Euclidean curvatures in the full modular group tessellation of the upper half-plane

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Abstract

The successive action of the generators of the full modular group $SL(2, \mathbb{Z})$ on the fundamental domain produces a tessellation of the upper half-plane $\mathbb{H}$. Each tile is a curved triangle whose boundaries are circular arcs. We will analyze the curvatures of the boundaries from a flat space point of view. All Euclidean curvatures are integer. We will show that these integer curvatures are either odd or multiples of 8, and that every odd number and every multiple of 8 occurs as a curvature.

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1 Introduction

A member γ of the special linear group $SL(2,\mathbb{Z})$ is a 2×2 matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

with a, b, c and $d \in \mathbb{Z}$, and determinant equal to one: $ad - bc = 1$. Each matrix γ is identified with the function $\gamma(z)$ which acts on a complex number z by a linear fractional (Móbius) transformation:

$$\gamma(z) = \frac{az+b}{cz+d}\,. \tag{2}$$

If $z \in \mathbb{H}$, then $\gamma(z) \in \mathbb{H}$.

It is well known that $\mathrm{SL}(2,\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, see for instance [1, 2]. Applying (2) gives

$$S(z) = \frac{-1}{z} \tag{3}$$

and

$$T(z) = z + 1\,. \tag{4}$$

Following common practice we start the tessellation with the standard fundamental domain $\mathcal{F} = \{z \in \mathbb{H} \,:\, |\mathrm{Re}(z)| \le \frac{1}{2},\ |z| \ge 1\}$, see Figure 1.

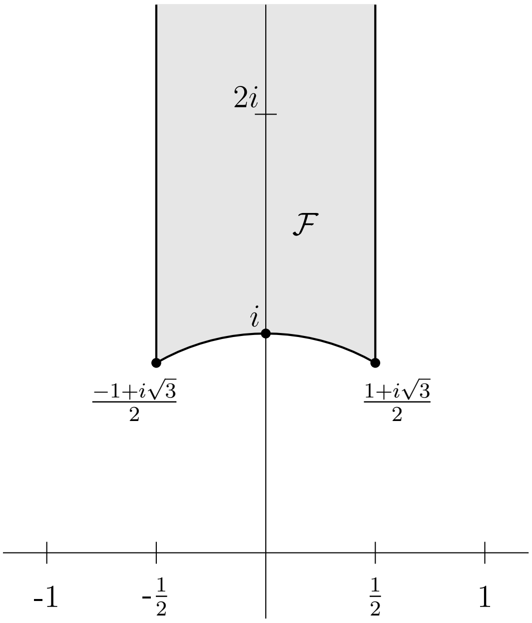

Figure 1: Standard fundamental domain $\mathcal{F}$.

The action of S and T on $\mathcal{F}$ leads to images of $\mathcal{F}$ which are also fundamental domains. In Figure 2 are shown some images of $\mathcal{F}$. Due to lack of space, the '$(\mathcal{F})$' is omitted in many image labels. For example, $TSTS(\mathcal{F})$ is abbreviated to $TSTS$.

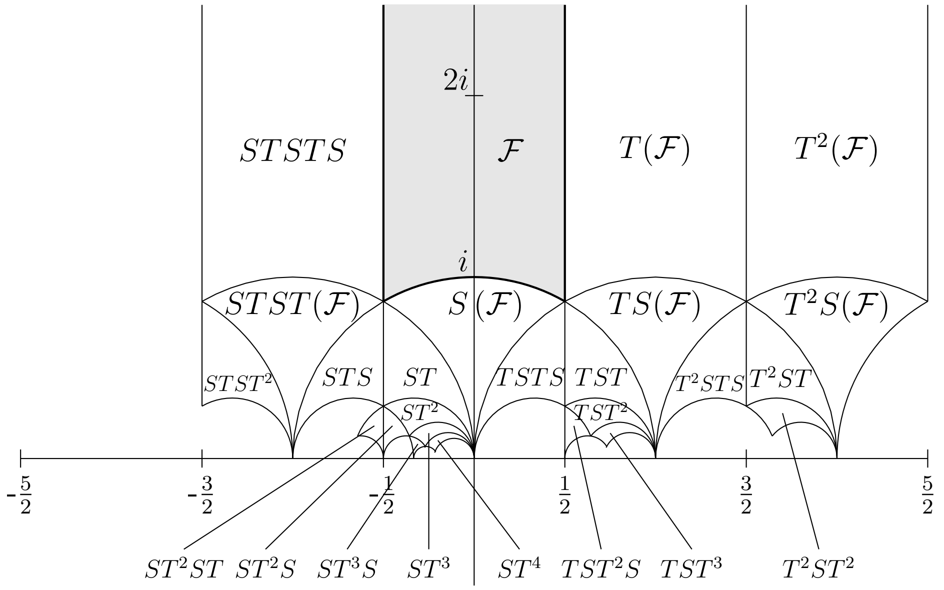

Figure 2: Some images of $\mathcal{F}$.

