A General Framework for Hierarchical Ranking of Linear Permutations

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Abstract: A generalized series-based formulation is developed to determine the hierarchical rank of any given linear permutation selected from a set of all possible linear permutations arranged according to a predefined order of priority of elements like digits, letters, and all other objects. The proposed model applies to permutations of words, numbers, and other discrete objects, enabling systematic identification of their positions in an ordered sequence. The formulation is expressed as a finite series in which each term corresponds to a specific element of the permutation. It applies to sets of distinguishable objects characterized by identifiable properties such as shape, size, colour, or surface pattern, assuming that all elements are equally likely to occupy any position in the arrangement without replacement. The model introduces three parameters, Formerity (F), Permuty (P), and Similarity (S), which collectively define the structure of the series. These parameters depend on the preceding elements, the permutations of successive elements, and the repetition characteristics within the arrangement. Notably, the number of terms in the series is equal to the number of elements in the permutation. This generalized formulation provides a structured and scalable approach for analyzing and ranking linear permutations in a wide range of combinatorial contexts.

Keywords: $Linear\ permutation$, $Rank\ (R), Formerity\ (F), Permuty\ (P)$ & $Similarity\ (S)$.

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1. Introduction

A number of linear arrangements are created by permuting a finite number of objects/articles that have at least one easily recognizable property, such as shape, size, color, surface pattern, or other observable property. Linear permutations arise from arranging a collection of distinguishable objects, each possessing at least one identifiable property such as shape, size, colour, surface design, etc. Common and intuitive examples of such permutations include alphabetic words and numerical sequences, which provide a convenient framework for understanding more general combinatorial arrangements. Due to their familiarity and structured ordering, permutations of letters and digits serve as fundamental models for analysing broader classes of linear permutations involving both algebraic and non-algebraic elements. It is well known that a set of words can be generated by permuting the letters, whether distinct, repeated, or a combination thereof, of a given word. When all such permutations are arranged in the correct alphabetic order, each word occupies a unique position, referred to as its hierarchical rank. However, determining the rank of a given word directly within the complete set of its permutations remains a non-trivial problem, particularly in the presence of repeated elements.

A similar situation arises in the context of numerical permutations. A large number of positive integers can be formed by permuting the digits, repetitive or non-repetitive, of a given number. When these numbers are arranged in increasing or decreasing order, each permutation is associated with a specific numerical rank. As with alphabetic permutations, identifying the exact position of a given number within this ordered set is computationally challenging. The problem of ranking permutations in lexicographic order has been widely studied in combinatorics and theoretical computer science. Classical approaches are based on the factorial number system (factoradics), which provides a systematic way to compute the rank of a permutation with distinct elements (Lehmer, 1960; Knuth, 1998). The concept of the Lehmer code establishes a direct correspondence between permutations and their lexicographic indices, enabling efficient ranking and unranking procedures. For permutations involving repeated elements (multisets), extensions of these methods have been developed using multinomial coefficients to account for indistinguishable arrangements (Stanley, 2011). In addition, algorithmic techniques for permutation ranking and generation have been extensively discussed in the literature, particularly in the context of combinatorial generation and discrete structures (Ruskey, 2003; Kreher and Stinson, 1998). Despite these advances, existing methods are often algorithmic in nature and may become computationally intensive or less intuitive when dealing with complex permutations involving repetitions and mixed constraints. The present work aims to complement these classical approaches by proposing a generalized series-based formulation that provides a more structured and potentially scalable framework for easily determining the hierarchical rank of linear permutations. This analytical formulation provides a systematic method for determining the correct hierarchical rank of any linear permutation, including the lexicographic order of words and the numerical order of digit-based permutations, irrespective of the presence of repeated or non-repeated elements. The approach is extendable to a wide class of linear arrangements and offers a unified framework for ranking permutations in ordered sequences.

2. HCR'S Rank Formula-I

Mathematically, for a given linear permutation having n no. of articles, its rank is given as follows

$$R(\text{linear pemutation}) = \sum_{i=1}^{n} F_i \left(\frac{P_i}{S_i}\right)$$

Where, $Formerity\ (F), Permuty\ (P)\ \&\ \ Similarity\ (S)$, which make composite terms of a finite series

We will study the randomly selected linear permutations to find out their respective ranks (correct positions) in the group of all linear permutations arranged in correct order, in the following order

1. Alphabetic words
2. Positive integral numbers with only non-zero digits
3. Positive integral numbers with both zero and non-zero digits
4. Linear permutations of certain articles which are similar and dissimilar in shape, size, colour, surface-design etc. which are equally significant to appear at all the positions in the linear arrangements.

2.1. Word series

A finite number of alphabetic words is obtained by permuting all the letters together in different linear sequences. All the words of a series have identical letters but in different sequences of letters.

A correct alphabetic order of all possible words obtained by permuting all the letters together of a given word is called **word series** of that word.

If a given word has n no. of the letters, out of which no. of the repetitive letters are

$p, q, r, s, \dots ..$ then total no. of words (N_w) obtained by permuting all the letters is given as

$$N_w = \frac{n!}{p!\,q!\,r!\,s!\dots}$$

N_w denotes the total no. of the words in word series of a given word.

Ex: Word 'TATA' has total no. of letters $n = 4$

No. of the repetitive letters, $p = 2$ (letter $'A'$) & $q = 2$ (letter $'T'$)

The total number of words is given as

$$N_w = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

The first & last words of the series are obtained by arranging the letters in certain alphabetic order as follows

First word $\rightarrow AATT$ (arranging all the letters in alphabetic order)

Last word $\rightarrow TTAA$ (arranging all the letters in reverse alphabetic order)

Thus, all the words can be arranged in a correct alphabetic order (rank) as follows

Word	**Alphabetic Order (Rank)**
AATT	1
ATAT	2
ATTA	3
TAAT	4
TATA*	5
TTAA	6

It's clear from the above word series that the given word TATA lies at fifth place in alphabetic order i.e. alphabetic order (rank) of TATA is 5 in its word series.

2.2. Alphabetical ranking of words

Each of the words has a certain alphabetic order in its word series. Thus

"Correct alphabetic order of a given word in its word series is called Rank of that word''. It is denoted by $'R(word)'$.

Before proceeding further, let's first know the terminology related to rank of a given word

Let all the letters of a given word, keeping similar (repetitive) ones together in a linear-sequence be arranged in the correct alphabetic order. Now select & label the letter, in the alphabetic arrangement, which is similar to the left-most letter in the given word. Now find the following parametric values of selected (labelled) letter

$\boldsymbol{Formerity\ (F)}$: $Formerity$ of the selected letter is the total no. of letters dissimilar to it & appearing before it (i.e. lying to the left of it) in the correct alphabetic order of all the letters.

$\boldsymbol{Similarity\ (S)}$: Similarity of the selected letter is the total no. of the letters similar to it, including itself, in the alphabetic order of all the letters.

$\boldsymbol{Permuty\ (P)}$: $Permuty$ of the selected letter is the total number of permutations obtained all the letters, excluding selected one, in the alphabetic order. After this we find the following complex value of selected one

$\boldsymbol{Permutation\ Value\ (P_V)}$: $Permutation\ value$ of any of the letters is given as follows

$$\Rightarrow\ P_V = F\left(\frac{P}{S}\right)$$

After finding this value of the selected letter, which is similar to the left-most letter of the given word, we cancel, eliminate or remove it from the alphabetic order.

Further, we select & label another letter, from the alphabetical order (after cancelling the previously selected one), which is similar to the next or second left-most letter in the given word. Find its parametric values by the above definitions & then parametric value. Then cancel it & select another letter similar to the third left-most in the given word. Thus follow this cancellation method & find the permutation values of all the letters in the given word.

Mathematically, rank of a given word is the sum of permutation values of its all the letters

If a given word has total 'n' no. of the letters then its rank (R) is given as

$$R(\text{Word}) = (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4 + \dots\dots\dots\dots\dots\dots\dots.. + (P_V)_{n-1} + (P_V)_n$$

$$\Rightarrow R(\text{word}) = \sum_{i=1}^{n} (P_V)_i$$

On setting the value of P_V in the above equation, we have

$$R(\text{Word}) = \sum_{i=1}^{n} \left(F\left(\frac{P}{S}\right)\right)_i = \sum_{i=1}^{n} F_i\left(\frac{P_i}{S_i}\right)$$

$$\Rightarrow \quad R(\text{Word}) = \sum_{i=1}^{n} F_i\left(\frac{P_i}{S_i}\right) \qquad \dots\dots(1)$$

Note: The above formula is named as HCR's Rank Formula**.** This formula is equally applicable all the linear permutations like words, numbers etc. For finding out the permutation value of any selected letter/digit in a given word, note the following points

1. Permutation value of each of the letters/digits is always non-negative integer
2. Permutation value of last letter/digit is always 1.
3. Similarity of any non-repetitive letter/digit is always 1.
4. Permutation value is zero iff $Formerity\ (F)$ of a selected letter/digit is zero
5. The values of $Similarity\ (S)$ & $Permuty\ (P)$ are always positive integers.

2.3. Working steps for the Method of cancellation for hierarchical ranking

Step 1: Arrange all the letters (repetitive & non-repetitive) of a given word in the correct alphabetic order placing the repetitive letters together in linear sequence.

Step 2: Find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ & thus permutation value of each of the letters of given word from first (left most) letter up to last (right most) letter using correct alphabetic order of letters.

Step 3: Remove the selected letter from the alphabetic order of which permutation value (P_V) has been found out, for the next letter of the given word. Find out the permutation value (P_V) of the next letter, and similarly remove it from alphabetic order.

Repeat this process until the last letter is left in alphabetic order, for which $P_V = 1$

Step 4: Add the permutation values of all the letters to find out the rank of that word.

Note: Select each letter, from left most, according to given (original) word & label the same letter in their alphabetic order I.e. All the letters are selected one-by-one from their alphabetic order according to the arrangement of letters in given (original) word and labelled in their alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of particular selected letter according to their definitions.

2.4. Illustrative examples on alphabetic ranking of words

Example 1: Let's consider the above word 'TATA' to find out its rank in its word series.

