Hodge classes

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Abstract

We show that on a complex projective manifold X, for $\mathbb{G} = \mathbb{R}$ or $\mathbb{Q}$, a class in $H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{G}$ is represented by a particular type of an infinite series of subvarieties.

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1 Introduction

Let X be a complex projective manifold. For $\mathbb{G} = \mathbb{R}$ or $\mathbb{Q}$ and a non-negative $p \leq dim(X)$, a class is called a Hodge class if u is in $H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{G}$. More specifically, a class in $H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{Q}$ is called a rational Hodge class, and a class in $H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{R}$ is called a real Hodge class. In this paper, we prove that the Hodge class is approximated by algebraic cycles in a particular way called an infinite cycle class. We recall the definition (see [4]).

Definition 1.1. *Let X be a compact complex manifold. Let $T_\bullet$ denote the integration current over a chain $\bullet$. Let $\mathbf{M}$ denote a mass of currents, based on an Hermitian metric. A class $u \in H^{2p}(X;\mathbb{R})$ is an infinite cycle class if it is represented by a closed current of an absolutely mass-convergent series of currents as*

$$\sum_{i=1}^{\infty} r_i T_{V_i} \tag{1.1}$$

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where V_i are irreducible subvarieties coupled with real coefficients r_i, and the absolute mass-convergence is defined to be

$$\lim_{N\to\infty}\sum_{i=N}^{N'}|r_i|\mathbf{M}(T_{V_i})=0. \tag{1.2}$$

for any $N'\geq N$. The currents in (1.1) are called infinitely complex cycles. It has positivity if $r_i>0$ for all i.

So, infinite cycle classes form a subspace and those with positivity form a convex cone.

The class is clearly independent of Hermitian metric. Then we show

Main theorem 1.2. *A Hodge class is an infinite cycle class.*

Note: the result combined with the results of

$$https://vixra.org/abs/2603.0069$$

and

$$https://vixra.org/abs/2603.0068$$

solves the Hodge conjecture (the one about the (p,p) classes). The sketched proof is also posted on the same website but with another number:

$$https://vixra.org/abs/2601.0080$$

The proof in this paper is based on the work in [3], in which we have shown that from a positive homology class, a complex analytic cycle with positive coefficients can be extracted, and the remaining part has strictly lower mass. In this paper, since X is projective, the analytic cycle is algebraic. Then we'll continue to extract to show that there is an infinite extraction squeezing the remaining mass to 0. Thus a positive homology class (or cohomology in Poincaré duality) is represented by the infinite sum of algebraic cycles. Since any Hodge class is a difference of two positive classes, we obtain that any Hodge class is an infinite cycle class.

2 Proof

Proof of Main theorem 1.2. The proof has 2 steps.

Step 1. First we assume that the non-zero rational class u is represented by a closed, positive current S_0 of bidegree (p,p). Changing it to homology, the current S_0 also represents a positive homology class $[S_0]_h$ in the homology group

$H_{2k}(X;\mathbb{Q})$ where $k = dim(X) - p$. By Theorem A.1 in appendix there exist an analytic cycle V_1 with positive rational coefficients, and a closed, positive current S_1 of bidimension (k,k) such that for the current T_{V_1}, the homology class $[S_0]_h$ is represented by the current $S_1 + T_{V_1}$, i.e.

$$S_0 = S_1 + T_{V_1} + d\Gamma_1 \tag{2.1}$$

where Γ_1 is a current of dimension $2k+1$. Since X is projective, V_1 is algebraic.

Next we state a fact on a compact Kähler manifold: for a positive current $\mathcal{T}$ of bidimension (k,k),

$$\mathbf{M}(\mathcal{T}) = \mathcal{T}[\frac{\omega^k}{k!}] \tag{2.2}$$

where ω is the Kähler form. This fact has been proved and used at multiple places. For the proof, see Theorem 2.2 and Remark 2.5 in [2].

