

# Riemannian Manifolds of Asymmetric Equilibria: Complete Geometric Structure and Curvature Signatures

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## Abstract

We develop the complete Riemannian geometry of Victoria–Nash asymmetric equilibrium manifolds (VNAE) for  $n$ -player games. The metric  $g_{ij} = \iota_i \iota_j \delta_{ij} + \varepsilon H_{ij}(V, \iota)$  yields explicit Levi-Civita connection  $\Gamma_{ij}^k$ , Riemann tensor  $R_{ijkl}^i$  with fourth-order  $V$ -derivative cancellation, Ricci tensor  $R_{ij} \approx \kappa(\iota_{i,j} \iota_i - \kappa \partial_i^2 \iota_i) \delta_{ij}$ , and scalar curvature  $K_s \approx \sum_{i < j} \iota_i \iota_j \det H_{ij}^s + O(\varepsilon^2)$ . Positive/negative/zero signatures classify stability geometrically. The Lyapunov–Morse functional  $\mathcal{L}$  satisfies  $\frac{d^2}{dt^2} \mathcal{L}|_{\text{VNAE}} \approx -2 \text{Ric}(\dot{s}^\perp, \dot{s}^\perp)$  along gradient flows, establishing Ricci curvature as the normal contraction rate. Classical Nash, von Neumann’s minimax theorem, and Lyapunov stability emerge as degenerate flat limits as  $\varepsilon \rightarrow 0$ .

## 1 Introduction

Classical game-theoretic equilibrium concepts, including Nash equilibria [30] and von Neumann’s minimax theorem [48], are fundamentally grounded on symmetry assumptions, either explicitly or through implicit homogeneity of strategic responses. While such assumptions enable analytical tractability, they significantly restrict the descriptive power of equilibrium theory in environments characterized by heterogeneous agents, unequal strategic inertia, or structurally asymmetric interactions. In many economic and strategic settings, asymmetry is not a perturbation but a defining feature, rendering classical equilibrium concepts either degenerate or insufficient to capture observed strategic stability.

The Victoria-Nash Asymmetric Equilibrium (VNAE) framework [36, 37] addresses this gap by embedding asymmetric strategic dynamics into a Riemannian manifold, where the equilibrium set  $\Gamma_{\text{VNAE}}(\theta)$  is no longer a discrete collection of points but a smooth submanifold endowed with a natural metric  $g_{ij}(s; \theta)$ . Previous work demonstrated that for quadratic potentials,  $\Gamma_{\text{VNAE}}$  exhibits constant positive curvature whenever asymmetry parameters differ ( $\theta_A \neq \theta_B$ ) and rigidity  $\beta > 0$ .

The theoretical development of this framework follows a systematic research lineage focused on the structural exploitation of stochastic asymmetries. This trajectory originated

with the ESBSD exponential smoothing model [33], which first proposed treating variance as an informative component in non-linear forecasting. In addition, it was presented the - Itamaracá - a novel pseudo-random number generator (PRNG) that leverages absolute value transformations to produce aperiodic sequences within a deterministic framework [34]. Similar to the dynamics of a Tent Map in chaotic systems, the Itamaracá algorithm utilizes simple functional iterations to generate high-entropy trajectories.

The culmination of these empirical observations led to the seminal formulation of the 'Predictable Random Component'  $\eta(X_t)$  in the Victoria framework [35], which established the phenomenological basis for the positive expected value for one of the players in asymmetric systems that this paper now rigorously formalizes through the lens of Riemannian geometry and Ricci-induced contraction.

The extension to nonlinear potentials [37] revealed that curvature can vary spatially and even become negative, opening the possibility of repulsive geometric behavior. This proposed framework lies in the fact that strategic stability is not an extrinsic property, but rather an intrinsic geometric consequence of the strategy space. In a manner analogous to the principles of General Relativity [12], where the curvature of spacetime governs the dynamics of geodesics, here the Ricci tensor dictates the rate of contraction or repulsion of decision flows.

In this paper, we complete the geometric foundation of the VNAE theory for  $n$ -player games. We derive the full Riemannian structure: the Levi-Civita connection [26], Christoffel symbols [9], Riemann curvature tensor, and Ricci tensor expressed directly in terms of the potential  $V$  and structural fields  $\phi_i$ . We show that the scalar curvature  $K$  [15] admits a leading-order expansion

$$K = \beta \sum_{i < j} |\theta_i - \theta_j| \det(H_{ij}) + O(\beta^2),$$

where  $H_{ij}$  are  $2 \times 2$  blocks of the Hessian  $\nabla^2(V + \phi)$ . Consequently,  $K$  can be positive, zero, or negative depending on the signs of these determinants. This provides a complete classification of equilibrium stability: positive curvature corresponds to spherical geometry and attraction, negative curvature to hyperbolic geometry and repulsion, and zero curvature recovers the flat Euclidean case of classical Nash theory.

We further establish a direct link between geometry and dynamics through the Lyapunov-Morse functional  $\mathcal{L}(s; \theta)$ . Along the gradient flow  $\dot{s} = -F(s; \theta)$ , we prove

$$\frac{d}{dt} \mathcal{L} = -\|F\|_g^2, \quad \frac{d^2}{dt^2} \mathcal{L}|_{\Gamma_{\text{VNAE}}} = -2\beta \text{Ric}(\dot{s}, \dot{s}) + O(\epsilon^2),$$

showing that the rate of convergence to the equilibrium manifold is governed by Ricci curvature in normal directions. This constitutes a geometric strengthening of Lyapunov's stability principle [27].

Recent developments illustrate the growing role of Riemannian geometry in machine learning and game theory. Kristály established connections between Nash equilibria and Hadamard manifolds via variational inequalities [22]. Davis introduced geometry-first detection frameworks with explicit error budgets for identity-preserving temporal domains [10]. Xie et al. developed manifold-constrained hyper-connections for stable signal propagation in large-scale neural networks [49]. Together with the Victoria-Nash framework presented herein, these works reflect a broader trajectory in which geometric principles are increasingly leveraged to address stability, generalization, and verifiability challenges across diverse scientific contexts.

The paper is organized as follows. Section 2 defines the  $n$ -player VNAE framework and the Riemannian metric. Section 3 computes the Christoffel symbols. Section 4 derives the Riemann and Ricci tensors, culminating in the curvature signature theorem. Section 5 introduces the Lyapunov-Morse functional and proves the Ricci-enhanced stability result. Section 6 presents a detailed nonlinear example with explicit negative curvature calculations. Section 7 states the main theorems for  $n$  players. Section 8 deals with the Geodesic Completeness of the VNAE Manifold. Section 9 discusses geometric interpretation and phase transitions. In section 10 we discuss the Structural Theorem for Geometric Game Dynamics. Finally, Section 11 concludes. Appendix A contains the tensor calculus details, and Appendix B contains the curvature derivation of the Hessian term. In addition, this study provides the complete R code for reproducibility.

While the VNAE framework originates from strategic motivations, this treatise focuses exclusively on its intrinsic geometric and dynamical foundations. The detailed economic mapping of the structural fields  $\phi_i$  is deferred to subsequent applied research, ensuring the current treatment maintains a rigorous focus on the manifold's global stability and curvature properties.

## 2 The $n$ -Player VNAE Framework

### 2.1 Strategy Space and Structural Fields

Let  $N = \{1, \dots, n\}$  denote the set of players. Each player  $i \in N$  has a compact  $C^\infty$  strategy space  $S_i \subset \mathbb{R}^{m_i}$ . The product space

$$S = \prod_{i=1}^n S_i$$

is compact and  $C^\infty$ . Asymmetry parameters  $\theta_i \in \mathbb{R}^+$  and structural weights  $\omega_i(\theta_i) > 0$  are  $C^1$  functions. The rigidity coefficient  $\beta \geq 0$  controls the geometric contribution; for  $\beta = 0$  the geometry reduces to Euclidean.

Each player possesses a structural field  $\phi_i : S_i \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , assumed  $C^2$  in the strategy variable and  $C^1$  in the parameter. These fields model player-specific inertia, rigidity, institutional constraints, or any form of asymmetric friction. The baseline strategic interaction is captured by a potential  $V : S \rightarrow \mathbb{R}$ ,  $C^2$  in all variables.

### 2.2 Asymmetric Expectation Field

The expectation field  $F : S \times \Theta \rightarrow \mathbb{R}^n$ , where  $\Theta \subset (\mathbb{R}^+)^n$ , is defined componentwise by

$$F_i(s; \theta) = \omega_i(\theta_i) \frac{\partial V}{\partial s_i}(s) + \frac{\partial \phi_i}{\partial s_i}(s_i; \theta_i), \quad i = 1, \dots, n. \quad (1)$$

The field  $F$  combines the gradient of the potential (weighted by  $\omega_i$ ) with the player-specific structural gradient. Zeros of  $F$  represent strategic equilibria where each player's marginal incentives vanish.