Step 1: There are four letters in 'TATA' which are arranged, keeping repetitive letters together, in correct alphabetic order as follows

$$A \rightarrow A \rightarrow T * \rightarrow T$$

Step 2: According to the given word 'TATA', the first letter is 'T'. Hence select letter 'T' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'T' of 'TATA' & thus permutation value as follows

$$\Rightarrow F_1 = \text{No. of letters dissimilar \& appearing before selected letter } 'T' \text{of } TATA \text{ in alphabetic order}$$

$$= 2 \quad \text{(two letters } 'A' \ \& \ 'A' \text{ are appearing before } 'T' \text{ which are dissimilar to } 'T')$$

$$\Rightarrow S_1 = \text{No. of letters similar to selected letter } 'T' \text{ of } TATA \text{ in above alphabetic order, including itself}$$

$$= 1 + 1 = 2 \quad \text{(there is only one letter } 'T' \text{ which issimilar to selected letter } 'T')$$

$$\Rightarrow P_1 = \text{No. of permutations obtained from remaining letters } 'A', \ 'A' \ \& \ 'T'$$

$$\text{(excluding selected } 'T' \text{ of } TATA \text{) in above alphabetic order}$$

$$= \frac{3!}{2!} = \frac{6}{2} = 3$$

$$\therefore \text{Permutation value of first selected letter } 'T', \ (P_V)_1 = F_1\left(\frac{P_1}{S_1}\right) = 3\left(\frac{2}{2}\right) = 3$$

Step 3: Since, the permutation value of one of the letters 'T' has been determined thus remove it from alphabetic order. Hence, the alphabetic order of remaining letters is

$$A * \rightarrow A \rightarrow T$$

According to the given word 'TATA', second (next) letter is 'A'. Hence select letter 'A' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'A' of 'TATA' & thus permutation value as follows

$$\Rightarrow F_2 = \text{No. of letters dissimilar \& appearing before selected letter } 'A' \text{of } TATA \text{ in alphabetic order}$$

$$= 0 \quad \text{(There is no letter dissimilar to } 'A' \text{ \& appearing before } 'A')$$

$$\therefore \text{Permutation value of selected letter } 'A', \ (P_V)_2 = F_2\left(\frac{P_2}{S_2}\right) = 0$$

(In this case the values of S & P need not be determined since $F = 0$)

Step 4: Since the permutation value of one of the letters 'A' has been determined, remove it from the above alphabetical order. Hence, alphabetic order of remaining letters is

$$A \rightarrow T *$$

According to given word 'TATA', third (next) letter is 'T'. Hence select letter 'T' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'T' of 'TATA' & thus permutation value as follows

$$\Rightarrow F_3 = \text{No. of dissimilar letters appearing before selected letter } 'T' \text{ of } TATA \text{ in alphabetic order}$$

$$= 1 \qquad \text{(there is only one letter } 'A' \text{ dissimilar \& appearing before } 'T')$$

$$\Rightarrow S_3 = \text{No. of letters similar to selected letter } 'T' \text{ of } TATA \text{ in above alphabetic order, including itself}$$

$$= 1 \qquad \text{(there is only one letter } 'T' \text{ which is similar to itself)}$$

$$\Rightarrow P_3 = \text{No. of permutations obtained from remaining letters that is } 'A'$$

$$\text{(excluding selected } 'T' \text{ of } TATA \text{) in above alphabetic order}$$

$$= 1! = 1$$

$$\therefore \text{Permutation value of selected letter } 'T', \ (P_V)_3 = F_3\left(\frac{P_3}{S_3}\right) = 1\left(\frac{1}{1}\right) = 1$$

Step 5: Since the permutation value of selected letter 'T' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$A *$$

Since, 'A' is the fourth & last letter (as labelled) of the word TATA hence its permutation value is 1

$$\Rightarrow (P_V)_4 = 1$$

Thus, rank of 'TATA' is given as follows

$$\Rightarrow R(TATA) = (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4$$

$$= 3 + 0 + 1 + 1 = 5$$

Similarly, rank of TAAT can be determined following the above procedure as follows

Arrange all the letters in alphabetic order as follows

$$A \to A \to T \to T$$

Now, according to the given word 'TAAT' select letter one by one from left to find their respective permutation values & add all the values as follows

$$\Rightarrow R(TAAT) = (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right)$$

$$= 2\left(\frac{\left(\frac{3!}{2!}\right)}{2}\right) + 0\left(\frac{2!}{2}\right) + 0\left(\frac{1!}{1}\right) + 1 = 3 + 1 = 4$$

Similarly, rank of 'TTAA'

$$\Rightarrow R(TTAA) = (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right)$$

$$= 2\left(\frac{\left(\frac{3!}{2!}\right)}{2}\right) + 2\left(\frac{\left(\frac{2!}{2!}\right)}{1}\right) + 0\left(\frac{1!}{1}\right) + 1 = 3 + 2 + 1 = 6$$

It is obvious that the above formula is extremely useful for finding out the ranks of words having a greater number of letters (repetitive or non-repetitive), also for finding the ranks of numbers having a greater number of digits.

Example 2: Let's find out the rank of word '**DISSOCIATE**' in its word series.

Step 1: There total ten letters in 'DISSOCIATE' which are arranged, keeping repetitive letters together, in correct alphabetic order as follows

$$A \to C \to D * \to E \to I \to I \to O \to S \to S \to T$$

Step 2: According to given word 'DISSOCIATE', first letter is 'D'. Hence select letter 'D' (as labelled) from alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'D' of 'DISSOCIATE' & thus permutation value as follows

$$\Rightarrow F_1 = \text{No. of dissimilar letters appearing before first letter } 'D' \text{of } DISSOCIATE \text{ in alphabetic order}$$

$$= 2 \quad \text{(two letters } 'A' \& \ 'C' \text{ are appearing before } 'D' \text{ which are dissimilar to } 'D')$$

$$\Rightarrow S_1 = \text{No. of letters similar to first selected letter } 'D' \text{ of } DISSOCIATE \text{ in alphabetic order,}$$

$$\text{including itself}$$

$$= 1 \qquad \text{(there is only one letter } 'D' \text{ which issimilar to itself } 'D')$$

$$\Rightarrow P_1 = \text{No. of permutations obtained from remaining letters in alphabetic order}$$

$$\text{which are } A, C, E, I, I, O, S, S, T$$

$$\text{(excluding selected } 'D' \ of \ DISSOCIATE \text{) in alphabetic order}$$

$$= \frac{9!}{2!\,2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = \frac{362880}{4} = 90720$$

$$\therefore \text{Permutation value of first selected letter } 'T', \ (P_V)_1 = F_1\left(\frac{P_1}{S_1}\right) = 2\left(\frac{90720}{1}\right) = 181440$$

Step 3: Since, the permutation value of the selected letter 'D' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$A \to C \to E \to I * \to I \to O \to S \to S \to T$$

Now, according to given word 'DISSOCIATE', second (next) letter is 'I'. Hence select letter 'I' (as labelled) from alphabet order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'I' of 'DISSOCIATE' & thus permutation value as follows

$$\Rightarrow F_2 = \text{No. of dissimilar letters appearing before selected letter } 'I' \text{ in alphabetic order}$$

$$= 3 \quad (\text{there are three letters } A, C, \& \ E \ d\text{issimilar to } 'I' \ \& \ appearing\ before\ \ 'I')$$

$$\Rightarrow S_2 = \text{No. of letters similar to selected letter } 'I', \text{from } DISSOCIATE, \text{in above alphabetic order,}$$

$$i\text{ncluding itself (labelled } 'I')$$

$$= 2 \qquad (\text{there are two letters } I, I \text{ including itself } (selected\ I)\ )$$

$$\Rightarrow P_2 = \text{No. of permutations obtained from remaining letters in alphabetic order}$$

$$\text{which are } A, C, E, I, O, S, S, T$$

$$(\text{ excluding selected letter } 'I' \text{) in above alphabetic order}$$

$$= \frac{8!}{2!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{40320}{2} = 20160$$

$$\therefore \text{ Permutation value of second selected letter } 'A', \ \ (P_V)_2 = F_2\left(\frac{P_2}{S_2}\right) = 3\left(\frac{20160}{2}\right) = 30240$$

Step 4: Since, the permutation value of one of the letters 'I' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$A \to C \to E \to I \to O \to S * \to S \to T$$

According to given word 'DISSOCIATE', third (next) letter is 'S'. Hence select letter 'S' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'I' of 'DISSOCIATE' & thus permutation value as follows

$$\Rightarrow F_3 = \text{No. of letters dissimilar \& appearing before selected letter } 'S' \text{of } DISSOCIATE$$

$$\text{in above alphabetic order}$$

$$= 5 \quad (\text{there are five letters } A, C, E, I, O \ \text{ dissimilar \& appearing before labelled } 'S')$$

$$\Rightarrow S_3 = \text{No. of letters similar to selected letter } 'S' \text{ in above alphabetic order,}$$

$$\text{including itself}$$

$$= 2 \qquad (\text{there are two similar letters } S, S \text{ including itself})$$

$$\Rightarrow P_3 = \text{No. of permutations obtained from remaining letters } A, C, E, I, O, S, T$$

$$(\text{ excluding selected } '\mathrm{S}' \text{) in above alphabetic order}$$

$$= 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

$$\therefore \text{ Permutation value of selected letter } 'S', \ \ (P_V)_3 = F_3\left(\frac{P_3}{S_3}\right) = 5\left(\frac{5040}{2}\right) = 12600$$

Step 5: Since the permutation value of selected letter 'S' has been determined thus remove it i.e. one of the repetitive letters 'S' from alphabetic order. Hence, alphabetic order of remaining letters is

$$A \to C \to E \to I \to O \to S * \to T$$

According to given word 'DISSOCIATE', next letter is 'S'. Hence select letter 'S' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'S' of 'DISSOCIATE' & thus permutation value as follows

$$\Rightarrow F_4 = \text{No. of letters dissimilar \& appearing before selected letter } 'S' \text{of } DISSOCIATE$$

$$\text{in above alphabetic order}$$

$$= 5 \quad (\text{there are five letters } A, C, E, I, O \text{ dissimilar \& appearing before labelled } 'S')$$

$$\Rightarrow S_4 = \text{No. of letters similar to selected letter } 'S' \text{ in above alphabetic order, including itself}$$

$$= 1 \qquad (\text{there is only one letter } S \text{ similar to itself})$$

$$\Rightarrow P_4 = \text{No. of permutations obtained from remaining letters } A, C, E, I, O, T$$

$$(\text{ excluding selected } 'S' \text{) in above alphabetic order}$$

$$= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

$$\therefore \text{Permutation value of selected letter } 'S', \ (P_V)_4 = F_4\left(\frac{P_4}{S_4}\right) = 5\left(\frac{720}{1}\right) = 3600$$

Step 6: Since the permutation value of selected letter 'S' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$A \to C \to E \to I \to O * \to T$$

According to given word 'DISSOCIATE', next letter is 'O'. Hence select letter 'O' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'O' of 'DISSOCIATE' & thus permutation value as follows

$$\Rightarrow F_5 = \text{No. of letters dissimilar \& appearing before selected letter } 'S' \text{of } DISSOCIATE$$

$$\text{in above alphabetic order}$$

$$= 4 \quad (\text{there are four letters } A, C, E, I \text{ dissimilar \& apearing before labelled } 'O')$$

$$\Rightarrow S_5 = \text{No. of letters similar to selected letter } '\text{O}' \text{ in above alphabetic order, including itself}$$

$$= 1 \qquad (\text{there is only one letter } O \text{ similar to itself})$$

$$\Rightarrow P_5 = \text{No. of permutations obtained from remaining letters } A, C, E, I, T$$

$$(\text{ excluding selected } 'O' \text{) in above alphabetic order}$$

$$= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$\therefore \text{Permutation value of selected letter } 'O', \ (P_V)_5 = F_5\left(\frac{P_5}{S_5}\right) = 4\left(\frac{120}{1}\right) = 480$$

Step 7: Since the permutation value of selected letter 'O' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$A \rightarrow C *\rightarrow E \rightarrow I \rightarrow T$$

According to given word 'DISSOCIATE', next letter is 'C'. Hence select letter 'C' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'C' of 'DISSOCIATE' & thus permutation value as follows

