does not correspond to any projective coordinate. Elements of the field $GF(N^2)$ can be represented by the following linear combinations [1]:

$$\Psi = \{x, x^{\mathrm{h}}\} = x\mathrm{A} + x^{\mathrm{h}} \cdot \mathbf{1}; \qquad \xi, \eta \in \mathbb{Z}_{\mathrm{N}} \qquad (1.4)$$

Here, A is a generator of the field (a primitive N^2-th root of unity of the field), and **1** is the multiplicative identity. Being finite, this field is homeomorphic to a flat Euclidean manifold with the topology of the torus $\mathbb{Z}_N^2$.

In the example under consideration, the ontological space Ω is given by the direct sum of the subject and object spaces: $x \oplus x^{h}$. This is the configuration space of states of the observer–object system. We shall treat it as a space of homogeneous coordinates. Proportional pairs $[x : x^{h}]$ form disjoint orbits:

$$x^{h} = R \cdot x \; mod \; N \qquad (1.5)$$

where R are rational coefficients. The orbit space (1.5) thus forms a projective space:

$$[x : x^{h}] \sim [\,1 : R]. \text{ Here: } R = \frac{x^{h}}{x};\; x \neq 0. \qquad (1.6)$$

Let us denote: $\theta = 2\pi \frac{x^{h}}{N}, \quad k = \frac{2\pi R}{N}$

where θ and k represent discrete phase and momentum (wave number), respectively. Then we obtain

$$\theta = kx \; mod \; 2\pi \qquad (1.7)$$

The affine coordinates k are equivalence classes of ontological states. They correspond to momenta (wave numbers), since they have the meaning of spatial frequencies. We observe that the periodic boundary conditions, imposed by the finite topology of the ontological space, generate harmonics in the projective physical space, with "wavelengths" $\lambda = 1/k$ interpreted as the distance between successive windings of the orbit on the torus (see Fig. 2).

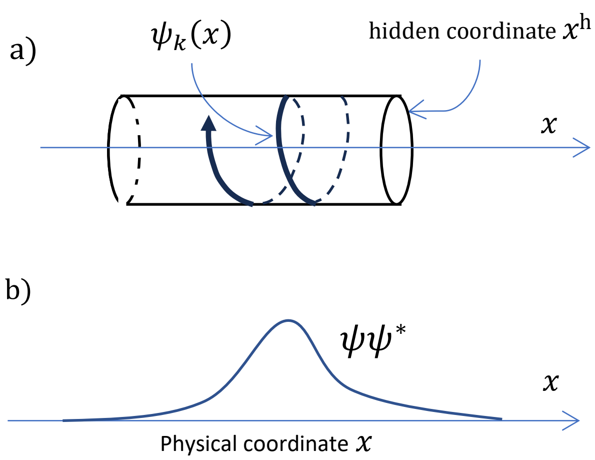

Fig.2

Taking into account the isomorphism between the additive group of residue classes and the multiplicative group of roots of unity, each orbit (1.7) can be represented by a complex function on the ring $\mathbb{Z}_\mathrm{N}$:

$$\psi_k(x) = 1 \cdot \mathrm{e}^{-\mathrm{i}kx} \qquad (1.8)$$

The set of functions (1.8) forms the momentum basis of the Hilbert space of states $\mathcal{H}^N$ in the coordinate representation. Each function $\psi_k(x) \in \mathcal{H}^N$, corresponding to a momentum state $k = k_n$, is degenerate with respect to the cosets (orbits parallel to it), so that the degree of degeneracy determines the amplitude of the given momentum state: $A_k = \sqrt{P(k)}$, where $P(k)$ is the normalized probability. The probability amplitude then takes the form:

$$\psi_k(x) = A_k \cdot \mathrm{e}^{-\mathrm{i}kx}; \qquad (1.9)$$

This, in particular, provides a justification of Born's rule (for more details, see [2,3]). Sets of orbits with different k "interfere" with each other, creating a probability distribution in projective space:

$$P(x) = \psi(x)\psi^*(x) \quad (1.10)$$

where: $\psi(x) = \frac{1}{\sqrt{N}}\sum \psi_k(x)$. Fig. 2a) shows one of the orbits $\psi_k(x)$, and Fig. 2b) shows a possible distribution function resulting from the interference of a set of such orbits.