**Definition 2.1.** The Victoria-Nash asymmetric equilibrium set is:

$$\Gamma_{\text{VNAE}}(\theta) = \{s \in S : F(s; \theta) = 0\}.$$

**Lemma 2.2.** Let  $F : M \rightarrow \mathbb{R}^k$  be a  $C^1$  map defined on a smooth manifold  $M$ . If  $0$  is a regular value of  $F$ , then the set

$$\Gamma_{VNAE} = \{s \in M : F(s) = 0\}$$

is a smooth submanifold of  $M$  with dimension

$$\dim(\Gamma_{VNAE}) = \dim(M) - k.$$

*Proof.* Since  $0$  is a regular value of  $F$ , the differential  $DF(s)$  has full rank  $k$  for all  $s \in F^{-1}(0)$ . By the regular value theorem, the level set  $F^{-1}(0)$  is a smooth submanifold of  $M$  with codimension  $k$ . Therefore  $\Gamma_{VNAE}$  is a smooth manifold of dimension  $\dim(M) - k$ .  $\square$

**Lemma 2.3.** Let  $s(t)$  be a trajectory generated by the dynamical system:

$$\dot{s} = F(s).$$

If  $s(0) \in \Gamma_{VNAE}$ , then

$$s(t) \in \Gamma_{VNAE}$$

for all  $t$ .

*Proof sketch.* By definition,  $\Gamma_{VNAE} = \{s \in M : F(s) = 0\}$ . Consider the derivative of  $F(s(t))$  along the trajectory:

$$\frac{d}{dt}F(s(t)) = DF(s(t))\dot{s}(t).$$

Using  $\dot{s}(t) = F(s(t))$ , we obtain

$$\frac{d}{dt}F(s(t)) = DF(s(t))F(s(t)).$$

If  $s(0) \in \Gamma_{VNAE}$ , then  $F(s(0)) = 0$ , which implies  $\frac{d}{dt}F(s(t)) = 0$  along the trajectory. Therefore  $F(s(t)) = 0$  for all  $t$ , and consequently  $s(t) \in \Gamma_{VNAE}$ .  $\square$

## 2.3 Riemannian Metric

Following the foundational construction [36], we endow  $S$  with a Riemannian metric that depends on the second derivatives of  $V$  and  $\phi_i$ :

$$g_{ij}(s; \theta) = \omega_i(\theta_i)\delta_{ij} + \beta \left( \frac{\partial^2 V}{\partial s_i \partial s_j}(s) + \frac{\partial^2 \phi_i}{\partial s_i^2}(s_i; \theta_i)\delta_{ij} \right), \quad i, j = 1, \dots, n. \quad (2)$$

The metric is symmetric and positive-definite for sufficiently small  $\beta$  (or under appropriate convexity conditions). The off-diagonal terms arise solely from mixed partials of  $V$ , capturing strategic interdependence. The parameter  $\beta$  scales the geometric distortion induced by the curvature of  $V$  and  $\phi_i$ .

**Remark 2.4.** It must be clear and understandable that the metric (2) is not an *ad hoc* choice. It arises as the first-order Taylor expansion of a general Riemannian structure on  $S$  that depends smoothly on the second derivatives of the game potential  $V$  and the structural fields  $\phi_i$ . This construction is analogous to the Fisher information metric

in information geometry, which originated with Fisher’s introduction of the information tensor [14] and was later recognized by Rao as defining a Riemannian metric on statistical manifolds [41]. The modern geometric framework was subsequently developed by Amari and others [1], where the metric is derived from the Hessian of a potential function. Here, the role of the potential is played by  $V + \phi$ , and the parameter  $\beta$  controls the coupling between players. Within the stability regime  $\beta < 1/\|\nabla^2(V + \phi)\|$ , this expansion captures the leading geometric effects of strategic asymmetry while preserving positive definiteness. Moreover, in the limit  $\beta \rightarrow 0$ , the metric reduces to the flat Euclidean case, recovering classical Nash theory. Thus, (2) is the natural metric for the VNAE framework, combining the baseline Euclidean structure with a curvature-dependent perturbation that reflects the strategic inertia of the agents.

**Remark 2.5.** In the limit  $\beta \rightarrow 0$ ,  $g_{ij} \rightarrow \omega_i \delta_{ij}$ , which is a Euclidean metric after a simple rescaling. Hence, classical game theory corresponds to the flat geometry limit.

### 3 Christoffel Symbols and Levi-Civita Connection

The geometric properties of the VNAE manifold are governed by the Levi-Civita connection  $\nabla$ , the unique torsion-free connection that preserves the metric  $g$ . Building upon the seminal works of Christoffel [9] and Levi-Civita [26], this connection is locally characterized by the Christoffel symbols of the second kind:

$$\Gamma_{ij}^m = \frac{1}{2} g^{mk} \left( \frac{\partial g_{jk}}{\partial s_i} + \frac{\partial g_{ik}}{\partial s_j} - \frac{\partial g_{ij}}{\partial s_k} \right), \quad (3)$$

which represent the acceleration components required to maintain parallel transport along the asymmetric strategy field. As established in the fundamental theorem of Riemannian geometry [43], these symbols encode how the strategic inertia of the agents warps the local coordinate system.

**Remark 3.1.** The geometric foundation of the VNAE finds solid support in the principles of Information Geometry. By defining the equilibrium through a Riemannian manifold where the metric tensor  $g_{ij}$  captures the structural sensitivity of the agents, the present framework aligns with Amari’s thesis that the curvature of a parameter space dictates convergence and optimization properties [1]. Thus, the stability classification via the scalar curvature signature presented herein provides a physical interpretation for dynamic efficiency in asymmetric decision systems.

**Proposition 3.2.** *For the VNAE metric (2), the partial derivatives of the metric are:*

$$\frac{\partial g_{jk}}{\partial s_i} = \beta \left( \frac{\partial^3 V}{\partial s_i \partial s_j \partial s_k} + \frac{\partial^3 \phi_j}{\partial s_i^2 \partial s_j} \delta_{jk} + \frac{\partial^3 \phi_k}{\partial s_k^2 \partial s_i} \delta_{ik} \right).$$

*Consequently, the Christoffel symbols are  $O(\beta)$  and vanish in the Euclidean limit  $\beta \rightarrow 0$ .*

*Proof.* Differentiate (2) with respect to  $s_i$ . The term  $\omega_i \delta_{ij}$  is constant. The remaining terms involve third derivatives of  $V$  and  $\phi$ . The symmetry in indices follows from the  $C^3$  regularity.  $\square$

Explicitly, the Christoffel symbols of the first kind are

$$\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial g_{jk}}{\partial s_i} + \frac{\partial g_{ik}}{\partial s_j} - \frac{\partial g_{ij}}{\partial s_k} \right).$$

Using the expression above, we obtain

$$\begin{aligned} \Gamma_{ijk} = \frac{\beta}{2} & \left( \frac{\partial^3 V}{\partial s_i \partial s_j \partial s_k} + \frac{\partial^3 V}{\partial s_j \partial s_i \partial s_k} - \frac{\partial^3 V}{\partial s_k \partial s_i \partial s_j} \right. \\ & + \left( \frac{\partial^3 \phi_j}{\partial s_j^2 \partial s_i} \delta_{jk} + \frac{\partial^3 \phi_k}{\partial s_k^2 \partial s_i} \delta_{ik} \right) \\ & + \left( \frac{\partial^3 \phi_i}{\partial s_i^2 \partial s_j} \delta_{ik} + \frac{\partial^3 \phi_k}{\partial s_k^2 \partial s_j} \delta_{jk} \right) \\ & \left. - \left( \frac{\partial^3 \phi_i}{\partial s_i^2 \partial s_k} \delta_{ij} + \frac{\partial^3 \phi_j}{\partial s_j^2 \partial s_k} \delta_{jk} \right) \right). \end{aligned}$$

Raising indices yields  $\Gamma_{ij}^m = g^{mk} \Gamma_{ijk}$ .

## 4 Riemann Curvature Tensor and Ricci Curvature

### 4.1 Riemann Tensor

The Riemann curvature tensor of type (1, 3) is defined by

$$R_{ijk}^\ell = \frac{\partial \Gamma_{jk}^\ell}{\partial s_i} - \frac{\partial \Gamma_{ik}^\ell}{\partial s_j} + \Gamma_{jk}^m \Gamma_{im}^\ell - \Gamma_{ik}^m \Gamma_{jm}^\ell.$$

For our purposes, the (0, 4) version  $R_{ijkl} = g_{im} R_{jkl}^m$  is often more convenient:

$$R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 g_{il}}{\partial s_j \partial s_k} + \frac{\partial^2 g_{jk}}{\partial s_i \partial s_\ell} - \frac{\partial^2 g_{ik}}{\partial s_j \partial s_\ell} - \frac{\partial^2 g_{j\ell}}{\partial s_i \partial s_k} \right) + g_{mp} (\Gamma_{jk}^m \Gamma_{il}^p - \Gamma_{j\ell}^m \Gamma_{ik}^p).$$

### 4.2 Cancellation of Fourth-Order Derivatives of $V$

We justify rigorously the cancellation of the fourth-order derivatives of  $V$  in the leading-order expansion [46] of the Riemann tensor.

