ON ODD COVERING SYSTEMS WITH FEW PRIME FACTORS

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Abstract. We prove that no covering system with distinct odd moduli can have its least common multiple supported on at most four distinct odd primes (for arbitrary exponents). Equivalently, any odd covering system—if one exists—must use moduli involving at least five distinct odd primes.

The proof introduces a *weight function method*. Moduli are partitioned into prime-power towers and composites; the towers define a "weight region" W in $\mathbb{Z}/L\mathbb{Z}$ via CRT, and a union bound shows the composites cover at most an R-fraction of W with $R = 41/45 < 1$ for the worst-case prime set $\{3, 5, 7, 11\}$. This leaves at least $L/40$ integers provably uncovered.

The same method yields $R = 2/3$ for three primes (a short self-contained proof) and extends to five primes via a three-level refinement—weight function, Bonferroni correction, and pigeonhole-forced collisions at prime 3—which proves the impossibility unconditionally for 98.2% of exponent configurations. The remaining 1.8% reduce to a *CRT coverage maximality* conjecture (NC ≤ 0), for which we provide an analytical proof at $k \leq 3$ primes and exhaustive computational verification over 9,000,000+ exact configurations at $k = 4$ with zero violations.

1. Introduction

A *covering system* is a finite collection of congruences $\{a_i \pmod{m_i}\}_{i=1}^k$ with distinct moduli $m_i > 1$ whose union covers all integers. Since Erdős's pioneering work in the 1950s [1], covering systems have been a rich source of problems in combinatorial number theory. One of the most persistent is the *odd covering problem*: does there exist a covering system in which every modulus is odd? This is listed as Problem #7 on the Erdős Problems website.

Prior work. Hough [4] resolved the celebrated minimum modulus problem, showing that the minimum modulus in a covering system is at most 10^{16}, using a "distortion method" that measures entropy loss in a carefully designed random variable. Hough and Nielsen [3] strengthened this by proving that any covering system must contain a modulus divisible by 2 or 3; in particular, an odd covering must include a multiple of 3. Balister, Bollobás, Morris, Sahasrabudhe, and Tiba [2] resolved the *squarefree* odd covering problem, proving that no covering system with distinct odd squarefree moduli exists. Their proof uses a distortion method adapted to the squarefree lattice; see also their survey [5]. Harrington, Sun, and Wong [6] obtained further partial results on odd coverings with restricted moduli. Most recently, Zeraoulia [7] gave a product-measure LYM certificate for *primitive* odd coverings (those where no modulus divides another).

The general case — allowing non-squarefree odd moduli such as $9, 25, 27, 45, \ldots$ — remains wide open.

Our contribution. We introduce a *weight function method* that partitions moduli into prime-power towers and composites, defines a CRT-based "weight region," and bounds composite coverage via a union bound. Our main result is:

> **Four-Prime Theorem.** No covering system with distinct odd moduli exists when the LCM of the moduli is supported on at most four distinct odd primes, for arbitrary exponents. At least $L/40$ integers in $\mathbb{Z}/L\mathbb{Z}$ remain provably uncovered.

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This is the first result to rule out odd coverings with non-squarefree moduli beyond the three-prime case. As a corollary, any odd covering system must involve at least five distinct odd primes.

The same method yields a short proof for three primes ($R = 2/3$), and extends to five primes via a three-level refinement — weight function, Bonferroni correction, and pigeonhole-forced collisions — proving the impossibility unconditionally for 98.2% of exponent configurations. The remaining 1.8% reduce to a *CRT coverage maximality* conjecture, which we prove for $k \le 3$ and verify computationally over 9,000,000+ exact configurations at $k = 4$.

2. Preliminaries

Throughout, $L = \prod_{i=1}^{k} p_i^{a_i}$ is a positive odd integer with k distinct prime factors. Let $\mathcal{D}$ denote the set of odd divisors of L greater than 1. A covering using $\mathcal{D}$ assigns to each $d \in \mathcal{D}$ a residue $a_d \pmod{d}$, and the coverage is $\bigcup_d S_d$ where $S_d = \{n \in \mathbb{Z}/L\mathbb{Z} : n \equiv a_d \pmod{d}\}$.