The configurational ontological space $\mathrm{x} \oplus x^{\mathrm{h}}$ can be regarded as a principal fiber bundle over the physical coordinate x, with the fiber being the orbit of the group of roots of unity $\mathbb{Z}_\mathrm{N}$ (in the limit $\mathbb{Z}_\mathrm{N} \in \mathrm{U}(1)$), representing the hidden cyclic coordinate x^{h}.

2. Conjugate Variables as Hidden Parameters

The functions (1.8) define the momentum basis in the xxx-representation. The corresponding dual space of linear functionals gives the p-representation:

$$\psi(k) = \langle e^{-ikx} | \cdot \rangle \qquad (2.1)$$

Let us consider the momentum state $\psi_k(k)$ in the momentum representation. By definition, a phase shift leaves the momentum state within its equivalence class:

$$\psi_k(k) \sim \psi_k(k) \cdot \mathrm{e}^{-\mathrm{i}\varphi} \qquad (2.2)$$

The meaning of this is precisely the inability of the internal observer (the subject) to distinguish between elements of the equivalence class $\{Subj, \sim\}$.
It is well known that the operator **X** of cyclic permutation of the coordinate basis and its dual operator $\boldsymbol{Z} = \boldsymbol{FXF}^{-1}$, where ***F*** is the Fourier transform matrix, together generate a projective unitary representation of the Heisenberg–Weyl group [4], acting on $\mathbb{Z}_N \times \widetilde{\mathbb{Z}_N}$. (Here the tilde denotes the conjugate space.)

In the finite-dimensional space of dimension N, the elements ***X*** and ***Z*** of the Heisenberg–Weyl group are defined as follows:

$\boldsymbol{X}|x_n\rangle = |x_{n+1}\rangle$ (translation)

$\boldsymbol{Z}|x_n\rangle = \omega^n |x_n\rangle$ (phase shift depending on $|x_n\rangle$)

Where: $\omega = exp(2\pi i \,/\, N)$ — primitive N-th root of unity

Accordingly, the action of the phase-shift operator on $\psi(x)$ results in a shift in the conjugate space, i.e.,

$$(\boldsymbol{F} \cdot \mathbf{Z})\psi(x) = \psi(p + \Delta p). \qquad (2.3)$$

From this follows, in particular, the well-known Fourier shifting theorem. It is easy to see that the operator **Z** of the Heisenberg–Weyl group acts as a local gauge transformation:

$$\psi(x) \to e^{-i\theta(x)}\psi(x). \qquad (2.4)$$

The action of $(\mathbf{F} \cdot \boldsymbol{Z})$ on a coordinate state leads to a displacement in the conjugate momentum space: $\psi(p) \to \psi(p + \Delta p)$. Suppose the phase varies in time but does not depend on x, i.e., $\theta \neq f(x)$. Such a transformation is called *global*. In this case, for an observer in both coordinate and momentum representations, the expectation values of observables do not change: $\langle\hat{x}\rangle \neq f(t)$, $\langle\hat{p}\rangle \neq f(t)$. However, if the phase depends not only on time but also on the coordinate, i.e., $\theta = f(x,t)$, then for an observer in the coordinate representation nothing changes: $\langle\hat{x}\rangle \neq f(t)$, while for an observer in the momentum representation the expectation value of momentum does change: $\langle\hat{p}\rangle = f(t)$, Hence, $\dot{p} \neq 0$, which means that a force arises: $F = \dot{p}$. The force *F* may be interpreted as the action of a compensating field that restores gauge symmetry.

This simple explanation of the idea of gauge fields does not appear in the literature. Apparently, this is because gauge theory developed from classical field theory and the geometry of wave equations, whereas the Heisenberg group arose from quantum mechanics and representation theory. For this reason, the two themes rarely intersect. Nevertheless, despite its simplicity, this “toy” interpretation of the gauge mechanism of interactions has a profound geometric meaning.