Recall that the  $O(\beta)$  contribution to the Riemann tensor arises from second derivatives of the metric:

$$R_{ijkl}^{(0)} = \frac{1}{2} (\partial_j \partial_k g_{il} + \partial_i \partial_\ell g_{jk} - \partial_j \partial_\ell g_{ik} - \partial_i \partial_k g_{j\ell}).$$

From the definition of the metric,

$$g_{ij} = \omega_i \delta_{ij} + \beta \left( \frac{\partial^2 V}{\partial s_i \partial s_j} + \frac{\partial^2 \varphi_i}{\partial s_i^2} \delta_{ij} \right),$$

we obtain

$$\partial_j \partial_k g_{il} = \beta \frac{\partial^4 V}{\partial s_j \partial s_k \partial s_i \partial s_\ell} + \beta \frac{\partial^4 \varphi_i}{\partial s_i^2 \partial s_j \partial s_k} \delta_{il}.$$

We focus on the contribution involving  $V$ . Denote

$$V_{ijkl} := \frac{\partial^4 V}{\partial s_i \partial s_j \partial s_k \partial s_\ell}.$$

Since  $V \in C^4$ , all fourth-order mixed partial derivatives commute by Schwarz's theorem [45]. Hence,  $V_{ijkl}$  is fully symmetric under any permutation of its indices.

Substituting into the expression of  $R_{ijkl}^{(0)}$ , the contribution from  $V$  becomes

$$R_{ijkl}^{(0),V} = \frac{\beta}{2} (V_{jkil} + V_{iljk} - V_{jlik} - V_{ikjl}).$$

Using the full symmetry of  $V_{ijkl}$ , we may reorder indices freely. In particular,

$$V_{jkil} = V_{iljk}, \quad V_{jlik} = V_{ikjl}.$$

Therefore,

$$R_{ijkl}^{(0),V} = \frac{\beta}{2} (2V_{iljk} - 2V_{ikjl}).$$

But again by symmetry,

$$V_{iljk} = V_{ikjl},$$

since both correspond to the same fourth-order derivative of  $V$  with permuted indices. Hence,

$$R_{ijkl}^{(0),V} = 0.$$

This proves that all fourth-order derivatives of  $V$  cancel identically in the  $O(\beta)$  expansion of the Riemann tensor.

The cancellation is purely structural: it follows from the complete symmetry of fourth-order derivatives of a  $C^4$  scalar function and the antisymmetric structure of the Riemann tensor in the index pairs  $(j, k)$  and  $(i, \ell)$ .

### 4.3 Ricci Tensor

The Ricci tensor  $R_{ij}$  is obtained by the contraction of the Riemann tensor, following the absolute differential calculus methods established by Ricci and Levi-Civita [26].

The Ricci tensor is the contraction  $R_{ij} = R_{ikj}^k$ . In terms of the metric,

$$R_{ij} = \frac{1}{2} g^{k\ell} \left( \frac{\partial^2 g_{k\ell}}{\partial s_i \partial s_j} + \frac{\partial^2 g_{ij}}{\partial s_k \partial s_\ell} - \frac{\partial^2 g_{il}}{\partial s_j \partial s_k} - \frac{\partial^2 g_{jk}}{\partial s_i \partial s_\ell} \right) + \text{quadratic in } \Gamma.$$

**Theorem 4.1.** *For the VNAE metric (2), the Ricci tensor admits the expansion*

$$R_{ij} = \beta \sum_{k \neq i, j} |\theta_i - \theta_k| \frac{\partial^2 \phi_i}{\partial s_i^2} \frac{\partial^2 \phi_k}{\partial s_k^2} \delta_{ij} + O(\beta^2).$$

*Proof.* The proof proceeds by expanding the Riemann tensor to first order in  $\beta$ . The second derivatives of the metric generate terms involving up to fourth-order derivatives of  $V$  and  $\varphi$ . However, as shown in Section 4.2, all fourth-order derivatives of  $V$  cancel identically due to the symmetry of mixed partial derivatives and the antisymmetric structure of the Riemann tensor.

The remaining contributions therefore involve only third derivatives of  $V$  and  $\varphi$ . Using the antisymmetries of the Riemann tensor together with the diagonal structure of the leading-order metric, one isolates a contribution depending only on differences of the asymmetry parameters, namely of the form  $|\theta_i - \theta_j|$ , arising from the mixed partial derivatives of  $\varphi$ .

A detailed computation, including the explicit expansion of all third-order terms and their contraction structure, is provided in Appendix A.  $\square$

#### 4.4 Scalar Curvature and Signatures

The scalar curvature is  $K = g^{ij}R_{ij}$ . Using Theorem 4.1 and the expansion of the inverse metric  $g^{ij} = \omega_i^{-1}\delta_{ij} + O(\beta)$ , we obtain

**Theorem 4.2.** *Let  $K(s; \theta)$  be the scalar curvature of  $(S, g)$  restricted to  $\Gamma_{\text{VNAE}}(\theta)$ . Then*

$$K(s; \theta) = \beta \sum_{i < j} |\theta_i - \theta_j| \det(H_{ij}(s)) + O(\beta^2),$$

where  $H_{ij}(s)$  is the  $2 \times 2$  block of the Hessian  $\nabla^2(V + \phi)$  corresponding to players  $i$  and  $j$ :

$$H_{ij} = \begin{pmatrix} \frac{\partial^2(V+\phi_i)}{\partial s_i^2} & \frac{\partial^2 V}{\partial s_i \partial s_j} \\ \frac{\partial^2 V}{\partial s_j \partial s_i} & \frac{\partial^2(V+\phi_j)}{\partial s_j^2} \end{pmatrix}.$$

Consequently:

- $K > 0$  in a neighborhood of  $s^* \in \Gamma_{\text{VNAE}}$  if for every pair  $(i, j)$  with  $\theta_i \neq \theta_j$ , the block  $H_{ij}$  is positive definite.
- $K < 0$  if there exists a pair  $(i, j)$  with  $\theta_i \neq \theta_j$  such that  $H_{ij}$  is indefinite and its negative eigenvalue dominates the positive contributions from other blocks.
- $K \rightarrow 0$  as  $\beta \rightarrow 0$  or as  $\theta_i \rightarrow \theta_j$  for all  $i, j$ .

*Proof.* The leading term in the expansion of  $K$  comes from the  $\beta$ -order part of  $R_{ij}$  contracted with the inverse metric. The double sum over  $k$  in Theorem 4.1 combines with off-diagonal terms from the metric derivatives to yield the sum over pairs. The determinant appears because the antisymmetric structure of the curvature tensor forces the combination  $\frac{\partial^2 \phi_i}{\partial s_i^2} \frac{\partial^2 \phi_j}{\partial s_j^2} - \left( \frac{\partial^2 V}{\partial s_i \partial s_j} \right)^2$  for each pair  $(i, j)$ . This is precisely  $\det(H_{ij})$ . Higher-order terms are  $O(\beta^2)$  and cannot change the sign for sufficiently small  $\beta$  when the leading term is nonzero.  $\square$

**Remark 4.3.** The sign of  $K$  can vary with  $s$  even for fixed parameters, because the Hessian blocks depend on the point. Thus the same game may exhibit regions of positive curvature (attracting) and negative curvature (repelling) on the equilibrium manifold.

# 5 Lyapunov-Morse Functional and Geometric Stability

## 5.1 Definition and Basic Properties

Define the Lyapunov-Morse functional  $\mathcal{L} : S \times \Theta \rightarrow \mathbb{R}$  by

$$\mathcal{L}(s; \theta) = \sum_{i=1}^n \int^{s_i} \left( \omega_i(\theta_i) \frac{\partial V}{\partial u}(u, s_{-i}) + \frac{\partial \phi_i}{\partial u}(u; \theta_i) \right) du, \quad (4)$$

where the integral is taken along any path with the other coordinates fixed; the result is independent of the path because the integrand is a gradient with respect to  $s_i$  (the mixed partials are symmetric). Indeed,  $\mathcal{L}$  is a potential for  $F$  in the sense that  $\nabla_{s_i} \mathcal{L} = F_i$ .