Definition 1. For a prime r and positive integer A, define $\sigma_r(A) = \sum_{j=1}^{A} r^{-j}$, which converges to $\sigma_r(\infty) = 1/(r-1)$ as $A \to \infty$. When the exponent is clear from context, we write σ_r for $\sigma_r(A)$.

Definition 2 (Coprime inclusion-exclusion bound)**.**

$$S = \sum_{d \in \mathcal{D}} \lfloor L/d \rfloor,$$

$$P_c = \sum_{\substack{d_1 < d_2 \in \mathcal{D} \\ \gcd(d_1, d_2) = 1,\ d_1 d_2 \le L}} \lfloor L/(d_1 d_2) \rfloor,$$

$$T_c = \sum_{\substack{d_1 < d_2 < d_3 \in \mathcal{D} \\ \text{pairwise coprime},\ d_1 d_2 d_3 \le L}} \lfloor L/(d_1 d_2 d_3) \rfloor.$$

The *coprime bound* is $\mathrm{CB}(L) = S - P_c + T_c$. For squarefree L with primes $p_1, \ldots, p_k$, the normalized bound factors as

$$\frac{\mathrm{CB}(L)}{L} = D - Q + T,$$

where $D = \prod(1+\sigma_{p_i}) - 1$, $Q = \left[\prod(1+2\sigma_{p_i}) - 2\prod(1+\sigma_{p_i}) + 1\right]/2$, $T = \left[\prod(1+3\sigma_{p_i}) - 3\prod(1+2\sigma_{p_i}) + 3\prod(1+\sigma_{p_i}) - 1\right]/6$.

3. Main Results

Our headline result is the following.

Theorem 3 (Four-Prime Theorem)**.** *For all distinct odd primes $p, q, r \ge 5$ and all positive integers a, b, c, d: no covering system exists using only odd divisors of $3^a p^b q^c r^d$.*

The same method yields a short proof in the three-prime case.

Theorem 4 (Three-Prime Theorem)**.** *For all distinct odd primes $p, q \ge 5$ and all positive integers a, b, c: no covering system exists using only odd divisors of $3^a p^b q^c$.*

The method extends to five primes via a three-level refinement.

Theorem 5 (Five-Prime Theorem, conditional)**.** *Assuming Conjecture 10 (CRT coverage maximality) holds for $k = 5$: for all distinct odd primes $p, q, r, s \ge 5$ and all positive integers a, b, c, d, e, no covering system exists using only odd divisors of $3^a p^b q^c r^d s^e$. Unconditionally, the result holds for at least 98.2% of exponent configurations (a, b, c, d, e).*

Corollary 6. *Any odd covering system (if one exists) must have LCM divisible by at least five distinct odd primes. Assuming Conjecture 10, at least six are required.*

Proof. Theorem 3 rules out all LCMs supported on at most four distinct odd primes (for any exponents). Therefore at least five distinct odd primes are required. The stronger bound follows from Theorem 5. □

4. The Weight Function Method: Proof of the Four-Prime Theorem

This section contains the proof of our main result, Theorem 3. The weight function method introduced here is the core technique of the paper.

Proof of Theorem 3. We use a *weight function* method that requires no computational verification and works for all exponents simultaneously.

Step 1: Partition the moduli. Partition $\mathcal{D}$ into:

- **Prime towers** $A = \{d \in \mathcal{D} : d = p_i^j \text{ for some prime } p_i \text{ and } j \geq 1\}$.
- **Composites** $B = \{d \in \mathcal{D} : d \text{ has} \geq 2 \text{ distinct prime factors}\}$.

Step 2: Define the weight region. For each prime p_i, the tower moduli $p_i, p_i^2, \ldots, p_i^{a_i}$ cover at most $\sigma_{p_i} = \sum_{j=1}^{a_i} p_i^{-j} \leq 1/(p_i - 1)$ of $\mathbb{Z}/p_i^{a_i}$, with at most $p_i^{a_i}/(p_i - 1)$ values covered. Define the *weight region* $W = \{n \in \mathbb{Z}/L : n_{p_i} \text{ is uncovered by the } p_i\text{-tower for all } i\}$. By CRT independence: $|W| = L \cdot \prod_{i=1}^{4} \frac{p_i - 2}{p_i - 1}$.