It should be emphasized that for an observer in the coordinate representation, the phase has no physically observable meaning, since only equivalence classes of phase states are distinguishable: $\psi(x) \cong e^{i\varphi}\psi(x)$. From (2.3) it follows that the vector of phase multipliers ω^n in front of $\psi(x)$, up to a global gauge, implicitly contains information inaccessible to the observer in the coordinate basis—namely, the displacement Δp in the momentum basis. The converse is also true. In this sense, the phase may be regarded as a hidden parameter encoding this displacement.

Since k, x, θ are related by (1.7), the ontological space $\{x, \theta\}$ for $x \neq 0$ is isomorphic to the phase space $\{x, k\}$. This means that the phase space $\{x, k\}$ is also complete. In this context, the well-known formulation of quantum mechanics on phase space [5] acquires the meaning of a complete quantum description from the standpoint of an external observer.

Thus, conjugate variables in quantum mechanics are “mutually hidden” parameters. The conjugate representation complements the description in the chosen basis to a complete, ontological one, and Bohr’s heuristic principle of complementarity is realized literally.

It is also worth emphasizing that the classes of momentum states are nonlocal, which is consistent with Bell's theorem on the impossibility of local hidden variables.

3. Duality and Noncommutativity

Mathematically, conjugate variables are pairs connected through a Fourier transform—or, more generally, by Pontryagin duality. Such dualities naturally give rise to the uncertainty principle. What is less often asked, however, is why duality exists in nature at all. Our proposal is that its foundation lies in the subject–object dichotomy. In quantum mechanics, the coordinate space |x⟩ is dual to the momentum space |k⟩. Yet the distinction between momentum and coordinate representations is purely conventional: changing representations does not alter the underlying physics. This situation resembles Plücker duality, where the line equation on a plane, $p_1x_1 + p_2x_2 = 0$, is symmetric in its arguments. Plücker observed that the coefficients p_1, p_2 can be treated as coordinates of the line, on equal footing with the point coordinates x_1, x_2. Exchanging x with p changes nothing—if one simultaneously replaces the notion of “point” with “line.” In the same way, dual spaces are isomorphic, governed by the same operator algebra, and the choice of representation in QM is largely a matter of convenience.

Following the pioneering works of Wigner and Weyl on phase-space quantum mechanics [6], Tom Kibble's recognition of geometric structures in quantum theory reignited interest in phase-space formulations (Heslot 1985; Anandan & Aharonov 1990) [7]. Yet a clear understanding that dual representations jointly provide a full ontological description remains absent.

Heslot [8] proposed an extension of classical mechanics in which quantum mechanics appears as a special case. He showed that the Schrödinger equation for the real and imaginary parts of the wavefunction, $\psi = x + ip$, can be rewritten as a Hamiltonian system. This allows the evolution of a quantum system to be described as the one-parameter unitary flow of Schrödinger dynamics, mapping $\mathcal{H} \in \mathbb{C}^{n+1}$ onto a quasi-classical phase space $\Gamma \cong \mathbb{R}^{2n}$, a projective symplectic manifold $P\mathcal{H} \in \mathbb{C}P^{n}$ with a Kähler structure:

$$U: \mathcal{H} \rightarrow P\mathcal{H} \qquad (3.1)$$

Thus, quantum mechanics regains a formal analogy with classical Hamiltonian mechanics, with its symplectic geometry and Hamiltonian flow. Crucially, however, this flow now acts in the space of states rather than in the physical space of particles. The link with the coordinate–action variables {x,s} was discussed in Section 2.

In our view, unifying dual spaces into a single symplectic manifold provides the ontological completion of quantum mechanics. In his debates with Bohr, Einstein speculated that QM might be extended through hidden-variable theories. A similar extension emerges when conjugate spaces are seen as a unified geometric structure.

The phase Hilbert space is neither strictly classical nor quantum. QM arises only after factorization, which, in our interpretation, reflects the transition from an external (objective) to an internal (subjective) observer—an observer "living" on the projective Hilbert manifold S^{2N-2}. The scalar curvature of this manifold gives rise to the noncommutativity of observables, a hallmark of quantum theory. As Dirac once remarked, "quantum mechanics is classical theory on a noncommutative algebra."