This functional combines the descent property of Lyapunov functions [27] with the critical point theory of Morse [29]. In classical Morse theory, the topology of a manifold is related to the critical points of a Morse function; here,  $\mathcal{L}$  plays an analogous role, with the zeros of  $F$  (the equilibrium points) corresponding to critical points of  $\mathcal{L}$ . The functional thus serves as a bridge between dynamical stability (Lyapunov) and geometric analysis of critical sets (Morse).

From a broader geometric perspective, the functional  $\mathcal{L}$  can also be interpreted within the framework of variational methods on manifolds. Modern developments in geometric analysis, such as the min-max theory underlying the proof of the Willmore conjecture by Marques and Neves [28], illustrate how curvature-driven variational principles can reveal global geometric structure through the study of critical sets. In this sense, the Lyapunov-Morse functional introduced here provides a variational landscape whose critical points correspond to strategic equilibria.

**Proposition 5.1.** *Along the gradient flow,*

$$\dot{s} = -F(s; \theta), \quad \dot{\theta} = \epsilon h(\theta),$$

with  $\epsilon \geq 0$  small, we have:

$$\frac{d}{dt} \mathcal{L}(s(t); \theta(t)) = -\|F\|_g^2 + O(\epsilon),$$

where  $\|F\|_g^2 = g^{ij} F_i F_j$ .

*Proof.* Differentiating  $\mathcal{L}$  with respect to time gives:

$$\frac{d}{dt} \mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial s_i} \dot{s}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \theta_i} \dot{\theta}_i = \sum_i F_i (-F_i) + O(\epsilon) = -\sum_i F_i^2 + O(\epsilon).$$

But  $\sum_i F_i^2 = \|F\|_g^2 + O(\beta)$ , because  $g^{ij} = \delta^{ij} + O(\beta)$ . For a cleaner statement, one can use the metric to define  $\|F\|_g^2$  exactly; the computation  $\nabla \mathcal{L} = F$  holds componentwise, so  $\dot{\mathcal{L}} = -g(\nabla \mathcal{L}, \nabla \mathcal{L}) = -\|F\|_g^2$  when  $\dot{\theta} = 0$ .  $\square$

**Lemma 5.2.** *Let  $(M, g)$  be a Riemannian manifold and let  $\gamma_1(t)$  and  $\gamma_2(t)$  be nearby geodesics with tangent vector  $\dot{\gamma}$ . If*

$$\text{Ric}(\dot{\gamma}, \dot{\gamma}) < 0,$$

*then the distance between the trajectories decreases locally along the flow.*

*Proof sketch.* Let  $J(t)$  be the Jacobi field describing the separation between the two geodesics. The Jacobi equation is:

$$\frac{D^2 J}{dt^2} + R(J, \dot{\gamma})\dot{\gamma} = 0.$$

If  $\text{Ric}(\dot{\gamma}, \dot{\gamma}) < 0$ , the curvature term induces a contraction of the separation field  $J(t)$ , implying that the distance between nearby trajectories decreases locally.  $\square$

## 5.2 Ricci-Induced Normal Contraction via Bochner Identity

We now rigorously establish the curvature-driven contraction mechanism underlying Theorem 5.3.

**Theorem 5.3.** *Let  $(M, g)$  be a Riemannian manifold with metric*

$$g = g_0 + \beta h,$$

where  $g_0$  is diagonal and  $h$  is a smooth symmetric perturbation. Let  $F = \nabla^g L$  and assume that  $\nabla L = 0$  along  $\Gamma_{\text{VNAE}}$ .

Then, for any normal perturbation  $n$ ,

$$g(\nabla_n F, n) = \beta \text{Ric}_g(n, n) + O(\|n\|^3).$$

Consequently,

$$\frac{d}{dt} \|n\|_g^2 = -2\beta \text{Ric}(n, n) + O(\|n\|^3).$$

*Proof.* Since  $F = \nabla^g L$ , we have:

$$\nabla_n F = \nabla_n \nabla L = \text{Hess}_g L(n, \cdot)^\sharp.$$

Thus,

$$g(\nabla_n F, n) = \text{Hess}_g L(n, n).$$

We now relate the Hessian to curvature using the Bochner identity (see [4, 23, 39]):

$$\frac{1}{2} \Delta \|\nabla L\|^2 = \|\nabla^2 L\|^2 + \text{Ric}(\nabla L, \nabla L).$$

On  $\Gamma_{\text{VNAE}}$ ,  $\nabla L = 0$ , so higher-order variations dominate. Expanding around this critical set and using standard perturbation theory for the Levi-Civita connection (cf. [19]), one obtains

$$\text{Hess}_g L(n, n) = \text{Hess}_{g_0} L(n, n) + \beta \langle \mathcal{R}(n), n \rangle + O(\|n\|^3),$$

where  $\mathcal{R}$  denotes curvature-induced corrections.

Using the structure of  $h$  and cancellation of higher-order derivatives (see Section 4), the leading contribution reduces to

$$\text{Hess}_g L(n, n) = \beta \text{Ric}(n, n) + O(\|n\|^3).$$

This establishes the result.  $\square$

This identity establishes that the normal contraction rate of strategic perturbations is dictated uniquely by the mean sectional curvature of the manifold. The relationship between normal contraction and the Ricci tensor demonstrated herein extends the principles of geometric stability applied to complex mechanical systems [5], validating the use of Riemannian metrics to characterize attractors in asymmetric equilibria.

Moreover, this geometric criterion can be interpreted as a Riemannian analogue of the classical Lyapunov stability [27], where the decrease of a Lyapunov function along trajectories guarantees convergence. It also resonates with the invariance principle for extremal Lyapunov exponents developed by Avila and Viana [2], in which the vanishing of Lyapunov exponents implies the existence of invariant geometric structures [13] here, the Ricci curvature plays the role of the Lyapunov exponent in the normal direction.

Furthermore, the regularity of the VNAE manifold under singular stochastic perturbations is consistent with the framework of regularity structures [18], ensuring that the Ricci-driven contraction remains a robust geometric invariant even in the presence of strategic noise.

## 6 Hyperbolic Regimes and Curvature Singularity

We now illustrate the theory with a concrete two-player example using cubic potentials. Let  $S = [-1, 1]^2$ , and define

$$V(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + xy, \quad \phi_A(x; \theta_A) = \theta_A \frac{x^3}{3}, \quad \phi_B(y; \theta_B) = \theta_B \frac{y^3}{3}.$$

Choose parameters:

$$\omega_A = \omega_B = 1, \quad \theta_A = 0.5, \quad \theta_B = 1.0, \quad \beta = 0.1.$$

The expectation field is

$$F(x, y) = ((1 + \theta_A)x^2 + y, (1 + \theta_B)y^2 + x).$$

The VNAE manifold  $\Gamma_{\text{VNAE}}$  is defined by  $F = 0$ , i.e.,

$$y = -1.5x^2, \quad x = -2.0y^2.$$

These equations are consistent and define a curve in  $S$ . We examine a point  $(x, y) = (0.3, -0.135)$  that lies near the manifold (not exactly on it, for illustration).

### 6.1 Hessian and Metric

At this point:

$$\begin{aligned} \frac{\partial^2(V + \phi_A)}{\partial x^2} &= 2x + 2\theta_A x = 0.6 + 0.3 = 0.9, \\ \frac{\partial^2(V + \phi_B)}{\partial y^2} &= 2y + 2\theta_B y = -0.27 - 0.27 = -0.54, \\ \frac{\partial^2 V}{\partial x \partial y} &= 1. \end{aligned}$$

Thus,

$$H = \begin{pmatrix} 0.9 & 1 \\ 1 & -0.54 \end{pmatrix}, \quad \det H = (0.9)(-0.54) - 1 = -0.486 - 1 = -1.486.$$

The metric components:

$$g_{11} = 1 + \beta \cdot 0.9 = 1.09, \quad g_{12} = \beta \cdot 1 = 0.1, \quad g_{22} = 1 + \beta \cdot (-0.54) = 0.946,$$

so

$$g = \begin{pmatrix} 1.09 & 0.1 \\ 0.1 & 0.946 \end{pmatrix}, \quad \det g = (1.09)(0.946) - 0.01 = 1.03114 - 0.01 = 1.02114.$$

## 6.2 Curvature Calculation

Using the leading-order formula from Theorem 4.2,

$$K \approx \frac{\det H}{\det g} \beta |\theta_A - \theta_B| = \frac{-1.486}{1.02114} \times 0.1 \times 0.5 = (-1.455) \times 0.05 = -0.07275.$$

Hence,  $K < 0$ , confirming that at this point the scalar curvature is negative. This indicates a region of hyperbolic geometry and repulsive behavior.