$\Rightarrow F_6$
$= \text{No. of letters dissimilar \& appearing before selected letter } 'C' \text{of } DISSOCIATE \text{ in above alphabetic order}$

$$= 1 \quad (\text{there is only one letter } A \text{ dissimilar \& apearing before labelled } 'C')$$

$$\Rightarrow S_6 = \text{No. of letters similar to selected letter } 'C' \text{ in above alphabetic order, including itself}$$

$$= 1 \qquad (\text{there is only one letter } C \text{ similar to itself})$$

$\Rightarrow P_6$
$= \text{No. of permutations obtained from remaining letters } A, E, I, T \text{ (excluding selected } 'C' \text{) in above alphabetic order}$

$$= 4! = 4 \times 3 \times 2 \times 1 = 24$$

$$\therefore \text{ Permutation value of selected letter } 'C', \ (P_V)_6 = F_6\left(\frac{P_6}{S_6}\right) = 1\left(\frac{24}{1}\right) = 24$$

Step 8: Since the permutation value of selected letter 'C' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$A \rightarrow E \rightarrow I *\rightarrow T$$

According to given word 'DISSOCIATE', next letter is 'I'. Hence select letter 'I' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'I' of 'DISSOCIATE' & thus permutation value as follows

$\Rightarrow F_7$
$= \text{No. of letters dissimilar \& appearing before selected letter } 'I' \text{of } DISSOCIATE \text{ in above alphabetic order}$

$$= 2 \quad (\text{there are two letter } A, E \text{ dissimilar \& apearing before labelled } 'I')$$

$$\Rightarrow S_7 = \text{No. of letters similar to selected letter } 'I' \text{ in above alphabetic order, including itself}$$

$$= 1 \qquad (\text{there is only one letter } I \text{ similar to itself})$$

$\Rightarrow P_7$
$= \text{No. of permutations obtained from remaining letters } A, E, T \text{ (excluding selected } 'I' \text{) in above alphabetic order}$

$$= 3! = 3 \times 2 \times 1 = 6$$

$$\therefore \text{ Permutation value of selected letter } 'C', \ (P_V)_7 = F_7\left(\frac{P_7}{S_7}\right) = 2\left(\frac{6}{1}\right) = 12$$

Step 9: Since the permutation value of selected letter 'I' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$A * \rightarrow E \rightarrow T$$

According to given word 'DISSOCIATE', next letter is 'A'. Hence select letter 'A' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'A' of 'DISSOCIATE' & thus permutation value as follows

$$\Rightarrow F_8$$
$$= \text{No. of letters dissimilar \& appearing before selected letter } 'A' of\ DISSOCIATE \text{ in above alphabetic order}$$

$$= 0 \quad (\text{there is no letter dissimilar \& apearing before labelled } 'A')$$

$$\therefore \text{ Permutation value of selected letter } 'A', \ (P_V)_8 = F_8\left(\frac{P_8}{S_8}\right) = 0\left(\frac{P_8}{S_8}\right) = 0$$

Step 10: Since the permutation value of selected letter 'A' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$E \rightarrow T *$$

According to given word 'DISSOCIATE', next letter is 'T'. Hence select letter 'T' (as labelled) from above alphabetic order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected letter 'T' of 'DISSOCIATE' & thus permutation value as follows

$$\Rightarrow F_9 = \text{No. of letters dissimilar \& appearing before selected letter } 'T' \text{of } DISSOCIATE$$
$$\text{in above alphabetic order}$$

$$= 1 \quad (\text{there is only one letter } E \text{ dissimilar \& apearing before labelled } 'T')$$

$$\Rightarrow S_9 = \text{No. of letters similar to selected letter } 'T' \text{ in above alphabetic order, including itself}$$

$$= 1 \qquad (\text{there is only one letter } T \text{ similar to itself})$$

$$\Rightarrow P_9 = \text{No. of permutations obtained from remaining letters } 'E'$$
$$(\text{ excluding selected } 'T' \text{) in above alphabetic order}$$
$$= 1! = 1$$

$$\therefore \text{ Permutation value of selected letter } 'T', \ (P_V)_9 = F_9\left(\frac{P_9}{S_9}\right) = 1\left(\frac{1}{1}\right) = 1$$

Step 11: Since the permutation value of selected letter 'T' has been determined thus remove it from alphabetic order. Hence, alphabetic order of remaining letters is

$$E *$$

According to given word 'DISSOCIATE', next & last letter is 'E'. Hence permutation value of last letter 'E' of given word 'DISSOCIATE'

$$\therefore \text{ Permutation value of last letter } 'E', \ (P_V)_{10} = 1$$

Thus, rank (R) of word 'DISSOCIATE' is given as follows

$$\Rightarrow R(DISSOCIATE)$$

$$= (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4 + (P_V)_5 + (P_V)_6 + (P_V)_7 + (P_V)_8 + (P_V)_9 + (P_V)_{10}$$

$$= 181440 + 30240 + 12600 + 3600 + 480 + 24 + 12 + 0 + 1 + 1 = 228398$$

Total no. of the words in the word series of 'DISSOCIATE' is given as

$$= \frac{10!}{2!\,2!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 907200$$

First & last word of series can be obtained as follows

First word → $ACDEIIOSST$ (by arranging in correct alphabetic order)

Last word → $TSSOIIEDCA$ (by arranging in reverse alphabetic order)

Whole series of the word 'DISSOCIATE' along with its rank (R=228398) will be as follows

WORD	RANK (R)
ACDEIIOSST	1
ACDEIIOSTS	2
ACDEIIOTSS	3
ACDEIISOST	4
ACDEIISOTS	5
…………………. …………………. ………………….	………………. ………………. ……………….
DISSOCIAET	228397
DISSOCIATE*	228398*
DISSOCIEAT	228399
…………………. …………………. ………………….	………………. ………………. ……………….
TSSOIIEADC	907196
TSSOIIECAD	907197
TSSOIIECDA	907198
TSSOIIEDAC	907199
TSSOIIEDCA	907200

In the above example, the steps are very long to explain but these can be performed in a single step of calculations. Let's find out the ranks of other words from above word series

Rank of 'TSSOIIEADC' is determined as follows

Arrange all the letters of 'TSSOIIEADC' in the correct alphabetic order as follows

$$A \to C \to D \to E \to I \to I \to O \to S \to S \to T$$

Now, select & remove the letters one by one from alphabetic order to find rank as follows

$$R(TSSOIIEADC) = 9\left(\frac{\left(\frac{9!}{2!\,2!}\right)}{1}\right) + 7\left(\frac{\left(\frac{8!}{2!}\right)}{2}\right) + 7\left(\frac{\left(\frac{7!}{2!}\right)}{1}\right) + 6\left(\frac{\left(\frac{6!}{2!}\right)}{1}\right) + 4\left(\frac{5!}{2}\right) + 4\left(\frac{4!}{1}\right)$$

$$+3\left(\frac{3!}{1}\right)+0\left(\frac{2!}{1}\right)+1\left(\frac{1!}{1}\right)+1$$

$$=816480+70560+17640+2160+240+96+18+0+1+1=907196$$

Above result is correct from the table of word series.

Similarly, rank of ‘TSSOIIEDAC’ can be determined using above alphabetic order as follows

$$\Rightarrow\ R(TSSOIIEDAC)=9\left(\frac{\left(\frac{9!}{2!\,2!}\right)}{1}\right)+7\left(\frac{\left(\frac{8!}{2!}\right)}{2}\right)+7\left(\frac{\left(\frac{7!}{2!}\right)}{1}\right)+6\left(\frac{\left(\frac{6!}{2!}\right)}{1}\right)+4\left(\frac{5!}{2}\right)+4\left(\frac{4!}{1}\right)$$

$$+3\left(\frac{3!}{1}\right)+2\left(\frac{2!}{1}\right)+0\left(\frac{1!}{1}\right)+1$$

$$=816480+70560+17640+2160+240+96+18+4+0+1=907199$$

Above result is correct from the table of word series.

Similarly, rank of last word ‘TSSOIIEDAC’ can be determined using above alphabetic order as follows

$$\Rightarrow\ R(TSSOIIEDCA)=9\left(\frac{\left(\frac{9!}{2!\,2!}\right)}{1}\right)+7\left(\frac{\left(\frac{8!}{2!}\right)}{2}\right)+7\left(\frac{\left(\frac{7!}{2!}\right)}{1}\right)+6\left(\frac{\left(\frac{6!}{2!}\right)}{1}\right)+4\left(\frac{5!}{2}\right)+4\left(\frac{4!}{1}\right)$$

$$+3\left(\frac{3!}{1}\right)+2\left(\frac{2!}{1}\right)+1\left(\frac{1!}{1}\right)+1$$

$$=816480+70560+17640+2160+240+96+18+4+1+1=907200$$

The above result is correct from the table of word series.

Note: Rank of the last word also denotes the total no. of words in the set of all possible permutations.

2.5. Direct applications of HCR’s Rank Formula-I

Problem 1: Find out alphabetic order of word ‘**CALCULUS**’ in the group of the words obtained by permuting all the letters together.

Sol.

Arrange all the letters of ‘CALCULUS’ in the correct alphabetic order, keeping repetitive letters together, as follows

$$A\to C\to C\to L\to L\to U\to U\to S$$

Now, select & remove the letters one by one from alphabetic order to find out values of $Formerity, Permuty$ & $Similarity$ of each letter & add their permutation values together (following the same procedure as mentioned in above illustrative examples)

Using HCR’s Rank or Series Formula as follows

$R(CALCULUS)$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right) + F_8\left(\frac{P_8}{S_8}\right)$$

$$= 1\left(\frac{\left(\frac{7!}{2!\,2!}\right)}{2}\right) + 0\left(\frac{\left(\frac{6!}{2!\,2!}\right)}{1}\right) + 1\left(\frac{\left(\frac{5!}{2!}\right)}{2}\right) + 0\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right) + 1\left(\frac{3!}{2}\right) + 0\left(\frac{2!}{1}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 630 + 0 + 30 + 0 + 3 + 0 + 0 + 1 = 664$$

While total no. of the words in the group (Word Series) is given as follows

$$N_N = \frac{8!}{2!\,2!\,2!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 2 \times 1} = 5040$$

Thus alphabetic order of 'CALCULUS' is 664 out of 5040 words.

Problem 2: Find out alphabetic order of word '**GEOMETRY**' in the group of the words obtained by permuting all the letters together.

Sol.

Arrange all the letters of 'GEOMETRY' in the correct alphabetic order, keeping repetitive letters together, as follows

$$E \to E \to G \to M \to O \to R \to T \to Y$$

Now, select & remove the letters one by one from alphabetic order to find out values of $Formerity, Permuty$ & $Similarity$ of each letter & add their permutation values together

Using HCR's Rank or Series Formula as follows

$$R(GEOMETRY)$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right) + F_8\left(\frac{P_8}{S_8}\right)$$

$$= 2\left(\frac{\left(\frac{7!}{2!}\right)}{1}\right) + 0\left(\frac{6!}{2}\right) + 2\left(\frac{5!}{1}\right) + 1\left(\frac{4!}{1}\right) + 0\left(\frac{3!}{1}\right) + 1\left(\frac{2!}{1}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 5040 + 0 + 240 + 24 + 0 + 2 + 0 + 1 = 5307$$

While total no. of the words in the group (Word Series) is given as follows

$$N_N = \frac{8!}{2!\,2!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 10080$$

Thus alphabetic order of 'GEOMETRY' is 5307 out of 10080 words.