This becomes evident in the projective representation of the Heisenberg–Weyl group, where the commutator of the operators ***Z*** and ***X*** is defined only up to a phase factor ω:

$$\boldsymbol{Z} \cdot \boldsymbol{X} = \omega \cdot \boldsymbol{X} \cdot \boldsymbol{Z} \qquad (3.2)$$

Here, traversing a closed loop across one elementary cell of the phase lattice $\mathbb{Z}_N \times \widetilde{\mathbb{Z}_N}$ acquires a direction-dependent global phase ω- a discrete analogue of the Berry phase. This corresponds to the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar\boldsymbol{I}$.

In this construction, the operators ***X*** and ***Z*** act within the same working basis—the basis available to the subjective observer. For such an observer, the conjugate observable is hidden. As noted earlier (Sec. 2.3), a phase shift in one basis is equivalent to a translation in its conjugate basis, and vice versa. From the standpoint of an objective observer, ***X*** and ***Z*** in their respective bases are simply translation operators, commuting automorphisms of the ontological phase lattice. The appearance of noncommutativity— "quantum magic"— is thus a perspectival effect of the subjective observer's position. The phase factor $\omega(p, q)$ plays the role of a connection on the underlying discrete bundle.

In the continuous setting, the same principle applies. The canonical commutator can be expressed in terms of successive translation operators along a closed contour:

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = \hat{T}_q \hat{T}_p \hat{T}_{-p} \hat{T}_{-q} = e^{-\frac{i}{\hbar}pq}, \qquad (3.3)$$

where $\hat{T}_p = e^{-\frac{i}{\hbar}p\hat{q}}$ and $\hat{T}_q = e^{-\frac{i}{\hbar}q\hat{p}}$ are the continuous analogues of ***Z*** and ***X***. Again, a closed loop in $P(\mathcal{H})$ lifts to the bundle, returning with an excess phase—the Berry phase. Its magnitude is set by the curvature of the enclosed region [9]. By Stokes' theorem, the circulation of the connection equals the flux of the symplectic form through the surface S bounded by the loop $L \in P(\mathcal{H})$:

$$\beta = \oint_L A = \int_S F \qquad (3.4)$$

Thus, the 1-form connection A and the 2-form curvature F on the projective Hilbert space are directly linked.

In the classical case—the perspective of the objective observer—the phase space is flat, and quantum effects collapse to Hamiltonian mechanics.

4. Hirschman–Everett Entropy as a Measure of Subjective Incompleteness

The noncommutativity of observables underlies Heisenberg's uncertainty principle. In 1929, Robertson [10] proved a relation connecting uncertainty with noncommutativity:

$$\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle [A,B] \rangle| \quad (4.1)$$

Here, ΔA is the standard deviation operator of the quantity A, and [·,·] denotes the commutator. This relation follows directly from the formalism of QM. However, as we noted above, noncommutativity arises from a deeper origin related to incompleteness.

In the 1950s, Everett (Hugh Everett III) and Hirschman (Isidore Isaac) derived the relation [11]:

$$I_x + I_p \geq log_2(e\pi) \quad (4.2),$$

where

$$-I_x = -\int |\psi_x|^2 \log\ |\psi_x|^2 \quad (4.3)$$

and similarly for I_p. In 1975, this inequality was rigorously proven by Beckner [12]. The Hirschman–Everett uncertainty (4.2) not only refines Robertson's relation (4.1) but also illuminates its nature, which we interpret as being tied to subjective incompleteness. We denote the quantity (4.3) as the hidden entropy, $\mathrm{H_{hide}}$, since the ontological structure of the state vector ψ_x, due to incompleteness, is inaccessible to the internal observer. Numerically, it is determined by the information obtained through measurement.
Thus, while the von Neumann entropy, $\mathrm{H_{Subj}} = -\mathrm{tr}(\rho_r \ln\ \rho_r$ reflects the number of quantum states realizing a given mixed state, the Hirschman–Everett entropy $\mathrm{H_{hide}}$ counts the number of ontological states realizing a given quantum state. Unlike the von Neumann entropy, it is nonzero for a pure state, and its lower bound is always strictly positive. The minimum is attained for Gaussians with standard deviations σ_x and σ_p satisfying the uncertainty relation $\sigma_\mathrm{x}\ \cdot\ \sigma_\mathrm{p}\ =\ \hbar\ /\ 2$.