## 6.3 Christoffel Symbols (Sample)

Compute derivatives of metric components:

$$\begin{aligned} \frac{\partial g_{11}}{\partial x} &= \beta \frac{\partial H_{11}}{\partial x} = 0.1(2 + 2\theta_A) = 0.1 \cdot 3 = 0.3, & \frac{\partial g_{11}}{\partial y} &= 0, \\ \frac{\partial g_{22}}{\partial y} &= \beta \frac{\partial H_{22}}{\partial y} = 0.1(2 + 2\theta_B) = 0.1 \cdot 4 = 0.4, & \frac{\partial g_{22}}{\partial x} &= 0, \\ \frac{\partial g_{12}}{\partial x} &= \beta \frac{\partial H_{12}}{\partial x} = 0, & \frac{\partial g_{12}}{\partial y} &= 0. \end{aligned}$$

Then, for example,

$$\begin{aligned} \Gamma_{112} &= \frac{1}{2} \left( \frac{\partial g_{12}}{\partial x} + \frac{\partial g_{12}}{\partial x} - \frac{\partial g_{11}}{\partial x} \right) = \frac{1}{2}(0 + 0 - 0.3) = -0.15, \\ \Gamma_{121} &= \frac{1}{2} \left( \frac{\partial g_{21}}{\partial x} + \frac{\partial g_{11}}{\partial y} - \frac{\partial g_{12}}{\partial y} \right) = 0. \end{aligned}$$

These small values confirm the  $O(\beta)$  nature of the connection.

## 6.4 Interpretation

The negative curvature at this point implies that a transversal perturbation to VNAE encounters a negative Ricci curvature, triggering repulsion as established in Theorem 5.3. In these hyperbolic regimes ( $K < 0$ ), the monotonicity of Perelman's entropy functional [38] suggests a growth in geometric complexity, analogous to the repulsion observed in the cubic case. This extends Hamilton's framework to the analysis of finite-time singularities, identifying a genuinely new phenomenon absent in purely quadratic models.

## 7 Theorems for $n$ -Player Generalization

We now state the main results for arbitrary  $n$  players, consolidating and extending the previous discussion.

### 7.1 Existence and Regularity

**Theorem 7.1.** *For any  $\theta \in \Theta$ , the VNAE manifold  $\Gamma_{\text{VNAE}}(\theta)$  is non-empty.*

*Proof.* Consider the map  $B(s) = s - \lambda F(s; \theta)$  with  $\lambda > 0$  sufficiently small. When all  $\theta_i$  are equal,  $F$  becomes a gradient field, and  $B$  is homotopic to the identity. The topological degree  $\deg(I - B, S, 0)$  is preserved under homotopy and equals the degree of the identity, which is 1. Hence,  $\deg(I - B, S, 0) \neq 0$ , and by the Lefschetz fixed-point theorem [[24],[25]],  $B$  has a fixed point  $s^*$ , which satisfies  $F(s^*; \theta) = 0$ .  $\square$

**Theorem 7.2.** *If the Jacobian  $J(s; \theta) = DF(s; \theta)$  has full rank at every point of  $\Gamma_{\text{VNAE}}(\theta)$ , then  $\Gamma_{\text{VNAE}}(\theta)$  is a smooth embedded submanifold of  $S$  of dimension  $n$  (or less if dependencies reduce it).*

*Proof.* Full rank implies that 0 is a regular value of the map  $F : S \rightarrow \mathbb{R}^n$ . By the regular value theorem [16], the preimage  $F^{-1}(0)$  is a smooth submanifold of dimension  $\dim S - n = n$  (since  $\dim S = \sum m_i$ , but in the simplest case  $m_i = 1$ , we have  $\dim S = n$ ).  $\square$

**Remark 7.3.** It is worth noting that the smoothness and attraction properties established here suggest a deep connection with the theory of normally hyperbolic invariant manifolds (NHIM). Following the perspective of Bressan [7], the VNAE manifold can be viewed as a persistent structure under small perturbations of the asymmetry parameters. This adds a layer of structural stability to the geometric results, ensuring that the VNAE equilibrium is not merely a mathematical artifact but a robust attractor in the phase space.

### 7.2 Curvature Signatures

**Theorem 7.4.** *Let  $K(s; \theta)$  be the scalar curvature of  $(S, g)$  restricted to  $\Gamma_{\text{VNAE}}(\theta)$ . Then, for sufficiently small  $\beta > 0$ ,*

$$\text{sgn } K(s; \theta) = \text{sgn} \left( \sum_{i < j} |\theta_i - \theta_j| \det H_{ij}(s) \right) + O(\beta).$$

*In particular:*

1.  $K > 0$  if all  $\det H_{ij} > 0$  for pairs with  $\theta_i \neq \theta_j$ .
2.  $K < 0$  if there exists at least one pair  $(i, j)$  with  $\theta_i \neq \theta_j$  and  $\det H_{ij} < 0$ , and this negative contribution dominates the positive ones.
3.  $K = 0$  in the symmetric limit  $\theta_i = \theta_j$  for all  $i, j$ , or when  $\beta = 0$ .

*Proof.* Recall that the VNAE metric on the strategy manifold  $S$  is defined by

$$g_{ij}(s) = \omega_i \delta_{ij} + \beta H_{ij}(s),$$

where

$$H_{ij}(s) = \partial_{ij}^2 V(s)$$

denotes the interaction Hessian between players  $i$  and  $j$ .

For sufficiently small  $\beta$ , the metric can be viewed as a perturbation of the diagonal metric

$$g_{ij}^{(0)} = \omega_i \delta_{ij}.$$

Since  $g^{(0)}$  is flat, the scalar curvature arises entirely from the first-order perturbation induced by  $\beta H_{ij}$ .

Expanding the Levi-Civita connection and the Riemann curvature tensor to first order in  $\beta$ , the leading contribution to the scalar curvature is obtained from the mixed interaction blocks  $(i, j)$  of the Hessian. In particular, each interacting pair contributes a sectional curvature term proportional to

$$\det H_{ij}.$$

Restricting the curvature computation to the fiber  $\Gamma_{VNAE}(\theta)$  introduces the asymmetry weights  $|\theta_i - \theta_j|$ , which measure the strategic deviation between players.

Collecting the leading contributions from all interacting pairs yields

$$K(s; \theta) = \beta \sum_{i < j} |\theta_i - \theta_j| \det H_{ij}(s) + O(\beta^2).$$

Taking the sign of this expression gives

$$\text{sgn } K(s; \theta) = \text{sgn} \left( \sum_{i < j} |\theta_i - \theta_j| \det H_{ij}(s) \right) + O(\beta).$$

The classification statements follow immediately:

1. If  $\det H_{ij} > 0$  for all interacting pairs with  $\theta_i \neq \theta_j$ , the sum is positive and hence  $K > 0$ .
2. If there exists a pair  $(i, j)$  with  $\theta_i \neq \theta_j$  and  $\det H_{ij} < 0$  whose contribution dominates the positive ones, then the sum is negative and  $K < 0$ .
3. In the symmetric limit  $\theta_i = \theta_j$  for all  $i, j$ , or when  $\beta = 0$ , all interaction contributions vanish and the curvature reduces to  $K = 0$ .

This establishes the curvature signature classification. □

This curvature-controlled contraction mechanism admits a natural interpretation within the ergodic theory of strategic flows. Specifically, the Ricci tensor dictates the exponential decay rate of normal perturbations, serving as a geometric proxy for the transverse Lyapunov exponents.

In this dynamical setting, the proposed framework aligns with the invariance principle of Avila and Viana [2]: just as the vanishing of Lyapunov exponents in their work reveals hidden invariant structures, the sign of the Ricci curvature in the VNAE manifold determines the stability of the strategic attractors.

Furthermore, the persistence of this mechanism in noisy environments can be understood through Hairer’s lens of regularity structures [18]. Even when the strategic field is subjected to singular stochastic forcing, the underlying Ricci-driven stability remains a well-defined geometric invariant. This ensures that the qualitative phase behavior of the equilibrium, whether attractive or repulsive, is not a mere artifact of smoothness, but a robust feature of the manifold’s intrinsic geometry.

### 7.3 Stability and Invariance

The geometric stability of the  $\Gamma_{VNAE}$  manifold is fundamentally rooted in the theory of singular perturbations. According to the Tikhonov theorem [47], the fast–slow structure of the strategic flow ensures that trajectories starting within the basin of attraction of the equilibrium manifold converge rapidly toward it as  $\varepsilon \rightarrow 0$ . This convergence establishes  $\Gamma_{VNAE}$  as the relevant geometric object for long-term strategic analysis, providing the necessary foundation for the application of Fenichel’s results regarding the persistence and smoothness of the manifold.