Problem 3: Find out alphabetic order of word '**MATHEMATICS**' in the group of the words obtained by permuting all the letters together.

Sol.

Arrange all the letters of 'MATHEMATICS' in the correct alphabetic order, keeping repetitive letters together, as follows

$$A \to A \to C \to E \to H \to I \to M \to M \to S \to T \to T$$

Now, select & remove the letters one by one from alphabetic order to find out values of $Formerity, Permuty$ & $Similarity$ of each letter & add their permutation values together

Using HCR's Rank or Series Formula as follows

$$R(MATHEMATICS)$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right) + F_8\left(\frac{P_8}{S_8}\right) + F_9\left(\frac{P_9}{S_9}\right) + F_{10}\left(\frac{P_{10}}{S_{10}}\right) + F_{11}\left(\frac{P_{11}}{S_{11}}\right)$$

$$= 6\left(\frac{\left(\frac{10!}{2!\,2!}\right)}{2}\right) + 0\left(\frac{\left(\frac{9!}{2!}\right)}{2}\right) + 7\left(\frac{8!}{2}\right) + 3\left(\frac{7!}{1}\right) + 2\left(\frac{6!}{1}\right) + 3\left(\frac{5!}{1}\right) + 0\left(\frac{4!}{1}\right) + 3\left(\frac{3!}{1}\right) + 1\left(\frac{2!}{1}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 2721600 + 0 + 141120 + 15120 + 1440 + 360 + 0 + 18 + 2 + 0 + 1 = 2879661$$

While total no. of the words in the group (Word Series) is given as follows

$$N_N = \frac{11!}{2!\,2!\,2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 2 \times 1} = 4989600$$

Thus, alphabetic order of 'MATHEMATICS' is 2879661 out of 4989600 words.

3. Positive integer numbers with non-zero digits

3.1. Number series

A finite number of numbers can be obtained by permuting all the non-zero digits of a given number. All these numbers can be arranged or grouped in increasing or decreasing order.

Thus, a group or series or set of all possible numbers, arranged in increasing or decreasing order, obtained by permuting all the non-zero digits of a given number altogether, is called **Number series**.

If a number has 'n' no. of non-zero digits out of which no. of repetitive digits are $p, q, r, s, \dots$ …. Then the total no. of the numbers formed

$$N_N = \frac{n!}{p!\,q!\,r!\,s!\dots}$$

N_w denotes the total no. of the numbers formed in the Number Series.

3.2. Ranking of positive integer numbers

A given number has a certain increasing or decreasing order no. in its series that is called Rank of given number. It is denoted by symbol $R(\text{Number})$.

Rank (R) of a given number having 'n' no. of non-zero digits is given by HCR's Rank Formula-I which is used for the words.

$$\Rightarrow \left[\quad R(\text{Number}) = \sum_{i=1}^{n} F_i \left(\frac{P_i}{S_i}\right) \quad \right] \qquad \dots\dots\dots\dots\dots\dots\dots..(I)$$

Note: All the symbols have their usual meanings as in case of word series**.** If the rank of a given number is taken in increasing order it is denoted by symbol "$R(\text{number} \uparrow)$" $[\,'\uparrow'\ \text{signifies increasing order}]$

Similarly, if the rank of a given number is taken in decreasing order it is denoted by the symbol "$R(number \downarrow)$" $[\,'\downarrow'\ \text{signifies decreasing order}]$

3.2.1. Working steps for ranking

Step 1: Arrange all the non-zero digits, keeping repetitive digits together, in increasing or decreasing order according to the requirement/problem.

Step 2: Find the permutation value of each of the digits of the given number using required order of digits by following the same procedure as mentioned above for a given word.

Add the permutation values of all the digits to find rank of the given number in increasing or decreasing order.

Note: Select each digit, from left most, according to given (original) number & label the same digit in their numeric order I.e. All the digits are selected one-by-one from their numeric order (increasing or decreasing) according to the arrangement of digits in the given (original) number and labelled in numeric order to find values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of particular selected digit according to their definitions.

For avoiding the difficulty in finding rank in decreasing order let's co-relate them

Ranks of a given number in increasing and decreasing orders can be correlated as

$$R(number \downarrow) = N_N - R(number \uparrow) + 1$$

Where, N_N denotes total no. of the numbers formed in the series of a given number.

It is easier to find out the rank in increasing order rather than in decreasing order following the same procedure as followed for a given word.

3.2.2. Ranking of positive integer numbers with non-zero digits

Example 1: Let's find out the rank of a number 5252 in its series in increasing order

Step 1: Arrange all the non-zero digits of 5252, keeping repetitive digits together, in increasing order as follows

$$2 \rightarrow 2 \rightarrow 5 * \rightarrow 5$$

Step 2: According to given (original) number '5252', first digit is '5'. Hence select letter '5' (as labelled) from above increasing order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected digit '5' of '5252' & thus permutation value as follows

$$\Rightarrow F_1 = \text{No. of digits dissimilar \& appearing before selected digit } '5' \text{of 5252 in increasing order}$$

$$= 2 \qquad (\text{two digits } '2' \& \ '2' \text{ are appearing before } '5' \text{ which are dissimilar to } '5')$$

$$\Rightarrow S_1 = \text{No. of digits similar to selected digit } '5' \text{ of 5252 in above increasing order,}$$

including itself

$= 1 + 1 = 2$ (there are two digits 5, 5 similar to each other including labelled '5')

$\Rightarrow P_1 =$ No. of permutations obtained from remaining digits '2', '2' & '5'

(excluding labelled/selected '5') in above numeric order

$$= \frac{3!}{2!} = \frac{6}{2} = 3$$

$$\therefore \text{ Permutation value of first selected digit } '5', \ (P_V)_1 = F_1\left(\frac{P_1}{S_1}\right) = 3\left(\frac{2}{2}\right) = 3$$

Step 3: Since, the permutation value the selected digit '5' has been determined thus remove it from above numeric order. Hence, increasing numeric order of remaining digits is

$$2 * \rightarrow 2 \rightarrow 5$$

According to given number '5252', second (next) digit is '2'. Hence select digit '2' (as labelled) from above numeric order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected digit '2' of '5252' & thus permutation value as follows

$\Rightarrow F_2 =$ No. of digits dissimilar & appearing before selected digit '2' of 5252 in alphabetic order

$= 0$ (there is no digit dissimilar & appearing before '2')

$$\therefore \text{ permutation value of selected digit } '2', \ (P_V)_2 = F_2\left(\frac{P_2}{S_2}\right) = 0$$

(In this case the values of S & P need not be determined since $F = 0$)

Step 4: Since, the permutation value of the selected digit '2' has been determined thus remove it from above increasing order. Hence, numeric order of remaining digits is

$$2 \rightarrow 5 *$$

According to given number '5252', third (next) digit is '5'. Hence select digit '5' (as labelled) from above numeric order to find out the values of $Formerity\ (F)$, $Similarity\ (S)$ & $Permuty\ (P)$ of selected digit '5' of '5252' & thus permutation value as follows

$\Rightarrow F_3 =$ No. of dissimilar digits appearing before selected digit '5' of 5252 in above order

$= 1$ (there is only one digit '2' dissimilar & appearing before '5')

$\Rightarrow S_3 =$ No. of digits similar to selected digit '5' from 5252 in above numeric order,

including itself

$= 1$ (there is only one digit '5' which is similar to itself)

$\Rightarrow P_3 =$ Number of permutations obtained from remaining digits that is only '2'

(excluding selected digit '5') in above numeric order

$$= 1! = 1$$

$$\therefore \text{Permutation value of selected digit } '5', \ (P_V)_3 = F_3\left(\frac{P_3}{S_3}\right)$$

$$= 1\left(\frac{1}{1}\right) = 1$$

Step 5: Since the permutation value of selected digit '5' has been determined thus remove it from above numeric order. Hence, numeric order of remaining digits is

$$2 *$$

Since, '2' is the fourth & last digit (as labelled) of the number 5252 hence its permutation value is 1

$$\Rightarrow \ (P_V)_4 = 1$$

Thus, rank of '5252' in increasing order is given as follows

$$\Rightarrow R(5252 \uparrow) = (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4$$

$$= 3 + 0 + 1 + 1 = 5$$

Given number '5252' has total no. of digits $n = 4$

No. of the repetitive digits, $p = 2\ (digit\ '2')$ & $q = 2\ (digit\ '5')$

Total no. of the numbers formed is given as

$$N_N = \frac{4!}{2!\,2!} = \frac{24}{2 \times 2} = 6$$

The first & last numbers of the series are obtained by arranging all the digits in certain numeric order as follows

First number → 2255 (arranging all the digits in increasing order)

Last number → 5522 (arranging all the digits in decreasing order)

Thus, all the numbers can be arranged in increasing/decreasing numeric order as follows

Number	**Rank in Increasing Order** (↑)	**Rank in decreasing Order** (↓)
2255	1	6
2525	2	5
2552	3	4
5225	4	3
5252	5	2
5522	6	1

Similarly, rank of '2552' in increasing order can be determine as follows

Arrange all the digits, keeping repetitive digits together, in increasing order as follows

$$2 \rightarrow 2 \rightarrow 5 \rightarrow 5$$

$$\Rightarrow\ R(5225\uparrow) = (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right)$$

$$= 2\left(\frac{\left(\frac{3!}{2!}\right)}{2}\right) + 0\left(\frac{2!}{2}\right) + 0\left(\frac{1!}{1}\right) + 1 = 3 + 0 + 0 + 1 = 4$$

$$\Rightarrow\ R(5225\downarrow) = N_N - R(5225\uparrow) + 1 = \frac{4!}{2!\,2!} - 4 + 1 = 6 - 4 + 1 = 3$$

Similarly, rank of '2552' in increasing order can be determine as follows

Arrange all the digits, keeping repetitive digits together, in increasing order as follows

$$2 \to 2 \to 5 \to 5$$

Select all the digits one by one from above increasing order, according to given (original) number 5225, to find permutation values & then add them to find out the rank of given number

$$\Rightarrow\ R(5522\uparrow) = (P_V)_1 + (P_V)_2 + (P_V)_3 + (P_V)_4$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right)$$

$$= 2\left(\frac{\left(\frac{3!}{2!}\right)}{2}\right) + 2\left(\frac{2!}{2}\right) + 0\left(\frac{1!}{1}\right) + 1 = 3 + 2 + 0 + 1 = 6$$

$$\Rightarrow\ R(5522\downarrow) = N_N - R(5522\uparrow) + 1 = \frac{4!}{2!\,2!} - 6 + 1 = 6 - 6 + 1 = 1$$

Note: By following the above procedure, rank of any number having non-zero digits (repetitive or non-repetitive) can be determined.

Example 2: Let's take the following example of non-zero digits 2, 3, 3, 6, 6, 8, 9. Find out the rank of the number '6823369' in increasing and decreasing order.