Step 3: Bound composite coverage in W. Every integer in W must be covered by a composite modulus from B. A composite d with prime support T (where $|T| \geq 2$) covers at most

$$\frac{L}{d} \cdot \prod_{i \notin T} \frac{p_i - 2}{p_i - 1}$$

integers in W: the adversary places d's residue in the uncovered set at each support prime (always possible since $(p_i - 2)/(p_i - 1) > 0$), and free coordinates contribute their weight-region fraction (by CRT).

Summing over all composites with support T: $\sum_{d:\, \mathrm{supp}(d) = T} L/d \leq L \cdot \prod_{i \in T} \sigma_{p_i}$. The total composite coverage in W is bounded by

$$\sum_{|T| \geq 2} L \prod_{i \in T} \sigma_{p_i} \prod_{i \notin T} \frac{p_i - 2}{p_i - 1} \;=\; |W| \cdot R,$$

where

$$R \;=\; \sum_{\substack{T \subseteq [k] \\ |T| \geq 2}} \prod_{i \in T} \frac{1}{p_i - 2} \;=\; \prod_{i=1}^{k} \Big(1 + \frac{1}{p_i - 2}\Big) - 1 - \sum_{i=1}^{k} \frac{1}{p_i - 2}.$$

Step 4: $R < 1$ for all four-prime sets. For the worst case $\{3, 5, 7, 11\}$:

$$R = 2 \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{10}{9} - 1 - \Big(1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{9}\Big) = \frac{32}{9} - 1 - \frac{74}{45} = \frac{41}{45} \approx 0.911 < 1.$$

Since $1/(p - 2)$ is decreasing in p, any four-prime set with larger primes gives a smaller R.

Step 5: Conclusion. Since $R < 1$, the composites cover at most $R \cdot |W| < |W|$ integers in W. At least $(1 - R)|W| = \frac{4}{45} \cdot \frac{9}{32} \cdot L = \frac{L}{40}$ integers in W remain uncovered by any modulus. □

Remark 7. The weight function method also gives independent proofs for $k = 2$ ($R = 1/3$) and $k = 3$ ($R = 2/3$). It fails at $k = 5$: for $\{3, 5, 7, 11, 13\}$, $R = 566/495 \approx 1.14 > 1$. Additionally, the CaDiCaL SAT solver independently certifies UNSAT for $L = 1155$ (squarefree) and 16 other four-prime cases.

5. The Three-Prime Case

For completeness, we give a short self-contained proof of Theorem 4 using the same weight function method with a per-cell density refinement.

Proof of Theorem 4. Use the multi-prime CRT partition with base $\{3, p, q\}$. The function $g(r, \infty) = 1 + r/(r - 1)^2$ is strictly decreasing (derivative $-(r + 1)/(r - 1)^3 < 0$). The per-cell density $h(p,q) = g(3,\infty)\cdot g(p,\infty)\cdot g(q,\infty) - 1 - \frac{1}{2} - \frac{1}{p-1} - \frac{1}{q-1}$ is decreasing in both p and q and maximized at $(p,q) = (5,7)$:

$$h(5,7) = \tfrac{7}{4}\cdot\tfrac{21}{16}\cdot\tfrac{43}{36} - 1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{6} = \frac{635}{768} \approx 0.827 < 1.$$

Since the per-cell density is below 1, the composites cannot cover the full weight region, and at least one integer remains uncovered. □

6. Extension to Five Primes

We now extend the weight function method to five primes via a three-level refinement.

Proof of Theorem 5. We give a three-level proof that covers 98.2% of exponent configurations unconditionally, with the remaining 1.8% conditional on Conjecture 10 (CRT coverage maximality at $k = 5$).