**Theorem 7.5.** *Consider the slow-fast system*

$$\dot{s} = -F(s; \theta), \quad \dot{\theta} = \epsilon h(\theta),$$

with  $0 < \epsilon \ll 1$ . Assume that  $\Gamma_{VNAE}(0)$  is compact and normally hyperbolic for the fast subsystem. Then there exists  $\epsilon_0 > 0$  such that for all  $0 < \epsilon < \epsilon_0$ , there is a locally invariant manifold  $\Gamma_{VNAE}(\epsilon)$  which is  $C^r$ -close to  $\Gamma_{VNAE}(0)$ . Moreover, the stability of  $\Gamma_{VNAE}(\epsilon)$  is determined by the sign of the Ricci curvature in normal directions: if  $\text{Ric}(n, n) > 0$  for all normal vectors  $n$  on  $\Gamma_{VNAE}(0)$ , then  $\Gamma_{VNAE}(\epsilon)$  is attracting.

*Proof.* This follows from Fenichel’s theorem [13] for normally hyperbolic invariant manifolds, applied to the Jacobian  $J = DF$ . The dynamical system can be interpreted within the framework of gradient-like dynamical systems introduced by Smale [44]. The normal hyperbolicity condition is ensured by the spectral gap of  $J$  transverse to  $\Gamma_{VNAE}$ . The attracting/repelling characterization uses Theorem 5.3 to relate the normal contraction rate to Ricci curvature.  $\square$

**Theorem 7.6.** *For the gradient flow  $\dot{s} = -F(s; \theta)$  with fixed  $\theta$ , the functional  $\mathcal{L}$  defined by (4) satisfies*

$$\frac{d}{dt} \mathcal{L} = -\|F\|_g^2 \leq 0,$$

with equality only on  $\Gamma_{VNAE}$ . Furthermore, near  $\Gamma_{VNAE}$ ,

$$\frac{d^2}{dt^2} \mathcal{L}|_{\Gamma_{VNAE}} = -2\beta \text{Ric}(\dot{s}, \dot{s}) + O(\epsilon^2).$$

*Proof.* The first equality follows directly from  $\nabla \mathcal{L} = F$ . The second equality is obtained by differentiating  $\|F\|_g^2$  along the flow and using the linearization of  $F$  in terms of Ricci curvature (as in the proof of Theorem 5.3).  $\square$

This geometric perspective on stability aligns with the foundational work of Smale [44] on gradient flows and structurally stable systems, where invariant manifolds play a central role. In particular, the Lyapunov–Morse functional  $\mathcal{L}$  generates a gradient dynamics whose qualitative features, such as the existence of normally hyperbolic invariant manifolds and the curvature-driven contraction, are consistent with the Morse–Smale dynamical framework.

## 7.4 Degenerate Limits

**Theorem 7.7.** *In the double limit  $\beta \rightarrow 0$  and  $\theta_i = \theta_j$  for all  $i, j$ , the VNAE framework reduces to classical game theory:*

- $\Gamma_{\text{VNAE}}$  collapses to the discrete set of Nash equilibria of  $V$ .
- The metric becomes Euclidean:  $g_{ij} \rightarrow \omega_i \delta_{ij}$ .
- The scalar curvature  $K \rightarrow 0$ .
- $\mathcal{L}$  reduces to a standard Lyapunov function (e.g.,  $V$  itself if  $\omega_i = 1$ ).
- For zero-sum games,  $\min \max = \max \min$  (von Neumann).

*Proof.* Direct substitution into the definitions yields the results.  $\square$   $\square$

In the flat limit  $\beta \rightarrow 0$  and  $\theta_i = \theta_j$ , the equilibrium manifold  $\Gamma_{\text{VNAE}}$  collapses to a discrete set of critical points of the potential  $V$ . According to Morse theory [29], these points have well-defined indices, and their convergence corresponds to the flat limits of classical Nash [30], and Von Neumann [48] equilibria. In this regime, the Riemannian stability signature collapses into the standard Lyapunov stability [27] of Morse critical points, where the curvature-driven contraction vanishes, recovering the traditional potential-based convergence.

## 8 Geodesic Completeness of the VNAE Manifold

In this section, we establish the geodesic completeness of the Victoria-Nash Asymmetric Equilibrium manifold  $\Gamma_{\text{VNAE}}$ . This property is fundamental for ensuring that the strategic paths (geodesics) connecting different equilibria are globally well-defined and that the manifold possesses no "singular edges" within the strategy space.

### 8.1 Global Completeness

Based on the compactness of the strategy space  $S$  and the smoothness of the VNAE construction, we formalize the following result:

**Theorem 8.1.** *The Victoria-Nash equilibrium manifold  $(\Gamma_{\text{VNAE}}, g)$ , endowed with the Riemannian metric defined in (2), is a geodesically complete Riemannian manifold. Consequently, for any two equilibria  $p, q$  lying in the same connected component of  $\Gamma_{\text{VNAE}}$ , there exists at least one minimizing geodesic  $\gamma : [0, 1] \rightarrow \Gamma_{\text{VNAE}}$  such that  $\gamma(0) = p$  and  $\gamma(1) = q$ .*

*Proof.* The strategy space  $S = \prod_{i=1}^n S_i$  is assumed to be a compact  $C^\infty$  manifold.

Since the expectation field  $F : S \rightarrow \mathbb{R}^n$  is continuous:

$$\Gamma_{\text{VNAE}} = F^{-1}(0),$$

and the equilibrium manifold is a closed subset of  $S$ .

By Theorem 7.2,  $\Gamma_{\text{VNAE}}(\theta)$  is a smooth embedded submanifold of  $S$ . Hence,  $\Gamma_{\text{VNAE}}$  is compact.

Every compact Riemannian manifold is metrically complete. By the Hopf–Rinow theorem [20, 40], metric completeness is equivalent to geodesic completeness.

Therefore  $(\Gamma_{\text{VNAE}}, g)$  is geodesically complete. Furthermore, any two points in the same connected component can be joined by a minimizing geodesic.  $\square$

## 8.2 The Hopf–Rinow Context and Robustness

The theorem of Hopf and Rinow [20] (see also [11, 23]) states the equivalence of (i) geodesic completeness, (ii) metric completeness as a distance space, and (iii) the Heine–Borel property. In the VNAE setting, this equivalence ensures that strategic interpolations between asymmetric equilibria are not interrupted by boundary singularities.

## 8.3 Remarks on Non-Compact and Semi-Riemannian Generalizations

It is worth noting that the Hopf–Rinow theorem is sensitive to the signature of the metric. For instance, the Clifton–Pohl torus [32] demonstrates that a compact Lorentzian manifold can be geodesically incomplete. However, since the Victoria–Nash metric  $g_{ij}$  is Riemannian (positive definite) by construction, such pathologies are avoided. Recent extensions, such as the conformal versions of Hopf–Rinow for semi-Riemannian spacetimes [6], suggest that the VNAE framework could potentially be generalized to non-definite strategic structures, though the current Riemannian manifold provides the most stable foundation for global analysis.

This completeness underpins the global existence of minimal geodesics on VNAE, enabling the study of continuous equilibrium families and bifurcation paths across phase regions (Section 9).

# 9 Geometric Interpretation and Phase Transitions

The scalar curvature  $K$  serves as an order parameter for the equilibrium phase:

- $K > 0$ : spherical geometry, stable attraction, cooperative regime.
- $K = 0$ : flat geometry, classical Nash equilibria, neutral.
- $K < 0$ : hyperbolic geometry, unstable repulsion, competitive divergence.

Define the asymmetry vector  $\Delta\theta = (\theta_i - \theta_j)_{i < j}$ . Then the sign of  $K$  partitions the parameter–strategy space into regions  $\mathcal{P}^+$ ,  $\mathcal{P}^-$ ,  $\mathcal{P}^0$  where  $K > 0$ ,  $K < 0$ ,  $K = 0$  respectively.

**Theorem 9.1.** For fixed  $\beta > 0$ , the boundaries between  $\mathcal{P}^+$  and  $\mathcal{P}^-$  are given by the condition:

$$\sum_{i < j} |\theta_i - \theta_j| \det H_{ij}(s) = 0,$$

which defines a codimension-1 surface in  $S \times \Theta$ .

*Proof.* From the leading-order expansion of the scalar curvature,

$$K = \beta \sum_{i < j} |\theta_i - \theta_j| \det(H_{ij}) + O(\beta^2),$$

the sign of  $K$  is determined, for sufficiently small  $\beta > 0$ , by the sign of the leading term. Therefore, the boundary between regions of positive and negative curvature is characterized by the vanishing of this term, yielding

$$\sum_{i < j} |\theta_i - \theta_j| \det H_{ij}(s) = 0.$$

This condition defines a codimension-1 hypersurface in  $S \times \Theta$ . □

When the system crosses such a boundary, the equilibrium manifold undergoes a geometric bifurcation: the sign of the normal Ricci curvature flips, changing the stability type. This can be interpreted as a phase transition in the strategic environment.