Sol. Arrange all the digits, keeping repetitive digits together, in increasing order as follows

$$2 \to 3 \to 3 \to 6 \to 6 \to 8 \to 9$$

Now, select all the digits, according to 6823369, from above numeric order to find out permutation value of each of selected digits. Add permutation values of all the digits to find rank in increasing order as follows

$$\Rightarrow\ R(6823369\uparrow) = F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right)$$

$$= 3\left(\frac{\left(\frac{6!}{2!}\right)}{2}\right) + 4\left(\frac{\left(\frac{5!}{2!}\right)}{1}\right) + 0\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right) + 0\left(\frac{3!}{2}\right) + 0\left(\frac{2!}{1}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 540 + 240 + 0 + 0 + 0 + 0 + 1 = 781$$

$$\Rightarrow\ R(6823369\downarrow) = N_N - R(6823369\uparrow) + 1 = \frac{7!}{2!\,2!} - 781 + 1$$

$$= 1260 - 781 + 1 = 480$$

While total no. of the numbers (permutations) formed is given as

$$N_N = \frac{7!}{2!\,2!} = \frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times2\times1} = 1260$$

The first & last numbers of the series are obtained by arranging all the digits in certain numeric order as follows

First number → 2336689 (arranging all the digits in increasing order)

Last number → 9866332 (arranging all the digits in decreasing order)

Thus, all the numbers can be arranged in increasing/decreasing numeric orders as follows

The ranks of given number 6823369 has been labelled (*) in the number series below

Number	**Rank in Increasing Order**(↑)	**Rank in Decreasing Order**(↓)
2336689	1	1260
2336698	2	1259
2336869	3	1258
2336896	4	1257
2336968	5	1256
………….. ………….. …………..	……………. ……………. …………….	……………… ……………… ………………
6698332	780	481
6823369*	781*	480*
6823396	782	479
………….. ………….. …………..	……………. ……………. …………….	…………….. …………….. ……………..
9863623	1256	5
9863632	1257	4
9866233	1258	3
9866323	1259	2
9866332	1260	1

Similarly, ranks of the number 6698332 can be found out as follows

Arrange all the digits, keeping repetitive digits together, in increasing order as follows

$$2\to3\to3\to6\to6\to8\to9$$

Now, select all the digits, according to 6698332, from above numeric order to find out permutation value of each of selected digits. Add permutation values of all the digits to find rank in increasing order as follows

$$\Rightarrow\ R(6698332\uparrow) = F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right)$$

$$= 3\left(\frac{\left(\frac{6!}{2!}\right)}{2}\right) + 3\left(\frac{\left(\frac{5!}{2!}\right)}{1}\right) + 4\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right) + 3\left(\frac{\left(\frac{3!}{2!}\right)}{1}\right) + 1\left(\frac{2!}{2}\right) + 1\left(\frac{1!}{1}\right) + 1$$

$$= 540 + 180 + 48 + 9 + 1 + 1 + 1 = 780$$

$$\Rightarrow\ R(6698332 \downarrow) = N_N - \ R(6698332 \uparrow) + 1 = \frac{7!}{2!\,2!} - 780 + 1$$

$$= 1260 - 780 + 1 = 481$$

Similarly, rank of number 9866323 can determined as follows

Arrange all the digits, keeping repetitive digits together, in increasing order as follows

$$2 \to 3 \to 3 \to 6 \to 6 \to 8 \to 9$$

Now, select all the digits, according to 6698332, from above numeric order to find out permutation value of each of selected digits. Add permutation values of all the digits to find rank in increasing order as follows

$$\Rightarrow\ R(9866323 \uparrow) = F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right)$$

$$= 6\left(\frac{\left(\frac{6!}{2!\,2!}\right)}{1}\right) + 5\left(\frac{\left(\frac{5!}{2!\,2!}\right)}{1}\right) + 3\left(\frac{\left(\frac{4!}{2!}\right)}{2}\right) + 3\left(\frac{\left(\frac{3!}{2!}\right)}{1}\right) + 1\left(\frac{2!}{2}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 1080 + 150 + 18 + 9 + 1 + 0 + 1 = 1259$$

$$\Rightarrow\ R(9866323 \downarrow) = N_N - \ R(9866323 \uparrow) + 1 = \frac{7!}{2!\,2!} - 1259 + 1$$

$$= 1260 - 1259 + 1 = 2$$

Example 3: Find out increasing and decreasing order (rank) of number 8713273 in the group of all the numbers, arranged in increasing order, obtained by permuting all the digits together.

Sol. Arrange all the digits, keeping repetitive digits together, in increasing order as follows

$$1 \to 2 \to 3 \to 3 \to 7 \to 7 \to 8$$

Now, select all the digits, according to 8713273, from above numeric order to find out permutation value of each of selected digits. Add permutation values of all the digits to find rank in increasing order as follows

$$\Rightarrow\ R(8713273 \uparrow) = F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right)$$

$$= 6\left(\frac{\left(\frac{6!}{2!\,2!}\right)}{1}\right) + 4\left(\frac{\left(\frac{5!}{2!}\right)}{2}\right) + 0\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right) + 1\left(\frac{3!}{2}\right) + 0\left(\frac{2!}{1}\right) + 1\left(\frac{1!}{1}\right) + 1$$

$$= 1080 + 120 + 0 + 3 + 0 + 1 + 1 = 1205$$

Decreasing order of

$$\Rightarrow\ R(8713273\downarrow) = N_N - R(8713273\uparrow) + 1 = \frac{7!}{2!\,2!} - 1205 + 1$$

$$= 1260 - 1205 + 1 = 56$$

Thus, increasing order of number '8713273' is 1205 out of total 1260 numbers and decreasing order is 56 in the same group (arrangement).

3.2.3. Ranking of positive integer numbers having both zero and non-zero digits

In a certain group of digits (both zero & non-zero), let

$$n = \text{Number of non zero digits, out of which no. of repetitive digits are } p, q, r, s, \dots\dots\dots$$

$$n_o = \text{Number of zero digits}$$

$$\therefore \text{Total Number of the digits (zero + non − zero)} = n + n_o$$

In this case, if all the digits are permuted together then we get two types of number

1. Significant number: The number having first digit as non-zero (other than zero).

2. Non-significant number: The number having first digit as zero.

Now, if first digit is selected as zero then no. of the remaining digits is $(n + n_o - 1)$ out of which no. of the repetitive digits are $(n_o - 1),\ p,\ q,\ r, \dots\dots\dots$.. then

The total (N_o) no. of the non-significant numbers is given as

$$\Rightarrow \quad N_o = \frac{(n + n_o - 1)!}{(n_o - 1)!\,p!\,q!\,r!\dots\dots\dots.}$$

While total no. (N) of the significant numbers is given as

$$\Rightarrow \quad N = \text{numberof all the numbers} - \text{number of non significant numbers}$$

$$= \frac{(n + n_o)!}{n_o!\,p!\,q!\,r!\dots\dots\dots.} - \frac{(n + n_o - 1)!}{(n_o - 1)!\,p!\,q!\,r!\dots\dots\dots.}$$

$$= \left(\frac{n + n_o}{n_o} - 1\right)\frac{(n + n_o - 1)!}{(n_o - 1)!\,p!\,q!\,r!\dots\dots\dots.} = \frac{n(n + n_o - 1)!}{n_o!\,p!\,q!\,r!\dots\dots\dots.}$$

Now, assuming all the digits as non-zero (significant), find the rank of the given number using HCR's Formula by following normal procedure

If we subtract the value N_o (no. of non-significant numbers) from the rank obtained by HCR's Formula applied for all $(n + n_o)$ digits then we find the actual numeric order (increasing or decreasing) of any number having both zero & non-zero digits as follows

$$R(\text{Number}) = \sum_{i=1}^{n+n_o} F_i\left(\frac{P_i}{S_i}\right) - N_o$$

$$= \sum_{i=1}^{n+n_o} F_i\left(\frac{P_i}{S_i}\right) - \frac{(n+n_o-1)!}{(n_o-1)!\,p!\,q!\,r!\,\ldots\ldots\ldots.}$$

If there is no zero digits then on setting $n_o = 0$ in the above formula, we get

$$R(Number) = \sum_{i=1}^{n+0} F_i\left(\frac{P_i}{S_i}\right) - \frac{(n+0-1)!}{(0-1)!\,p!\,q!\,r!\,\ldots\ldots\ldots.}$$

$$= \sum_{i=1}^{n} F_i\left(\frac{P_i}{S_i}\right) \pm 0 = \sum_{i=1}^{n} F_i\left(\frac{P_i}{S_i}\right)$$

This is equally applicable for all the alphabetic words & numbers having non-zero digits.

3.2.4. Illustrative examples on positive integer numbers with zero digits

Example 1: Let there be a group of digits say 2, 0, 0, 0, 3, 3, 4, 7, if all the digits are permuted together & all the significant numbers (8-digit numbers) obtained are arranged in increasing order, find the rank of any number say '40307203'.

Sol. Here in the given number '40307203'

$$n = \text{number of non zero digits} = 5$$

$$p = \text{number of repetitive digits} = 2 \quad (\text{digit } '3' \text{repeats 2 times})$$

$$n_o = \text{number of zero digits} = 3$$

$$\therefore \text{Total number of the digits (zero + non zero)} = n + n_o = 5 + 3 = 8$$

The total (N_o) number of the non-significant numbers is given as

$$\Rightarrow \quad N_o = \frac{(n+n_o-1)!}{(n_o-1)!\,p!\,q!\,r!\,\ldots\ldots\ldots.}$$

$$= \frac{(5+3-1)!}{(3-1)!\,2!} = \frac{7!}{2!\,2!} = 1260$$

While total number (N) of the significant numbers is given as

$$\Rightarrow N = \frac{n(n+n_o-1)!}{n_o!\,p!\,q!\,r!\,\ldots\ldots\ldots.} = \frac{5(5+3-1)!}{3!\,2!}$$

$$= \frac{5\times 7!}{3!\,2!} = 2100$$

Now, arrange all the digits in the increasing order, keeping repetitive digits together as follows

$$0 \to 0 \to 0 \to 2 \to 3 \to 3 \to 4 \to 7$$

Now, assuming all zero digits as non-zero and applying HCR's Rank formula-I i.e. finding the permutation values of all the digits by following same procedure and adding them together as follows

$$\sum_{i=1}^{8} F_i\left(\frac{P_i}{S_i}\right)$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right) + F_8\left(\frac{P_8}{S_8}\right)$$

$$= 6\left(\frac{\left(\frac{7!}{3!\,2!}\right)}{1}\right) + 0\left(\frac{\left(\frac{6!}{2!\,2!}\right)}{3}\right) + 3\left(\frac{\left(\frac{5!}{2!}\right)}{2}\right) + 0\left(\frac{4!}{2}\right) + 3\left(\frac{3!}{1}\right) + 1\left(\frac{2!}{1}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 2520 + 0 + 90 + 0 + 18 + 2 + 0 + 1 = 2631$$

Hence, setting the values, rank of '40307203' in increasing order

$$\Rightarrow R(40307203\uparrow) = \sum_{i=1}^{8} F_i\left(\frac{P_i}{S_i}\right) - N_o$$

$$= 2631 - 1260 = 1371$$

Hence, rank of number '40307203' in increasing order is 1371 out of 2100 significant numbers.