Extending Figure 2 to all images of $\mathcal{F}$ yields the full tessellation of the upper half-plane. As a consequence of circle inversion, the full tessellation of the upper half-plane consists of semi-circles with their centers and endpoints on the real axis, and with vertical half-lines with one end at the real axis and the other end at infinity. To investigate the curvatures of the boundaries we will analyze the curvatures of the semi-circles.

2 Initial half-line

The right side of $\mathcal{F}$ is a segment of the half-line with one endpoint at $x = \frac{1}{2}$ on the real axis and the other endpoint at infinity. We denote this half-line as $\mathcal{L}$. The images $T^{m_1}(\mathcal{L})$, $m_1 \in \mathbb{Z}$, are half-lines with one endpoint at infinity and the other endpoint at $x = m_1 + \frac{1}{2}$ on the real axis. The Euclidean curvature, denoted as κ, is the reciprocal of the Euclidean radius, denoted as r, which is half the Euclidean distance between the endpoints of a semi-circle or a half-line. All half-lines have zero curvature:

$$\kappa\left(T^{m_1}(\mathcal{L})\right) = 0\,. \tag{5}$$

The images $ST^{m_1}(\mathcal{L})$ are semi-circles with both endpoints on the real axis. One endpoint is at the origin and the x-coordinate of the other endpoint is

$$S\left(m_1 + \frac{1}{2}\right) = \frac{-2}{2m_1 + 1}\,. \tag{6}$$

The semi-circles $ST^{m_1}(\mathcal{L})$ therefore have radius $\left|2m_1+1\right|^{-1}$. The curvature of $ST^{m_1}(\mathcal{L})$ is

$$\kappa\left(ST^{m_1}(\mathcal{L})\right) = \left|2m_1+1\right|. \tag{7}$$

Since $m_1 \in \mathbb{Z}$ it follows that all odd numbers occur as curvatures of $ST^{m_1}(\mathcal{L})$.

Half-line $\mathcal{L}$ and some of its images are shown in Figure 3.

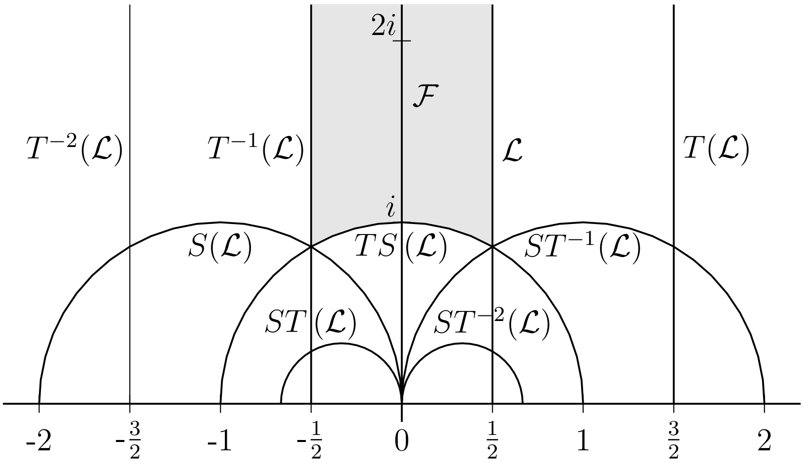

Figure 3: Half-line $\mathcal{L}$ and some of its images.

All images of the half-lines $T^{m_1}(\mathcal{L})$ are either $ST^{m_1}(\mathcal{L})$ or images thereof. The curvatures of $T^{m_1}(\mathcal{L})$ and $ST^{m_1}(\mathcal{L})$ are already known. For the investigation of other curvatures it suffices to restrict to orbits that start at $ST^{m_1}(\mathcal{L})$ and do not pass through a half-line. Alternatively, we only have to investigate the evolution of the curvature when a semi-circle is mapped to a semi-circle.