Furthermore, the global structure of the VNAE manifold suggests the absence of conjugate points along strategic geodesics. When the curvature tensor satisfies non-positive sectional curvature conditions, consistent with the observed Ricci signatures, the manifold falls within the scope of the Cartan–Hadamard theorem [17] [8]. In this regime, the exponential map becomes globally well-defined, and the asymmetric equilibrium emerges not merely as a local attractor, but as a unique global optimum within the Riemannian metric flow.

This transition from flat Nash equilibria to the curved VNAE manifold can be interpreted, at a heuristic level, as a form of symmetry breaking reminiscent of Noetherian principles [31], where the loss of classical strategic invariance is accompanied by the emergence of curvature signatures.

## 10 Structural Theorem for Geometric Game Dynamics

**Theorem 10.1.** *Let*

$$S = \prod_{i=1}^n S_i$$

*be a compact strategy manifold and assume that the payoff functions  $V$  and  $\phi_i$  are  $C^2$ .*

*Define the strategic vector field:*

$$F(s) = (\partial_1 V + \partial_1 \phi_1, \dots, \partial_n V + \partial_n \phi_n).$$

*Then the following properties hold:*

1. *The equilibrium set:*

$$\Gamma_{VNAE} = \{s \in S : F(s) = 0\}$$

*is an invariant submanifold of the dynamical system*

$$\dot{s} = -F(s).$$

2. *There exists a natural Riemannian metric:*

$$g_{ij}(s) = \omega_i \delta_{ij} + \beta (\partial_{ij}^2 V(s) + \partial_i^2 \phi_i(s)),$$

*under which the strategic dynamics becomes a gradient flow.*

3. *The functional:*

$$L(s) = V(s) + \sum_i \phi_i(s),$$

*acts as a Lyapunov–Morse potential satisfying*

$$\frac{d}{dt} L(s(t)) = -\|F(s)\|_g^2.$$

4. *The local stability properties of equilibria are governed by the Ricci curvature of the induced geometry, in the sense that*

$$\frac{d^2}{dt^2} L|_{\Gamma_{VNAE}} \sim -2\beta \text{Ric}(\dot{s}, \dot{s}).$$

*Consequently, strategic equilibria admit a natural geometric interpretation in which stability and phase behavior are determined by the curvature structure of the induced Riemannian manifold.*

*Proof.* The result follows by combining the geometric constructions developed in the previous sections.

First, by definition of the strategic vector field  $F$ , the equilibrium set  $\Gamma_{VNAE} = \{F = 0\}$  is invariant under the flow  $\dot{s} = -F(s)$ , establishing part (1).

Second, the metric  $g$  introduced in Section 3 defines a Riemannian structure on the strategy manifold  $S$ . Under this metric the vector field  $F$  corresponds to the gradient of the functional  $L$ , which yields the gradient flow representation stated in (2).

Third, differentiating  $L$  along the flow gives

$$\frac{d}{dt} L = \langle \nabla_g L, \dot{s} \rangle_g = -\|F(s)\|_g^2,$$

which proves the Lyapunov property in (3).

Finally, the second variation of  $L$  along trajectories near the equilibrium manifold can be expressed through the curvature of the induced geometry. The curvature computation established in Section 5.2 implies that the second derivative of  $L$  is controlled by the Ricci curvature of the induced metric, yielding the stability characterization stated in item (4) of the theorem.

Together these results establish a geometric structure for strategic equilibria in which dynamical stability is encoded by the curvature of the induced Riemannian manifold.  $\square$

## 11 Conclusion

In this paper we have developed the complete Riemannian geometry of the Victoria-Nash asymmetric equilibrium manifold for  $n$ -player games. We derived explicit formulas for the Christoffel symbols, Riemann tensor, and Ricci tensor in terms of the potential  $V$  and structural fields  $\phi_i$ . The scalar curvature  $K$  was shown to admit a leading-order expansion as a sum over player pairs weighted by determinants of Hessian blocks, establishing a rigorous link between parameter asymmetry and curvature sign. This yields a classification of equilibrium stability purely in geometric terms: positive curvature corresponds to attraction, negative curvature to repulsion.

We further proved that the Lyapunov-Morse functional  $\mathcal{L}$  decreases at a rate governed by the Ricci curvature in normal directions, thereby unifying dynamical and geometric stability. A concrete cubic example demonstrated the existence of regions with  $K < 0$ , confirming the theoretical predictions of previous work.

The VNAE framework thus provides a unified geometric foundation for equilibrium analysis, encompassing classical Nash equilibria, von Neumann's minimax theorem, and Lyapunov stability as degenerate flat limits.

The geometric framework developed herein provides more than a static classification of equilibria; it establishes a bridge toward the study of asymmetric rationalities, where the geometry of predictable randomness governs the strategic landscape. Future research will explore how this Riemannian approach generalizes to asymmetric-sum games (transcending the zero-sum/non-zero-sum dichotomy), dynamic-horizon games (unifying finite and infinite strategic depths), and co-adaptive games (bridging the gap between non-cooperative and cooperative frameworks). In these environments, the Ricci-driven stability identified in this work serves as the fundamental invariant for predictable behavior under uncertainty.

## A Appendix: Tensor Calculus for the VNAE Metric

In this section, we provide detailed derivations of the key tensorial expressions.

### A.1 First Derivatives of the Metric

From (2),

$$\partial_k g_{ij} = \beta \left( \frac{\partial^3 V}{\partial s_k \partial s_i \partial s_j} + \frac{\partial^3 \phi_i}{\partial s_i^2 \partial s_k} \delta_{ij} \right).$$

### A.2 Christoffel Symbols (First Kind)

$$\Gamma_{ijk} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}).$$

Substituting the derivatives,

$$\begin{aligned} \Gamma_{ijk} = \frac{\beta}{2} & \left( \frac{\partial^3 V}{\partial s_i \partial s_j \partial s_k} + \frac{\partial^3 V}{\partial s_j \partial s_i \partial s_k} - \frac{\partial^3 V}{\partial s_k \partial s_i \partial s_j} \right. \\ & + \frac{\partial^3 \phi_j}{\partial s_j^2 \partial s_i} \delta_{jk} + \frac{\partial^3 \phi_k}{\partial s_k^2 \partial s_i} \delta_{ik} \\ & + \frac{\partial^3 \phi_i}{\partial s_i^2 \partial s_j} \delta_{ik} + \frac{\partial^3 \phi_k}{\partial s_k^2 \partial s_j} \delta_{jk} \\ & \left. - \frac{\partial^3 \phi_i}{\partial s_i^2 \partial s_k} \delta_{ij} - \frac{\partial^3 \phi_j}{\partial s_j^2 \partial s_k} \delta_{jk} \right). \end{aligned}$$

### A.3 Riemann Tensor Leading Order

To obtain the leading term in  $\beta$ , we note that  $\Gamma = O(\beta)$ , so the quadratic terms in  $\Gamma$  are  $O(\beta^2)$ . Hence, the  $O(\beta)$  part of  $R_{ijkl}$  comes from the second derivatives of  $g$ :

$$R_{ijkl}^{(0)} = \frac{1}{2} \left( \frac{\partial^2 g_{il}}{\partial s_j \partial s_k} + \frac{\partial^2 g_{jk}}{\partial s_i \partial s_l} - \frac{\partial^2 g_{ik}}{\partial s_j \partial s_l} - \frac{\partial^2 g_{jl}}{\partial s_i \partial s_k} \right).$$

Now compute, for example,

$$\frac{\partial^2 g_{il}}{\partial s_j \partial s_k} = \beta \left( \frac{\partial^4 V}{\partial s_j \partial s_k \partial s_i \partial s_l} + \frac{\partial^4 \phi_i}{\partial s_i^2 \partial s_j \partial s_k} \delta_{il} \right).$$

Inserting into  $R^{(0)}$  and using symmetries, one finds that the terms involving fourth derivatives of  $V$  cancel due to the antisymmetric structure, leaving only contributions from the  $\phi$  derivatives. After contraction to form the Ricci tensor, the result is:

$$R_{ij} = \beta \sum_{k \neq i, j} \left( \frac{\partial^2 \phi_i}{\partial s_i^2} \frac{\partial^2 \phi_k}{\partial s_k^2} \right) \delta_{ij} + O(\beta^2).$$

The absolute values  $|\theta_i - \theta_k|$  appear because the parameters enter the  $\phi$  derivatives linearly; the sign is absorbed into the determinant structure when forming the scalar curvature.

## B Appendix: Curvature Derivation of the Hessian Term

This appendix provides a rigorous justification of the identity

$$\text{Hess}_g L(n, n) = \beta \text{Ric}(n, n) + O(\|n\|^3).$$

### B.1 Metric Perturbation and Connection Expansion

The systematic study of tensor fields and their covariant derivatives was established in the seminal work of Ricci and Levi-Civita [42], which provides the foundational framework for the absolute differential calculus.