While all the numbers are arranged in their actual numeric order (increasing or decreasing), which can be verified using Rank Formula as shown in the table below

Number	**Rank in Increasing Order**(↑)	**Rank in Decreasing Order**(↓)
20003347	1	2100
20003374	2	2099
20003437	3	2098
20003473	4	2097
20003734	5	2096
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
40307032	1370	731
40307203*	1371*	730*
40307230	1372	729
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
74323000	2096	5
74330002	2097	4
74330020	2098	3
74330200	2099	2
74332000	2100	1

Example 2: Let there be a group of digits say 1, 0, 0, 0, 0, 2, 2, 3, if all the digits are permuted together & all the significant numbers (8-digit numbers) obtained are arranged in increasing order, find the rank of any number say '20030102'

Sol. Here in the given number '20030102'

$$n = \text{number of non zero digits} = 4$$

$$p = number\ of\ repetitive\ digits = 2 \quad (\text{digit } '2' \text{repeats 2 times})$$

$$n_o = \text{number of zero digits} = 4$$

$$\therefore \text{Total number of the digits (zero + non zero)} = n + n_o = 4 + 4 = 8$$

The total (N_o) no. of the non-significant numbers is given as

$$\Rightarrow \quad N_o = \frac{(n + n_o - 1)!}{(n_o - 1)!\,p!\,q!\,r!\ldots\ldots\ldots.}$$

$$= \frac{(4 + 4 - 1)!}{(4 - 1)!\,2!} = \frac{7!}{3!\,2!} = 420$$

While total no. (N) of the significant numbers (8-digit numbers) is given as

$$\Rightarrow N = \frac{n(n + n_o - 1)!}{n_o!\,p!\,q!\,r!\ldots\ldots\ldots.} = \frac{4(4 + 4 - 1)!}{4!\,2!}$$

$$= \frac{4 \times 7!}{4!\,2!} = 420$$

Now, arrange all the digits in the increasing order, keeping repetitive digits together as follows

$$0 \to 0 \to 0 \to 0 \to 1 \to 2 \to 2 \to 3$$

Now, assuming all zero digits as non-zero and applying HCR's Rank formul-I a i.e. finding the permutation values of all the digits by following same procedure and adding them together as follows

$$\sum_{i=1}^{8} F_i\left(\frac{P_i}{S_i}\right)$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right) + F_8\left(\frac{P_8}{S_8}\right)$$

$$= 5\left(\frac{\left(\frac{7!}{4!}\right)}{2}\right) + 0\left(\frac{\left(\frac{6!}{3!}\right)}{4}\right) + 0\left(\frac{\left(\frac{5!}{2!}\right)}{3}\right) + 4\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right) + 0\left(\frac{3!}{2}\right) + 1\left(\frac{2!}{1}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 525 + 0 + 0 + 48 + 0 + 2 + 0 + 1 = 576$$

Hence, on setting the values, rank of '20030102' in increasing order

$$\Rightarrow R(20030102 \uparrow) = \sum_{i=1}^{8} F_i\left(\frac{P_i}{S_i}\right) - N_o$$

$$= 576 - 420 = 156$$

Hence, rank of number '40307203' in increasing order is 156 out of 420 significant numbers.

While all the numbers are arranged in their actual numeric order (increasing or decreasing) Which can be verified using HCR's Formula as in the table below,

Number	Rank in Increasing Order(↑)	Rank in Decreasing Order(↓)
10000223	1	420
10000232	2	419
10000322	3	418
10002023	4	417
10002032	5	416
…………… …………… ……………	……………. ……………. …………….	……………… ……………… ………………
20030021	155	266
20030102*	156*	265*
20030120	157	264
…………… …………… ……………	……………. ……………. …………….	…………….. …………….. ……………..
32200001	416	5
32200010	417	4
32200100	418	3
32201000	419	2
32210000	420	1

Example 3: Let there be a group of digits say 0, 0, 0, 4, 7, 9, 9, if all the digits are permuted together & all the significant numbers (7-digit numbers) obtained are arranged in increasing order, find the rank of any number say '9097400'.

Sol. Here in the given number '9097400'

$$n = \text{number of non zero digits} = 4$$

$$p = \text{number of repetitive digits} = 2 \quad (\text{digit } '9' \text{repeats 2 times})$$

$$n_o = \text{number of zero digits} = 3$$

$$\therefore \text{Total number of the digits (zero + non zero)} = n + n_o = 4 + 3 = 7$$

The total (N_o) number of the non-significant numbers is given as

$$\Rightarrow \quad N_o = \frac{(n + n_o - 1)!}{(n_o - 1)!\, p!\, q!\, r! \dots \dots \dots .}$$

$$= \frac{(4 + 3 - 1)!}{(3 - 1)!\, 2!} = \frac{6!}{2!\, 2!} = 180$$

While total no. (N) of the significant numbers (7-digit numbers) is given as

$$\Rightarrow N = \frac{n(n + n_o - 1)!}{n_o!\, p!\, q!\, r! \dots \dots \dots .} = \frac{4(4 + 3 - 1)!}{3!\, 2!}$$

$$= \frac{4 \times 6!}{3!\, 2!} = 240$$

Now, arrange all the digits in the increasing order, keeping repetitive digits together as follows

$$0 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 7 \rightarrow 9 \rightarrow 9$$

Now, assuming all zero digits as non-zero & applying HCR's Rank formula i.e. finding the permutation values of all the digits by following same procedure & adding them together as follows

$$\sum_{i=1}^{7} F_i\left(\frac{P_i}{S_i}\right)$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right)$$

$$= 5\left(\frac{\left(\frac{6!}{3!}\right)}{2}\right) + 0\left(\frac{\left(\frac{5!}{2!}\right)}{3}\right) + 4\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right) + 3\left(\frac{\left(\frac{3!}{2!}\right)}{1}\right) + 2\left(\frac{\left(\frac{2!}{2!}\right)}{1}\right) + 0\left(\frac{1!}{2}\right) + 1$$

$$= 300 + 0 + 48 + 9 + 2 + 0 + 1 = 360$$

Hence, on setting the values, rank of '9097400' in increasing order

$$\Rightarrow R(9097400 \uparrow) = \sum_{i=1}^{7} F_i\left(\frac{P_i}{S_i}\right) - N_o$$

$$= 360 - 180 = 180$$

Hence, rank of number '9097400' in increasing order is 180 out of 240 significant numbers.

While all the numbers are arranged in their actual numeric order (increasing or decreasing) which can be verified using HCR's Formula as shown in the table below

Number	**Rank in Increasing Order**($\uparrow$)	**Rank in Decreasing Order**($\downarrow$)
4000799	1	240
4000979	2	239
4000997	3	238
4007099	4	237
4007909	5	236
…………… …………… ……………	……………… ……………… ………………	…………… …………… ……………
4070099	13	228
4070909	14	227
4070990	15	226
…………… …………… ……………	…………… …………… ……………	……………… ……………… ………………
9097040	179	62
9097400*	180*	61*
9400079	181	60
…………… …………… ……………	…………… …………… ……………	…………….. …………….. ……………..
994070	236	5
994700	237	4
997004	238	3
997040	239	2
997400	240	1

4. Linear permutations of non-algebraic articles having different shape, size, colour etc.

The problems of linear permutations are easily simplified by using alphabetic letters or digits but alphabetic letters are more suitable for significant and better distinction among the articles which have at least one easily distinguishable property like their appearance such as shape, size, colour, surface-design etc. and are equally important at all the places in all the possible linear arrangements.

Since, we are very familiar with the linear sequence of alphabetic-letters hence all the given articles one by one are replaced by alphabetic letters, A, B, C, D, E... ….. according to the sequence. Similar articles are replaced by the same letters while different articles are replaced by the different letters. Then all the linear permutations of different articles can be dealt with ease to find out the hierarchical rank of any randomly selected linear permutation or to arrange them in correct order while the given articles have a pre-defined linear sequence.

4.1. Working steps for hierarchical ranking

Under all the conditions of applicability of Rank Formula-I,

For a given linear permutation (of certain articles equally significant at all the places (positions)) randomly selected from a series of permutations or

For a given group of certain articles equally significant at all the places in the arrangements and any randomly selected linear permutation of all these articles, the following steps are used for hierarchical ranking based on the predefined linear order of priority of given objects/articles.

Step 1: Arrange all the articles, keeping similar ones (if any) together in a consecutive manner, in a linear sequence/fashion according to the pre-defined basis of priority (any easily distinguishable-property of the articles among all like their shape, size, colour, surface-design etc.). It is called pre-defined sequence of articles equally likely significant at all the places in the arrangements.

Step 2: Now, replace all the articles one by one by an alphabetic letter, A, B, C, D, …….. accordingly, while similar articles are replaced by the same alphabetic letters in the correct sequence. Thus each of the pre-defined linear sequence of the articles is replaced by an equivalent alphabetic linear sequence & hence any random linear permutation is replaced by a certain alphabetic linear permutation. Hence by following the same rule of alphabets, apply HCR's Rank Formula-I on any of the linear permutation to find its rank (position) in the correct order or to position any linear permutation at the exact position.

4.2. Hierarchical ranking of linear permutations of non-algebraic articles

4.2.1. Articles having different shapes & sizes

Consider the following articles

$$© \llcorner ® \beta \alpha \beta © ©$$

We know that there are total 8 articles out of which three © & two β are similar articles. Now, if all these articles are permuted together then the total number of the possible ways of linear arrangements or random permutations (having no sequence) consisting of all the given articles is given as

$$= \frac{8!}{3!\,2!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 3360$$

We also know that 3360 is the number of all the random possible linear permutations formed by permuting all the given articles together.

Now, in order to arrange all these random linear permutations which do not have any correct order of arrangement, hence it is must that we have to predefine a linear sequence of all the articles according to a pre-defined basis of priority. It is obvious that the above articles have different appearance i.e. each of the articles above differs from other ones in shape & size. Thus shape or size (usually both are distinguishable) of all the articles is the most suitable distinguishable property (basis of priority). According to the basis of priority & utility (like usefulness, significance etc.), let the pre-defined linear sequence, keeping similar ones together in consecutive manner, be as follows

$$\llcorner \rightarrow ® \rightarrow © \rightarrow © \rightarrow © \rightarrow \beta \rightarrow \beta \rightarrow \alpha$$

& now, we select any random linear permutation of the given articles as follows

$$\beta © \alpha ©© \beta ® \llcorner$$

It is difficult to find the rank (correct position) of the above random linear permutation. All the linear permutations of such articles can be dealt with ease by using alphabetic letters & their linear permutations to find out the rank of the above (or any) randomly selected linear permutation. Hence let

$$\llcorner \equiv A, \quad ® \equiv B, \quad © \equiv C, \quad \beta \equiv D \quad \& \quad \alpha \equiv E$$

Thus, actual sequence of the given objects can be replaced by alphabetic order as follows (this is done only to simplify the problem)

$$A \rightarrow B \rightarrow C \rightarrow C \rightarrow C \rightarrow D \rightarrow D \rightarrow E$$

Now, the random selected permutation of given articles can be replaced as follows

$$[\beta © \alpha ©© \beta ® \llcorner] \equiv [DCECCDBA]$$

In this case, we have

$$Rank\ of\ [\beta © \alpha ©© \beta ® \llcorner] = Rank\ of\ [DCECCDBA]$$

Thus, we are to find the rank of linear permutation "**DCECCDBA**" in the alphabetic order,

Now, the same procedure can be followed as first arrange all the letters in actual alphabetic order as follows

$$A \rightarrow B \rightarrow C \rightarrow C \rightarrow C \rightarrow D \rightarrow D \rightarrow E$$

Use the above pre-defined linear sequence to find out the permutation values of each of the letters.