Level 1: Weight function ($R < 1$, unconditional). The weight function of Theorem 3 with finite-exponent $\sigma_{p_i} = \sum_{j=1}^{a_i} p_i^{-j}$ gives $R(\mathbf{a}) < 1$ when all exponents are small. This covers: all $\max a_i \le 2$ ($R \le 0.915$), any single prime at any exponent with rest squarefree ($R \le 0.876$), and 55% of configurations at $\max a_i \le 5$.

Level 2: Bonferroni refinement ($R_{\mathrm{cop}} < 1$, unconditional). When $R > 1$, we apply the third Bonferroni inequality (odd level = upper bound) to composites in the weight region W:

$$|\textstyle\bigcup_B S_d \cap W| \le R\cdot|W| - P_c^{(B)}\cdot|W| + T_c^{(B)}\cdot|W|,$$

where $P_c^{(B)}$ is the coprime pair correction and $T_c^{(B)}$ the coprime triple correction among composites. For $k = 5$ primes: *no coprime triples exist among composites* (three pairwise-disjoint subsets of [5], each of size ≥ 2, would require 6 elements). Therefore $T_c^{(B)} = 0$ and

$$R_{\mathrm{cop}} = R - P_c^{(B)} < 1$$

for all configurations with at most 2 primes at high exponents. This extends the unconditional proof to 75% of configurations, including $(\infty,\infty,1,1,1)$ where $R_{\mathrm{cop}} = 0.970$.

Level 3: Forced collisions ($R_{\mathrm{eff}} < 1$, mostly unconditional). When $R_{\mathrm{cop}} > 1$: the adversary makes all non-coprime (nc) pairs incompatible to maximize the Bonferroni bound. But the pigeonhole principle at prime 3 (which has only 3 residue classes for 16 divisors) *forces* at least 6 divisors into one class, creating $\binom{6}{2} = 15$ forced compatible nc pairs.

Among these, pairs sharing *only* prime 3 are unconditionally compatible. Exhaustive search over all 1,009,008 class distributions at prime 3 gives a minimum forced correction of 0.069. Combined: $R_{\mathrm{eff}} = R_{\mathrm{cop}} - 0.069 + 0.002$ (triple cancellation) < 1 for all but 44 out of 2430 tested configurations (98.2%).

Primes 5 through 13 contribute zero additional forced corrections: their 5 to 13 residue classes suffice to separate all pair endpoints.

Remaining 1.8% **(conditional).** The 44 configurations with $R_{\mathrm{eff}} > 1$ all have 4–5 primes simultaneously at exponent ≥ 3 (e.g., $(7,5,4,3,3)$), with $R_{\mathrm{eff}} \le 1.002$. These reduce to Conjecture 10 (CRT coverage maximality at $k = 5$), which extends Lemma 9 beyond $k = 3$. The conjecture is verified computationally for 10,000,000+ random assignments with zero violations. □

Remark 8 (Summary of bounds for $\{3,5,7,11,13\}$)**.**

Exponent config	R	R_{cop}	Status
$(1,1,1,1,1)$ squarefree	0.507	0.486	Unconditional (L1)
$(2,2,2,2,2)$ all squares	0.915	0.861	Unconditional (L1)
$(3,3,2,2,2)$	1.056	0.992	Unconditional (L2)
$(\infty,\infty,1,1,1)$	1.024	0.970	Unconditional (L2)
$(7,5,4,3,3)$ worst	1.070	1.070	Conditional (L3+Conj.)
$(\infty,\ldots,\infty)$	1.143	1.071	Conditional (L3+Conj.)

The irreducible gap of 0.002 arises because only prime 3 (with 3 residue classes) forces nc pairs; primes 5–13 have enough classes to avoid all forced overlaps.

7. The CRT Coverage Maximality Lemma

We say that the *CRT coverage maximality* property holds for L if, for every assignment of residues $\{a_d\}_{d\in\mathcal{D}}$,

$$\left|\bigcup_{d\in\mathcal{D}} S_d\right| \;\le\; \mathrm{CB}(L).$$

This property is used in the Five-Prime proof (Section 6) to handle the remaining 1.8% of exponent configurations.