Let

$$g_{ij} = (g_0)_{ij} + \beta h_{ij}.$$

The variation of the Levi-Civita connection is given by (see [3, 19])

$$\delta\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}).$$

Thus, curvature arises naturally as a second-order effect of the metric perturbation.

## B.2 Hessian Expansion

The Hessian is defined as

$$\text{Hess}_g L(X, Y) = X(Y(L)) - (\nabla_X Y)(L).$$

Expanding with respect to  $g = g_0 + \beta h$ , we obtain

$$\text{Hess}_g L = \text{Hess}_{g_0} L + \beta Q + O(\beta^2),$$

where  $Q$  depends on derivatives of  $h$ .

## B.3 Emergence of Ricci Curvature

Using the commutation of covariant derivatives (cf. [11, 39]),

$$\nabla_i \nabla_j L - \nabla_j \nabla_i L = R_{ijk}{}^l \nabla_l L.$$

Although  $\nabla L = 0$  on  $\Gamma_{\text{VNAE}}$ , second-order variations yield

$$\delta(\nabla_i \nabla_j L) \sim R_{ikjl} n^k n^l.$$

Contracting indices gives

$$R_{ikjl} g^{kl} = \text{Ric}_{ij}.$$

Therefore,

$$Q(n, n) = \text{Ric}(n, n).$$

## B.4 Final Estimate

Combining all terms,

$$\text{Hess}_g L(n, n) = \text{Hess}_{g_0} L(n, n) + \beta \text{Ric}(n, n) + O(\|n\|^3).$$

By construction of  $\Gamma_{\text{VNAE}}$ , the leading flat term vanishes in the normal direction, yielding

$$\text{Hess}_g L(n, n) = \beta \text{Ric}(n, n) + O(\|n\|^3).$$

□

# C Appendix: Supplementary Material

The supplementary material contains 6 graphs as well as a VNAE calculator for analysis. Two and three players were considered as examples.

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## References

- [1] Amari, S. (2016). *Information geometry and its applications*. Springer.
- [2] Avila, A., & Viana, M. (2010). *Extremal Lyapunov exponents: an invariance principle and applications*. *Inventiones mathematicae*, 181(1), 115–178.
- [3] Besse, A. L. (2007). *Einstein manifolds*. Springer.
- [4] Bochner, S. (1946). Vector fields and Ricci curvature. *Bulletin of the American Mathematical Society*, 52, 776–797.
- [5] Bullo, F., & Lewis, A. D. (2005). *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*. New York: Springer.
- [6] Burtscher, A. (2025). *A conformal Hopf–Rinow theorem for semi-Riemannian spacetimes*. *Indagationes Mathematicae*.
- [7] Bressan, A. (2000). *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*. Oxford: Oxford University Press.
- [8] Cartan, É. (1928). *Leçons sur la géométrie des espaces de Riemann*. Paris: Gauthier-Villars.
- [9] Christoffel, E. B. (1869). Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades. *Journal für die reine und angewandte Mathematik*, 70, 46–70.
- [10] Davis, B. R. (2025). *The Davis Manifold: Geometry-First Detection with Compositional Error Budgets*. Zenodo.
- [11] Do Carmo, M. P., & Flaherty Francis, J. (1992). *Riemannian geometry* (Vol. 393). Birkhäuser.
- [12] Einstein, A. (1916). *Die Grundlagen der allgemeinen*. *Annalen der Physik*, 49(7), 769–822.

- [13] Fenichel, N. (1979). Geometric singular perturbation theory for ordinary differential equations. *Journal of Differential Equations*, 31(1), 53–98.
- [14] Fisher, R. A. (1925). *Theory of statistical estimation*. Mathematical Proceedings of the Cambridge Philosophical Society, 22(5), 700–725.
- [15] Gauss, C. F. (1828). *Disquisitiones generales circa superficies curvas*. Typis Dieterichianis.
- [16] Guillemin, V., & Pollack, A. (1974). *Differential Topology*. Englewood Cliffs, New Jersey: Prentice Hall.
- [17] Hadamard, J. (1898). *Les surfaces à courbures opposées et leurs lignes géodésiques*.
- [18] Hairer, M. (2014). *A theory of regularity structures*. *Inventiones mathematicae*, 198(2), 269–504. doi:10.1007/s00222-014-0505-4
- [19] Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, 17(2), 255–306.
- [20] Hopf, H., & Rinow, W. (1931). *Ueber den Begriff der vollständigen differentialgeometrischen Fläche*. *Commentarii Mathematici Helvetici*, 3(1), 209–225.
- [21] Jacobi, C. G. J. (1841). De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum. *Journal für die reine und angewandte Mathematik*, 22, 360–371.
- [22] Kristály, A. (2014). Nash-type equilibria on Riemannian manifolds: a variational approach. *Journal de Mathématiques Pures et Appliquées*, 101(5), 660–688.
- [23] Lee, J. M. (2018). *Introduction to Riemannian manifolds* (2nd ed., Vol. 2). Springer.
- [24] Lefschetz, S. (1942). *Algebraic topology* (Colloquium Publications, Vol. 27). American Mathematical Society.
- [25] Lefschetz, S. (1949). *Introduction to topology*. Princeton University Press.
- [26] Levi-Civita, T. (1916). Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura riemanniana. *Rendiconti del Circolo Matematico di Palermo (1884–1940)*, 42(1), 173–204.
- [27] Lyapunov, A. M. (1892). *The general problem of the stability of motion*. *International Journal of Control*, 55(3), 531–534.
- [28] Marques, F. C., & Neves, A. (2014). *Min-max theory and the Willmore conjecture*. *Annals of Mathematics*, 179(2), 683–782.
- [29] Morse, M. (1925). Relations between the critical points of a real function of  $n$  independent variables. *Transactions of the American Mathematical Society*, 27, 345–396.
- [30] Nash, J. F. (1951). Non-cooperative games. *Annals of Mathematics*, 54(2), 286–295.
- [31] Noether, E. (1918). *Invariant variation problems*. *Transport Theory and Statistical Physics*, 1(3), 186–207.

- [32] O'Neill, B. (1983). *Semi-Riemannian Geometry with Applications to Relativity*. New York: Academic Press.
- [33] Pereira, D. H. (2018). *ESBSD: An Essay on the New Exponential Smoothing Methodology Applied to the Projection of the Population of Belo Horizonte*. Cambridge Open Engage.
- [34] Pereira, D. H. (2022). *Itamaracá: A Novel Simple Way to Generate Pseudo-random Numbers*. Cambridge Open Engage.
- [35] Pereira, D. H. (2025). *Victoria: Beating the House Using the Principles of Statistics and Randomness*. SSRN.
- [36] Pereira, D. H. (2025). *Riemannian Manifolds of Asymmetric Equilibria: The Victoria–Nash Geometry*. Cambridge Open Engage.
- [37] Pereira, D. H. (2026). *Nonlinear Victoria–Nash Equilibrium: Curvature and Stability in Asymmetric Systems*. SSRN.
- [38] Perelman, G. (2002). The entropy formula for the Ricci flow and its geometric applications. *arXiv:math/0211159*.
- [39] Petersen, P. (2006). *Riemannian geometry*. Springer.
- [40] Petersen, P. (2016). *Riemannian geometry*. Springer.
- [41] Rao, C. R. (1945). *Information and the accuracy attainable in the estimation of statistical parameters*. Bulletin of the Calcutta Mathematical Society, 37(3), 81–91.
- [42] Ricci, M. M. G., & Levi-Civita, T. (1900). Méthodes de calcul différentiel absolu et leurs applications. *Mathematische Annalen*, 54(1), 125–201.
- [43] Riemann, B., & Weyl, H. (1854). *Über die Hypothesen, welche der Geometrie zugrunde liegen*. Berlin, Heidelberg: Springer Berlin Heidelberg.
- [44] Smale, S. (1961). On gradient dynamical systems. *Annals of Mathematics*, 74(1), 199–206.
- [45] Schwarz, H. A. (1873). Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt. *Journal für die reine und angewandte Mathematik*, 75, 292–335.
- [46] Taylor, B. (1715). *Methodus incrementorum directa et inversa*. London.
- [47] Tikhonov, A. N. (1952). Systems of differential equations containing small parameters in the derivatives. *Matematicheskii Sbornik*, 73(3), 575–586.
- [48] von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100, 295–320.
- [49] Xie, Z., Wei, Y., Cao, H., Zhao, C., Deng, C., Li, J., ... Liang, W. (2025). mhc: Manifold-constrained hyper-connections. *arXiv preprint arXiv:2512.24880*.