Continuing from (2.1), we obtain

$$S_0[\frac{\omega^k}{k!}] = S_1[\frac{\omega^k}{k!}] + T_{V_1}[\frac{\omega^k}{k!}] + (d\Gamma_1)[\frac{\omega^k}{k!}]. \tag{2.3}$$

Since V_1 has positive coefficients, T_{V_1} is a positive current. Hence S_0, S_1, T_{V_1} are all positive currents of bidimension (k,k). By the mass formula (2.2), (2.3) can be written as

$$\mathbf{M}(S_0) = \mathbf{M}(S_1) + \mathbf{M}(T_{V_1}).$$

Since T_{V_1} is positive,

$$\mathbf{M}(S_0) > \mathbf{M}(S_1). \tag{2.4}$$

Since S_1 is positive of bidimension (k,k) with rational homology in $H_{2k}(X;\mathbb{Q})$, applying Theorem A.1, we can iterate the decomposition (2.1) for the positive current S_1, then afterwards iterate it for the similar positives currents $S_2, S_3, \cdots$. With finitely many such iterations, we obtain

$$S_0 = S_N + \sum_{i=1}^{N} T_{V_i} + d\Gamma_N$$

where N is a natural number, and V_i are algebraic cycles with positive rational coefficients. Write it as

$$S_N = S_0 - \left(\sum_{i=1}^{N} T_{V_i} + d\Gamma_N\right). \tag{2.5}$$

Then similarly

$$S_N[\frac{\omega^k}{k!}] = S_0[\frac{\omega^k}{k!}] - \left(\sum_{i=1}^{N} T_{V_i}\right)[\frac{\omega^k}{k!}] \tag{2.6}$$

It implies that

$$\mathbf{M}(S_0) = \mathbf{M}(S_N) + \sum_{i=1}^{N} \mathbf{M}(T_{V_i}). \tag{2.7}$$

Since V_i are holomorphic chains with positive rational coefficients, all $\mathbf{M}(T_{V_i})$ are positive. Hence the mass inequality (2.4) is extended to the decreasing sequence

$$\mathbf{M}(S_0) > \mathbf{M}(S_1) > \cdots \mathbf{M}(S_N) > \cdots \tag{2.8}$$

Since the cone of positive currents is closed and $\mathbf{M}(S_0) < +\infty$, the limit $\lim\limits_{N\to\infty} \mathbf{M}(S_N)$ must be zero (otherwise the iteration could continue). By (2.5)

$$\lim_{N\to 0} \left(\sum_{i=1}^{N} T_{V_i} + d\Gamma_N \right) = S_0 \ (\mathit{in\ mass}) \tag{2.9}$$

Notice that by (2.5) and (2.7), it is clear that both currents $\sum\limits_{i=1}^{N} T_{V_i}$ and $d\Gamma_N$ have bounded mass for all N. Thus there are sub-sequences such that

$$\begin{aligned} \lim_{N_j\to\infty} \sum_{i=1}^{N_j} T_{V_i} &= F_\infty \ (\mathit{weakly}) \\ \lim_{N_j\to\infty} d\Gamma_{N_j} &= d\Gamma_\infty \ (\mathit{weakly}). \end{aligned} \tag{2.10}$$

Note: Γ_∞ is not $\lim\limits_{N\to\infty} \Gamma_N$, but it is obtained through the boundaries $\lim\limits_{N_j\to\infty} d\Gamma_{N_j}$. Once those weak limits are well-defined, we obtain the infinite series for currents

$$S_0 = \sum_{i=1}^{\infty} T_{V_i} + d\Gamma_\infty \ (\mathit{in\ mass})$$

where V_i are algebraic cycles with positive rational coefficients, and Γ_∞ is some current of dimension $2k+1$. By the positivity of the currents T_{V_i} and the formula (2.7), the convergence of (2.10) further becomes the absolute mass-convergence. Changing it back to cohomology, the class u is an infinitely (algebraic) cycle class with positivity. This completes the proof of step 1 for weakly positive, rational homology classes.

Step 2. In general, we may write

$$u = \sum_{finite\ j} b_j u_j \tag{2.11}$$

where b_j are real numbers, and u_j are integeral classes of (p,p) type. For each j, we write

$$u_j = a\omega^p + u_j - a\omega^p \tag{2.12}$$

where a is a real number. Since u_j is of (p, p) type, the idea of Demailly in [1], is that for a sufficiently large rational number a, the cohomology class

$$a\omega^p + u_j$$

is represented by a strongly positive current with rational homology. By step 1, the class $a\omega^p + u_j$ is an infinite cycle class. So is u_j.

□

Appendix A Extraction of algebraic cycles

Theorem A.1. *Let X be a compact Kähler manifold. Let $\tau \in H_{2k}(X, \mathbb{Q})$ be weakly positive, i.e. τ is represented by a closed positive current. Then there exist a complex analytic cycle V in X with positive rational coefficients and a closed, positive (k, k) current S in X such that*

$$\tau = [T_V + S] \tag{A.1}$$

in $H_{2k}(X, \mathbb{Q})$, where $[\bullet]$ denotes a homology class.

Proof. It follows from Theorem 2.8, [3]. □

References

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