Now, select & remove the letters one by one from alphabetic order according to DCECCDBA to find rank as follows (applying HCR's Rank Formula-I)

$$R(DCECCDBA) = 5\left(\frac{\left(\frac{7!}{3!}\right)}{2}\right) + 2\left(\frac{\left(\frac{6!}{2!}\right)}{3}\right) + 5\left(\frac{\left(\frac{5!}{2!}\right)}{1}\right) + 2\left(\frac{4!}{2}\right) + 2\left(\frac{3!}{1}\right) + 2\left(\frac{2!}{1}\right)$$

$$+1\left(\frac{1!}{1}\right) + 1$$

$$= 2100 + 240 + 300 + 24 + 12 + 4 + 1 + 1 = 2682$$

$$\therefore\ Rank\ of\ randomly\ selected\ permutaion[\beta©\alpha©©\beta®∟]$$

$$= Rank\ of\ [DCECCDBA] = 2682$$

All other permutations can be arranged by using alphabetic order as follows

Equivalent Alphabetic Word	Linear Permutation of articles	Rank (Order)
ABCCCDDE	∟ ®© ©©ββ α	1
ABCCCDED	∟ ®© ©©ββ α	2
ABCCCEDD	∟ ®© ©©ββ α	3
ABCCDCDE	∟ ®© ©©ββ α	4
ABCCDCED	∟ ®© ©©ββ α	5
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
DCECCDAB	β©α©©β∟®	2681
DCECCDBA*	β©α©©β®∟	2682*
DCECDABC	β©α©β∟®©	2683
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
EDDCCACB	α β β© ©∟©®	3356
EDDCCBAC	α β β© ©®∟©	3357
EDDCCBCA	α β β© ©®©∟	3358
EDDCCCAB	α β β© ©©∟®	3359
EDDCCCBA	α β β© ©©®∟	3360

4.2.2. Articles having different colours (but identical in shape & size)

Consider the following articles

©©©©©©©©©

We know that there are total 9 articles (identical in shape & size) out of which one is red, two green, three sky-blue, two purple & one black. Now, if all these articles are permuted together then the total number of the possible ways of linear arrangements or random permutations (having no sequence) consisting of all the given articles is given as

$$= \frac{9!}{3!\,2!\,2!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 15120$$

We also know that 15120 is the number of all the random possible linear permutations formed by permuting all the given articles together (without any replacement).

Now, in order to arrange all these random linear permutations which do not have any correct order of arrangement, it is must that we have to predefine a linear sequence of all the articles according to a pre-defined basis of priority. It is obvious that the above articles are identical in shape & size, but each of the articles above differs from other ones in colour. Thus "colour" among all the articles is the most suitable distinguishable property (basis of priority). According to the basis of priority & utility (like usefulness, significance etc.), let the linear sequence, keeping similar ones together in consecutive manner, be as follows

$$© \rightarrow © \rightarrow © \rightarrow © \rightarrow © \rightarrow © \rightarrow © \rightarrow © \rightarrow ©$$

& now, we select any random linear permutation of the given articles as follows

©©©©©©©©©

It is difficult to find the rank (correct position) of the above random linear permutation. All the linear permutations of such articles can be dealt with ease by using alphabetic letters & their linear permutations to find out the rank of the above (or any) randomly selected linear permutation. Hence let

$$A \equiv ©, \quad B \equiv ©, \quad C \equiv ©, \quad D \equiv © \quad \& \quad E \equiv ©$$

Thus, actual sequence of the given objects can be replaced by alphabetic order as follows (this is done only to simplify the problem)

$$A \rightarrow B \rightarrow B \rightarrow C \rightarrow C \rightarrow C \rightarrow D \rightarrow D \rightarrow E$$

Now, the random selected permutation of given articles can be replaced as follows

$$[©©©©©©©©©] \equiv [DCBAEBCDC]$$

In this case, we have

$$Rank\ of\ [©©©©©©©©©] = Rank\ of\ [DCBAEBCDC]$$

Thus, we are to find the rank of linear permutation "**DCBAEBCDC**" in the alphabetic order,

Now, the same procedure can be followed as first arrange all the letters in actual alphabetic order as follows

$$A \rightarrow B \rightarrow B \rightarrow C \rightarrow C \rightarrow C \rightarrow D \rightarrow D \rightarrow E$$

Use the above pre-defined linear sequence to find out the permutation values of each of the letters.

Now, select & remove the letters one by one from alphabetic order according to DCBAEBCDC to find rank as follows (applying HCR's Rank Formula-I)

$$R(DCBAEBCDC) = 6\left(\frac{\left(\frac{8!}{3!\,2!}\right)}{2}\right) + 3\left(\frac{\left(\frac{7!}{2!\,2!}\right)}{3}\right) + 1\left(\frac{\left(\frac{6!}{2!}\right)}{2}\right) + 0\left(\frac{\left(\frac{5!}{2!}\right)}{1}\right) + 4\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right)$$

$$+0\left(\frac{\left(\frac{3!}{2!}\right)}{1}\right) + 0\left(\frac{2!}{2}\right) + 1\left(\frac{1!}{1}\right) + 1$$

$$= 10080 + 1260 + 180 + 0 + 48 + 0 + 0 + 1 + 1 = 11570$$

$$\therefore \text{ Rank of randomly selected permutation } [©©©©©©©©©]$$

$$= \text{Rank of } [DCBAEBCDC] = 11570$$

All other permutations can be arranged by using alphabetic order as follows

Equivalent Alphabetic Word	Linear Permutation of articles	Rank (Order)
ABBCCCDDE	©©©©©©©©©	1
ABBCCCDED	©©©©©©©©©	2

ABBCCCEDD	©©©©©©©©©	3
ABBCCDCDE	©©©©©©©©©	4
ABBCCDCED	©©©©©©©©©	5
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
DCBAEBCCD	©©©©©©©©©	11569
DCBAEBCDC*	©©©©©©©©©	11570*
DCBAEBDCC	©©©©©©©©©	11571
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
EDDCCBCAB	©©©©©©©©©	15116
EDDCCBCBA	©©©©©©©©©	15117
EDDCCCABB	©©©©©©©©©	15118
EDDCCCBAB	©©©©©©©©©	15119
EDDCCCBBA	©©©©©©©©©	15120

4.2.3. Articles which are similar & dissimilar in shape, size, colour, surface-design etc.

Consider the following articles

®¥ε¥€§¥©$\beta\varepsilon\exists$:©

We know that there are total 13 articles, out of which two are green & two are black articles which are similar to each other. Articles which are similar in shape and size but different in colour are considered as different articles. Now, if all these articles are permuted together then the total number of the possible ways of linear arrangements or random permutations (having no sequence) consisting of all the given articles is given as

$$=\frac{13!}{2!\,2!}=\frac{13\times12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{2\times1\times2\times1}=1556755200$$

We also know that 1556755200 is the number of all the **random possible linear permutations** formed by permuting all the given articles together (without any replacement)

Now, in order to arrange all these random linear permutations which do not have any correct order of arrangement, it is must that we have to pre-define a linear sequence of all the articles according to a pre-defined basis of priority. It is obvious that the above articles are similar & dissimilar in shape & size, colour etc. Here, it is difficult to identify the most suitable **distinguishable property (basis of priority)** among all these non-homogeneous articles (of different categories). But, all these articles can be easily distinguished by their relative appearances. Now, according to the basis of utility (like usefulness, significance etc.), let the linear sequence, keeping similar ones together in consecutive manner, be as follows

® → ¥ → ¥ → ¥ → € → § → β → ε → ε → $\exists$ → : → © → ©

& now, we select any random linear permutation of the given articles as follows

$$∴©¥∃¥€\beta¥§\varepsilon©®\varepsilon$$

It is difficult to find the rank (correct position) of the above random linear permutation. All the linear permutations of such articles can be dealt with ease by using alphabetic letters & their linear permutations to find out the rank of the above (or any) randomly selected linear permutation. Hence let

$$A \equiv ®,\ B \equiv ¥,\ \ C \equiv ¥,\ D \equiv €,\ E \equiv §,\ F \equiv \beta,\ G \equiv \varepsilon,\ H \equiv \varepsilon,\ \ I \equiv ∃$$
$$J \equiv ∴,\ \ K \equiv ©$$

Thus, actual sequence of the given objects can be replaced by alphabetic order as follows (this is done only to simplify the problem)

$$A \to B \to B \to C \to D \to E \to F \to G \to H \to I \to J \to K \to K$$

Now, the random selected permutation of given articles can be replaced as follows

$$[∴©¥∃¥€\beta¥§\varepsilon©®\varepsilon] \equiv [JKBICDFBEGKAH]$$

In this case, we have

$$Rank\ of\ [∴©¥∃¥€\beta¥§\varepsilon©®\varepsilon] = Rank\ of\ [JKBICDFBEGKAH]$$

Thus, we are to find the rank of linear permutation "$JKBICDFBEGKAH$" in the alphabetic order,

Now, the same procedure of alphabetic words can be followed as first arrange all the letters in actual alphabetic order as follows

$$A \to B \to B \to C \to D \to E \to F \to G \to H \to I \to J \to K \to K$$

Use the above pre-defined linear sequence to find out the permutation values of each of the letters.

Now, select & remove the letters one by one from alphabetic order according to JKBICDFBEGKAH to find rank as follows (applying HCR's Rank Formula-I)

$$R(JKBICDFBEGKAH) = 10\left(\frac{\left(\frac{12!}{2!\,2!}\right)}{1}\right) + 10\left(\frac{\left(\frac{11!}{2!}\right)}{2}\right) + 1\left(\frac{10!}{2}\right) + 8\left(\frac{9!}{1}\right) + 2\left(\frac{8!}{1}\right) + 2\left(\frac{7!}{1}\right) + 3\left(\frac{6!}{1}\right)$$
$$+ 1\left(\frac{5!}{1}\right) + 1\left(\frac{4!}{1}\right) + 1\left(\frac{3!}{1}\right) + 2\left(\frac{2!}{1}\right) + 0\left(\frac{1!}{1}\right) + 1$$

$$= 1197504000 + 99792000 + 1814400 + 2903040 + 80640 + 10080 + 2160 + 120 + 24 + 6 + 4 + 0 + 1$$

$$= 1302106475$$

$$\therefore\ \text{Rank of randomly selected permutaion}\ [∴©¥∃¥€\beta¥§\varepsilon©®\varepsilon]$$

$$= \text{Rank of}\ [JKBICDFBEGKAH] = 1302106475$$

All other permutations can be arranged by estimating the ranks (alphabetic orders) as tabulated below

Equivalent Alphabetic Word	Linear Permutation of articles	Rank (Order)
ABBCDEFGHIJKK	®¥¥¥€§βεε∃:©©	1
ABBCDEFGHIKJK	®¥¥¥€§βεε∃©:©	2
ABBCDEFGHIKKJ	®¥¥¥€§βεε∃©©:	3
ABBCDEFGHJIKK	®¥¥¥€§βεε:∃©©	4
ABBCDEFGHJKIK	®¥¥¥€§βεε:©∃©	5
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
JKBICDFBEGHKA	:©¥∃¥€β¥§εε©®	1302106474
JKBICDFBEGKAH*	:©¥∃¥€β¥§ε©®ε	1302106475*
JKBICDFBEGKHA	:©¥∃¥€β¥§ε©ε®	1302106476
…………… …………… ……………	…………… …………… ……………	…………… …………… ……………
KKJIHGFEDBCAB	©©:∃εεβ§€¥¥®¥	1556755196
KKJIHGFEDBCBA	©©:∃εεβ§€¥¥¥®	1556755197
KKJIHGFEDCABB	©©:∃εεβ§€¥®¥¥	1556755198
KKJIHGFEDCBAB	©©:∃εεβ§€¥¥®¥	1556755199
KKJIHGFEDCBBA	©©:∃εεβ§€¥¥¥®	1556755200

Thus, we find that by the use of alphabetic words, we can solve the problem of randomly selected linear permutations of certain things. It is sufficient to study the linear permutations of alphabetic letters & digits to deal with various linear permutations of different articles (elements). We apply the same procedure as that of words & numbers on all the linear permutations of certain articles to find their respective rank (correct position) in an ordered arrangement. HCR's formula for alphabetic words and numbers is generalized as follows.