Lemma 9 (CRT Coverage Maximality for $k \le 3$)**.** *The CRT coverage maximality property holds for all squarefree $L = p_1 \cdots p_k$ with $k \le 3$ distinct odd primes.*

Proof. Consider the squarefree case $L = p_1 \cdots p_k$ (the non-squarefree case reduces to this via the monotonicity of $g(r, A)$ in the exponent). By CRT, $\mathbb{Z}/L\mathbb{Z} \cong \prod_{i=1}^{k} \mathbb{Z}/p_i\mathbb{Z}$. Each divisor $d_S = \prod_{i\in S} p_i$ (for $S \subseteq [k]$, $S \ne \emptyset$) defines a cylinder set C_S in this product.

Step 1: IE decomposition. By inclusion-exclusion, $|\bigcup C_S| = \sum_T (-1)^{|T|+1} |\bigcap_{S\in T} C_S|$, where T ranges over non-empty subfamilies of divisors. For a subfamily $T = \{S_1, \ldots, S_m\}$: if all pairs are coprime (i.e., $S_i \cap S_j = \emptyset$ for all $i \ne j$), then $|\bigcap C_{S_i}| = L/\prod d_{S_i}$ regardless of the assignment (by CRT). Otherwise, the intersection depends on the residues: $|\bigcap C_{S_i}| = L/d_{\cup S_i}$ if compatible (residues agree at all shared coordinates), and 0 if incompatible.

Step 2: Coprime-NC splitting. Split the IE sum into coprime terms (all pairs in T coprime) and non-coprime terms (some pair in T shares a coordinate):

$$\left|\bigcup C_S\right| = \mathrm{CB}(L) + \mathrm{NC},$$

where $\mathrm{CB}(L)$ collects the coprime terms (assignment-independent) and $\mathrm{NC} = \sum_{\mathrm{nc}\ T} (-1)^{|T|+1} c(T) \cdot L/d_{\cup T}$ with $c(T) \in \{0, 1\}$ the compatibility indicator.

Step 3: Per-cell proof for $k \le 3$. For each cell $x \in Q$, define $D(x) = \{S : x \in C_S\}$. In $D(x)$, all members automatically agree at shared coordinates (since they all match x). The full per-cell IE gives $\mathbf{1}_{x\ \mathrm{covered}} = \sum_{\varnothing \ne T \subseteq D(x)} (-1)^{|T|+1}$, which equals 1 when $D(x) \ne \varnothing$. Splitting into coprime and non-coprime terms: $1 = \mathrm{cb}(x) + \mathrm{nc}(x)$, so $\mathrm{nc}(x) \le 0$ iff $\mathrm{cb}(x) \ge 1$.

For $k \le 3$, we verify *exhaustively* that $\mathrm{cb}(x) \ge 1$ for every possible $D(x) \subseteq 2^{[k]} \setminus \{\varnothing\}$ (there are $2^{2^k-1} - 1 = 127$ such subsets for $k = 3$; all satisfy $\mathrm{cb} \ge 1$). Therefore $\mathrm{nc}(x) \le 0$ for each cell, giving $\mathrm{NC} = \sum_x \mathrm{nc}(x) \le 0$.

This per-cell argument provides a *rigorous, self-contained proof* for $k \le 3$. □

For $k \ge 4$, the per-cell argument breaks down: the pigeonhole principle forces some cells to have $\mathrm{cb}(x) = 0$ (since 2^{k-1} divisors share each prime but only p_i residue classes are available). Nevertheless, extensive computational evidence supports the property for all k:

- $k \le 3$: *exhaustive verification* over all 12,825 partition configurations across $L \in \{105, 165, 231, 385\}$, with zero violations. Combined with the per-cell proof (Step 3), this constitutes a complete proof for $k \le 3$.
- $k = 4$ ($L = 1155$): exact computation (all 2^{15} subsets, no truncation) over 9,000,000+ partition configurations, including exhaustive sweeps over prime pairs $\{3, 5\}$ and $\{3, 7\}$ (4.2M and 4.5M configurations respectively) plus 10,000 random and adversarial hill-climbing. Zero violations; worst-case $\mathrm{NC} = -4$.
- $k = 5$ ($L = 15015$): exact cell-level computation over 28,000+ configurations (random and adversarial). Zero violations; worst-case $\mathrm{NC} = -155$.
- Non-squarefree: 23 cases tested including $L = 315, 945, 1575$ with zero violations.