Relation (4.2) represents the sum of the objective entropies of the subject (observer) and the object. The objective entropy of the world as a whole, $H_W = H_x\ +\ H_p$ is composed of the Hirschman–Everett entropies of the conjugate observables, since relative to each other they play the roles of the subject's and the object's entropies. In a finite discrete model, the analogous relation is the Maassen–Uffink inequality (1988) [13]:

$$H_x\ +\ H_p\ \geq\ -2 \cdot log(c), \qquad (4.4)$$

where c = max $\langle u_i v_j \rangle$ — is the maximum modulus of the scalar product between elements of two orthonormal bases. If the bases are mutually unbiased (MUBs), for instance determined by a discrete Fourier transform, then $c = N^{-\frac{1}{2}}$, where N is the dimension of the basis (in our interpretation, the number of states of consciousness). In this case, the lower bound becomes exactly $log(N)$:

$$H_x + H_p \geq log(N) \qquad (4.5)$$

The notion of the world's entropy H_{World} is not strictly relevant in the subject–object model, since entropy is inherently a subjective concept, reflecting the degree of incomplete information accessible to the subject. Nevertheless, the sum: $\mathrm{H}_{World} = H_{Subject} + H_{hide}$ can be interpreted as the total information of the world. The extreme case corresponds to zero subject entropy and maximal object entropy, equal to $log(N)$. This represents the maximal entropy gradient and the emergence of entropic "forces" tending to increase the subject's entropy. Hereafter, we will denote entropies in general form, keeping in mind that they refer to distributions in conjugate bases:

$$H_{Subj} + H_{hide} = H_W \qquad (4.6)$$

The Hamiltonian flow preserves volume in the ontological phase space, which, in the context of a discrete ontological space, corresponds to the conservation of the total information of the world.

The question of the origin of the second law of thermodynamics, $\dot{\mathrm{H}}_{\mathrm{Subj}} \geq 0$, and the thermodynamic arrow of time remains open. As early as the 1950s, Landau [14] pointed to quantum measurement as a potential source of entropy increase. Since then, numerous studies have addressed this problem, including a recent paper by physicists at the University of Vienna [15]. Yet a proper understanding of the subjective nature of entropy is still lacking. Equation (4.6) implies that the subjective (thermodynamic) entropy grows solely due to the inflow of entropy from hidden ontological degrees of freedom into the emergent layer of physical reality, $H_{hide} \rightarrow H_{Subj}$. The driving force behind this flow is the "motion" of the observer's consciousness along the gradient of the ontological state density.

This "motion" should be understood in the context of **indexical uncertainty**. For example, it implies that the probability of experiencing oneself in a more complex world, with a larger number of possibilities, is higher than in a simpler world. This "motion" of the observer in the direction of increasing entropy is what we interpret as the flow of time.

Conclusions

1. We considered the hypothesis that the dual structure of observables in quantum mechanics is induced by the subject–object structure of reality. We showed that the structure of the extended Galois field $GF(N^2)$ adequately describes this framework, which plays a fundamental role in quantum mechanics.

2. It was shown that the hidden parameters in the representation of an observable operator $\hat{\lambda}$ are the eigenvalues of its canonical conjugate. For example, in the momentum representation the hidden parameters are the coordinates.

3. It was demonstrated that conjugate observables complement each other to form a complete ontological description, thereby clarifying and formalizing the meaning of Bohr's principle of complementarity.

4. The role of the Hirschman–Everett entropic uncertainty relation in the mechanism of subjective entropy growth has been elucidated.

The basis of our conclusions and hypotheses is the notion of a subject–object structure of the world, where the subject (observer) is part of the world that they observe. This ontological situation is analogous to the incompleteness of closed axiomatic systems studied in mathematical logic. Gödel's theorem asserts the existence of contentually true statements in such systems that are unprovable within the system itself. The idea of transferring the concept of incompleteness to physics is not new. It is appropriate to recall the formal argument by David Hilton Wolpert, who asserted that for any intelligence, it is in principle impossible to know everything about the universe of which it is a part [16].

However, subjective physical incompleteness not only produces cognitive limitations akin to Gödelian constraints, but also induces physical reality itself. In this article, we have shown that the key mathematical constructs underlying the formalism of quantum mechanics can be justified within the framework of the concept of subjective incompleteness.

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