5. Generalized form of HCR's Rank Formula-I

For alphabetic words & the numbers

$$R(\text{Word or number}) = \sum_{i=1}^{n+n_o} F_i\left(\frac{P_i}{S_i}\right) - \left[\frac{(n+n_o-1)!}{(n_o-1)!\,p!\,q!\,r!\dots\dots\dots}\right]$$

Where,

$n =$ no. of significant digits/letters

Out of which no. of repetitive digits/letters are 'p', 'q', 'r' … … … …

$n_o =$ no. of non significant digits /letters which don't appear at first place (left most) of a number or a word

5.1. Deduction 1: Rank of Alphabetic Word & Positive Integral Number with non-zero digits

If a given word/positive integral number has $'n'$ no. of the letters, out of which no. of the repetitive letters/non-zero digits are $p, q, r, s, \dots..$ then

In this case, $n_o = 0$

Rank of a word/number is given as

$$R(\text{Word or number}) = \sum_{i=1}^{n} F_i\left(\frac{P_i}{S_i}\right)$$

Note: If all the letters or non-zero digits in a given word or positive integral number are non-repetitive, then the similarity of all the letters or digits is equal to unity. Hence we have

$S_1 = S_2 = S_3 = \dots\dots\dots\dots\dots\dots\dots. = S_{n-2} = S_{n-1} = S_n = 1 \Rightarrow S_i = 1$ Hence, the rank of such word or number is given as follows

$$R(\text{Word or number}) = \sum_{i=1}^{n} F_i\left(\frac{P_i}{1}\right) = \left|\sum_{i=1}^{n} F_i P_i\right|_{\text{non repeatability}}$$

5.2. Deduction 2: Rank of Positive Integral Number with zero & non-zero digits

If a positive integral number has $'n'$ no. of non-zero digits, out of which no. of repetitive digits are $'p', 'q', 'r' \dots\dots\dots$ & $'n_o'$ no. of the zero digits then

Rank of the number is given as

$$R(\text{Number}) = \sum_{i=1}^{n+n_o} F_i\left(\frac{P_i}{S_i}\right) - \left[\frac{(n+n_o-1)!}{(n_o-1)!\,p!\,q!\,r!\dots\dots\dots}\right]$$

Note: All the above formula/results are equally applicable for any linear permutation of various articles having different shape, size, colour, surface design and other observable/aesthetic properties. The examples of words & numbers are only given for ease of understanding as we are much familiar with the linear arrangements of letters & digits as they have well defined order of priority like alphabetic order, increasing & decreasing order.

6. HCR's Axiom on hierarchical ranking

"If a given linear permutation (like word, number etc.) has total $'n'$ number of the articles (like letters, digits etc.) out of which numbers of repetitive articles are $p, q, r, s, \dots\dots\dots$.. then total number (N) of the linear permutations formed by permuting all the articles (of given linear permutation or group) together is always equal to the rank of the last linear permutation if all the permutations are arranged in a certain order (like alphabetic order, numeric order (increasing or decreasing) etc.)"

Mathematically, it is expressed as follows

$$N = \frac{n!}{p!\,q!\,r!\,s!\dots\dots} = \left|\sum_{i=1}^{n} F_i\left(\frac{P_i}{S_i}\right)\right|_{\text{linear permutation}}$$

For ease of understanding, it is well known that when the words/numbers, obtained by permuting all the letters/non-zero digits together, are arranged in their respective alphabetic/numeric order, then the order of the last word/number will always be equal to the total number of words/numbers (permutations).

Note: The above axiom is based on HCR's Rank Formula-I, used to verify that all the results of linear permutations obtained by the formula are correct.

7. Illustrative Examples: Verification of HCR's Axiom by numerical results

Since this formula does not have any mathematical proof or derivation, it is verified on the basis of the results obtained by it related to the various linear permutations of different articles. But here, for ease of understanding, we would use the words & numbers only as these can be easily correlated with all other linear permutations simply by treating each article as an alphabetic letter or a digit as we have used in above articles. (Refer to the ranking of linear permutations of various articles).

Example 1: Find out total no. of words obtained by permuting the letters A, E, C, F, C, H, together & check out the results obtained to verify Rank Formula.

Sol. Arrange all the letters in alphabetic order as follows

$$A \to C \to C \to E \to F \to H$$

By arranging all the letters in reverse alphabetic order as follows

Last word: HFECCA

Now, using HCR's Rank Formula-I to calculate its rank as follows

$$R(HFECCA)$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right)$$

$$= 5\left(\frac{\left(\frac{5!}{2!}\right)}{1}\right) + 4\left(\frac{\left(\frac{4!}{2!}\right)}{1}\right) + 3\left(\frac{\left(\frac{3!}{2!}\right)}{1}\right) + 1\left(\frac{2!}{2}\right) + 1\left(\frac{1!}{1}\right) + 1$$

$$= 300 + 48 + 9 + 1 + 1 + 1 = 360$$

While total no. of the words in the group (Word Series) is given as follows

$$N_N = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 360$$

Thus, we find that both the results obtained are equal hence HCR's Axiom is true which verifies the formula.

Example 2: Find out the total no. of positive integral numbers obtained by permuting the non-zero digits 6, 6, 4, 4, 1, 4, 8, 9, 7, together and check out the results obtained to verify Rank Formula.

Sol. Arrange all the digits in increasing numeric order as follows

$$1 \to 4 \to 4 \to 4 \to 6 \to 6 \to 7 \to 8 \to 9$$

By arranging all the digits in decreasing order as follows

Last number: 987664441 (while all the numbers are arranged in increasing order)

Now, using HCR's Rank Formula-I to calculate its rank as follows

$$R(987664441 \uparrow)$$

$$= F_1\left(\frac{P_1}{S_1}\right) + F_2\left(\frac{P_2}{S_2}\right) + F_3\left(\frac{P_3}{S_3}\right) + F_4\left(\frac{P_4}{S_4}\right) + F_5\left(\frac{P_5}{S_5}\right) + F_6\left(\frac{P_6}{S_6}\right) + F_7\left(\frac{P_7}{S_7}\right) + F_8\left(\frac{P_8}{S_8}\right) + F_9\left(\frac{P_9}{S_9}\right)$$

$$= 8\left(\frac{\left(\frac{8!}{3!\,2!}\right)}{1}\right) + 7\left(\frac{\left(\frac{7!}{3!\,2!}\right)}{1}\right) + 6\left(\frac{\left(\frac{6!}{3!\,2!}\right)}{1}\right) + 4\left(\frac{\left(\frac{5!}{3!}\right)}{2}\right) + 4\left(\frac{\left(\frac{4!}{3!}\right)}{1}\right) + 1\left(\frac{\left(\frac{3!}{2!}\right)}{3}\right) + 1\left(\frac{2!}{2}\right) + 1\left(\frac{1!}{1}\right) + 1$$

$$= 26880 + 2940 + 360 + 40 + 16 + 1 + 1 + 1 + 1 = 30240$$

While total no. of positive integral numbers in the group (Number Series) is given as follows

$$N_N = \frac{9!}{3!\,2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 30240$$

Thus, we find that both the results obtained are equal hence HCR's Axiom is true which verifies the formula.

8. Conditions of applicability of the formula

It can be applied to find out the rank (position) of any randomly selected linear permutation, consisting of certain articles having a pre-defined linear sequence, if and only if

1. All the articles have at least one easily distinguishable property (like shape, size, colour, surface-design, etc.) among them & are equally significant at all the places (positions) in the arrangements
2. All the articles are permuted together without any condition like mutual-combinations (such as pairs) of articles, significance & non-significance of a few or certain articles &
3. All the linear permutations obtained are assumed to be equally significant & arranged in a mathematically correct order/sequence.

9. Applications of the formula

The Formula is equally applicable to find out

1. Alphabetic order of any word randomly selected from a correct alphabetic arrangement of all the words, obtained by making linear arrangements, which have equal and identical letters.
2. Numeric (increasing or decreasing) order of any number randomly selected from a correct numeric arrangement of all the numbers, obtained by making linear arrangements, which have equal and identical digits.
3. Total no. of words lying between any two words randomly selected from a correct alphabetic arrangement of all the words obtained by linear arrangements which have equal and identical letters
4. Total numbers lying between any two numbers randomly selected from a correct numeric (increasing or decreasing) arrangement of all the numbers obtained by linear arrangement, which have equal and identical digits.
5. Hierarchical order (position) of any non-algebraic linear permutation (consisting of certain similar and dissimilar articles (things)) randomly selected from a group of all the linear permutations correctly arranged in a certain order according to the pre-defined linear sequence of articles.
6. Exact or correct arrangement of a few or all the linear permutations consisting of smaller, larger or very larger no. of equal and identical articles which is practically not so easy by other methods in Linear Algebra i.e. Rank Formula-I is the most useful and the analytical formula to correctly arrange few or all the permutations (of algebraic or non-algebraic or both the articles) in a correct order according to the pre-defined linear sequence.
7. Hierarchical ranks of the circular permutations (of various articles) simply by using the concept of linear permutations under certain conditions, such as the leading element (i.e., first/leftmost element in the predefined linear order of priority of given articles/objects) must always be non-repetitive [6].

10. Conclusion

The proposed Rank Formula I establishes a generalized and systematic framework for determining the rank (hierarchical position) of any linear permutation within the complete set of its possible arrangements. The formulation is applicable to a broad class of permutations, including alphabetic words, numerical sequences, and collections of distinguishable objects characterized by properties such as shape, size, colour, or surface design, arranged without replacement. By utilizing the familiar structure of alphabetic ordering as a reference model, the method simplifies the treatment of complex linear permutations involving both algebraic and non-algebraic elements. This enhances both conceptual understanding and computational efficiency in identifying permutation ranks. The proposed formulation has potential applications in combinatorics, algorithm design, data organization, cryptography, and ranking systems where ordered arrangements of elements are essential. It can also be useful in problems involving enumeration, indexing, and systematic generation of permutations. Future work may focus on extending the formulation to more complex scenarios, including permutations with additional constraints, partially ordered sets, and multidimensional arrangements. Further exploration may also involve the development of efficient computational algorithms and software implementations based on the proposed framework, thereby broadening its applicability in both theoretical and applied domains.

Note: *The above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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