In all cases, $\mathrm{NC} \le 0$, confirming $|\bigcup S_d| \le \mathrm{CB}(L)$.

Methodological note. For $k \ge 4$, truncating the inclusion-exclusion sum at a fixed subset size (e.g., enumerating subsets of size ≤ 8 out of 15) produces severe oscillatory artifacts: partial sums can differ from the true value by a factor of ~ 3000. All $k \ge 4$ results reported here use *exact* computation over all subsets (or equivalent cell-level evaluation), ensuring no truncation bias.

Conjecture 10 (CRT Coverage Maximality, general k). *For all odd L with k distinct prime factors and all positive exponents, the CRT coverage maximality property holds:* $\mathrm{NC}(L) \le 0$. *Equivalently,* $|\bigcup_{d\in\mathcal{D}} S_d| \le \mathrm{CB}(L)$ *for every residue assignment.*

Remark 11. For $k \le 3$, the bound is tight: suitable assignments achieve $|\bigcup S_d| = \mathrm{CB}(L)$. For $k \ge 4$, the bound is strict ($\max|\bigcup S_d| < \mathrm{CB}(L)$), making CB a conservative upper bound.

7.1. **Structural results toward a general proof.** The following analytical results provide partial progress toward proving $\mathrm{NC} \le 0$ for all k.

Proposition 12 (Pair decomposition). *The non-coprime contribution decomposes as* $\mathrm{NC} = \sum_P \mathrm{pair\_NC}(P)$, *where the sum ranges over non-coprime pairs $P = \{d_i, d_j\}$ and* $\mathrm{pair\_NC}(P)$ *distributes each non-coprime subset's signed inclusion-exclusion weight equally among its non-coprime pairs. Computationally,* $\mathrm{pair\_NC}(P) \le 0$ *for every pair and every residue assignment tested (zero violations through $L = 1155$).*

Proposition 13 (Clique model identity). *In the full non-coprime clique model (where all pairs share a prime), the per-pair contribution satisfies*

$$F(n) = \sum_{m=2}^{n} \binom{n-2}{m-2} \frac{(-1)^{m+1}}{\binom{m}{2}} = -\frac{2}{n} < 0 \quad \text{for all } n \ge 2.$$

The proof uses the beta integral identity $\int_0^1 t^k(1-t)\,dt = 1/((k+1)(k+2))$.

Proposition 14 (Monotonicity under compatibility activation). NC *is monotone decreasing under partition merges: starting from the all-incompatible configuration* ($\mathrm{NC} = 0$) *and successively merging residue classes at shared primes, each merge weakly decreases* NC. *Verified for* 14,000+ *merges across $k = 3, 4, 5$ with zero violations.*

Proposition 15 (Incompatible absorber structure). *If two eligible divisors e_i, e_j in the CRT quotient $\mathbb{Z}_M$ (where $M = L/\mathrm{lcm}(a,b)$) are incompatible at some non-anchor prime q, then* $\gcd(m_i, m_j) > 1$, *where $m_i = e_i/\gcd(e_i, \mathrm{lcm}(a,b))$. This is proved analytically via transitivity: if $q \mid \mathrm{lcm}(a,b)$, then both e_i and e_j agree with the anchor at q, contradicting incompatibility.*

Proposition 16 (Coverage bound). *In the CRT decomposition $\mathbb{Z}_M \cong \prod \mathbb{Z}_{q_i}$, elements with all CRT components nonzero are never covered by any absorber (each absorber requires at least one component to be zero). This gives at least $\prod(q_i - 1) \ge 2$ uncovered elements, providing a lower bound on the "positive cell" count that compensates any negative contributions at the origin.*

8. Computational verification

- **SAT (CaDiCaL):** UNSAT certified for: $L = 105$ (3-prime, squarefree), $L = 945$ ($3^3 \times 5 \times 7$), $L = 1155$ (4-prime, squarefree), $L = 1575$ ($3^2 \times 5^2 \times 7$), and 17 of 35 four-prime squarefree cases. Symmetry breaking (fixing base residues to 0 via CRT coordinate permutation) reduces $L = 1155$ from 7 minutes to under 2 seconds.
- **Gap scaling (4-prime):** For $L = 3^a \times 5 \times 7 \times 11$ with $a = 1, 2, 3, 4$: the coverage gap is 26.2%, 18.2%, 16.4%, 15.8% respectively (2,000,000+ random assignments per case). The gap grows absolutely ($302 \to 629 \to 1707 \to 4928$) while the ratio converges to $\sim$15%.
- **CRT maximality ($k = 4$, exact):** For $L = 1155$, exact computation over all 2^{15} subsets (no truncation) across 9,000,000+ partition configurations, including exhaustive sweeps over prime pairs $\{3, 5\}$ and $\{3, 7\}$ (4.2M and 4.5M each) plus adversarial hill-climbing. Zero violations; worst-case $\mathrm{NC} = -4$.
- **CRT maximality ($k = 5$, exact):** For $L = 15015$, exact cell-level computation over 28,000+ configurations (random and adversarial). Zero violations; worst-case $\mathrm{NC} = -155$.

- **CRT maximality (general):** NC ≤ 0 verified exhaustively for $k \le 3$ (12,825 configs across four L values), by exact computation for $k = 4$ (9M+ configs), and by exact cell-level computation for $k = 5$ (28K+ configs), plus 23 non-squarefree L values. Zero violations in all tests.
- **Construction search:** Simulated annealing on L up to 363,825 (density 1.33): coverage gap always $\ge 10\%$. No covering found.

9. Discussion

The proof status of our three theorems:

Theorem	Status	R_{eff}	Method
Three-Prime (Thm 4)	Unconditional	2/3	Weight function
Four-Prime (Thm 3)	Unconditional	41/45	Weight function
Five-Prime (98.2%) (Thm 5)	Unconditional	< 1	Weight fn + Bonferroni + forced nc
Five-Prime (1.8%)	Conditional*	1.002	CRT maximality

*The 44 remaining configurations (all with 4–5 primes at exponent ≥ 3) depend on Conjecture 10 at $k = 5$, which extends the proved Lemma 9 ($k \le 3$). The conjecture is verified for 10,000,000+ random assignments with zero violations.

The three-level structure. The Five-Prime proof reveals a hierarchy of bounds:

(1) **Weight function** ($R < 1$): handles 55% of configurations. This is a first-moment bound on composite coverage.
(2) **Bonferroni refinement** ($R_{\text{cop}} < 1$): subtracting coprime pair overlaps (with zero coprime triples for $k = 5$) handles 75%. This uses the IE at level 3.
(3) **Forced collisions** ($R_{\text{eff}} < 1$): the pigeonhole principle at prime 3 forces nc pairs that create coverage redundancy, handling 98.2%.

The irreducible gap of 0.002 arises because only prime 3 contributes forced collisions (its 3 residue classes cannot accommodate 16 divisors without overlap). Primes 5 through 13, with 5 to 13 classes respectively, have enough room to avoid all forced pairs.

Comparison with the primitive certificate. Zeraoulia [7] independently observes that for *primitive* odd coverings (where no modulus divides another), a product-measure LYM inequality gives $S(L) := \sum_{p^a \| L} a/p \ge 1$. This certificate eliminates $\sim 94\%$ of abundant candidates with $v_3(L) = 1$ at scale $L \le 10^{10}$, but becomes ineffective when $v_3(L) \ge 2$. Our results are strictly stronger: Theorems 3 and 4 apply to *all* distinct-moduli coverings (not just primitive ones), handle arbitrary exponents (including $v_3(L) \ge 2$), and provide an unconditional proof rather than a certificate that must be checked per L.

Beyond five primes. For $k = 6$, the subset decomposition fails (3-moduli density exceeds the five-prime gap). Extending to six or more primes requires either a proof of the CRT maximality conjecture (Conjecture 10) for general k, or an adaptation of the distortion method of Balister et al. to non-squarefree moduli.

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