3 Fractions

From now on we denote the x-coordinate of an endpoint of a semi-circle always as a fraction. The smallest fraction, corresponding to the left endpoint, is denoted as $\dfrac{t}{u}$ and the largest fraction, corresponding to the right endpoint, is denoted as $\dfrac{v}{w}$, where t, u, v and w are integers. Furthermore, a fraction is always written in lowest terms: t and u are co-prime, and v and w are co-prime. A left endpoint with integer x-coordinate is denoted as $\dfrac{t}{1}$ and a right endpoint with integer x-coordinate is denoted as $\dfrac{v}{1}$. The sign of a fraction is determined by its numerator; u and w are positive integers. From the integers t, u, v and w we create an endpoint matrix M as follows

$$M = \begin{pmatrix} v & t \\ w & u \end{pmatrix}. \tag{8}$$

Its determinant is $\det(M) = vu - tw$. The action of S or T on a semi-circle leads to a new semi-circle for which the endpoint matrix is

$$M' = \begin{pmatrix} v' & t' \\ w' & u' \end{pmatrix} , \tag{9}$$

where

$$\frac{t'}{u'} = \min\left(T\left(\frac{t}{u}\right), T\left(\frac{v}{w}\right) \right) \quad , \quad \frac{v'}{w'} = \max\left(T\left(\frac{t}{u}\right), T\left(\frac{v}{w}\right) \right) \tag{10}$$

in case of generator T, and

$$\frac{t'}{u'} = \min\left(S\left(\frac{t}{u}\right), S\left(\frac{v}{w}\right) \right) \quad , \quad \frac{v'}{w'} = \max\left(S\left(\frac{t}{u}\right), S\left(\frac{v}{w}\right) \right) \tag{11}$$

in case of generator S. The determinant of M' is $\det(M') = v'u' - t'w'$. We will show that T and S preserve the determinant of the endpoint matrix. That is, we will show for semi-circles mapped to semi-circles that

$$\det(M') = \det(M) . \tag{12}$$

For the action of S we have three cases:

1. If $0 < t < v$ then $t' = -u$, $u' = t$, $v' = -w$ and $w' = v$. Since $v'u' - t'w' = -wt - -uv = vu - tw$, equation (12) is satisfied. Alternatively, since here $M' = SM$ and $\det(S) = 1$ it follows that equation (12) is satisfied.

2. If $t < v < 0$ then $t' = u$, $u' = -t$, $v' = w$ and $w' = -v$. Since $v'u' - t'w' = w \cdot -t - u \cdot -v = vu - tw$, equation (12) is satisfied. Alternatively, since here $M' = -SM$ and $\det(S) = 1$ it follows that equation (12) is satisfied.

3. If $t < 0 < v$ then $t' = -w$, $u' = v$, $v' = u$ and $w' = -t$. Since $v'u' - t'w' = uv - -w \cdot -t = vu - tw$, equation (12) is satisfied. Alternatively, since here $M' = -SMS$ and $\det(S) = 1$ it follows that equation (12) is satisfied.

For the action of T we have one case.

4. Since both endpoints are on the real axis we have $t' = t + u$, $u' = u$, $v' = v + w$ and $w' = w$. Since $v'u' - t'w' = (v + w)u - (t + u)w = vu - tw$, equation (12) is satisfied. Alternatively, since here $M' = TM$ and $\det(T) = 1$ it follows that equation (12) is satisfied.

In all four cases the determinant is preserved under the action of S as well as T. Its value will therefore be the same for every image. To determine the value of $\det(M)$ it suffices to take just one semi-circle. For instance, semi-circle $TS(\mathcal{L})$ (which is the single member of case 3) has $\frac{-1}{1}$ and $\frac{1}{1}$ as the x-coordinates of its left and right endpoints respectively. Hence,

$$\det(M) = 2\,. \tag{13}$$

For an image with x-coordinate $\frac{t}{u}$ for the left endpoint and x-coordinate $\frac{v}{w}$ for the right endpoint, the radius is given by

$$r = \frac{1}{2}\left(\frac{v}{w} - \frac{t}{u}\right) = \frac{\det(M)}{2uw}\,. \tag{14}$$

Substitution of equation (13) gives

$$r = \frac{1}{uw}\,. \tag{15}$$

As a consequence, the curvature of an image is given by

$$\kappa = uw\,. \tag{16}$$

The latter implies the curvature is always an integer.

4 Even curvatures

Since the fractions $\frac{t}{u}$ and $\frac{v}{w}$ are in their simplest form, at most two of the entries $\{t, u, v, w\}$ are even. Explicitly, equation (13) reads

$$vu - tw = 2\,. \tag{17}$$

The latter rules out the possibility of exactly one even entry. If all four entries are odd then (16) implies that the curvature is odd. To study even curvatures of semi-circles we therefore can restrict to the case in which exactly two entries are even. The subcases in which both v and w are even, or both t and u are even, can be excluded because they violate the assumption of the simplest form. The subcases in which both v and u are even, or both t and w are even, violate equation (17). The subcase in which both v and t are even, yields an odd curvature by (16). Hence, for the investigation of even curvatures it suffices to consider the subcase in which both u and w are even and v and t both odd. To this end we write u and w as $u = 2^\alpha(2A-1)$ and $w = 2^\beta(2B-1)$ respectively, where A, B, α and β are positive integers. Then equation (17) becomes

$$v\,2^\alpha(2A-1) - t\,2^\beta(2B-1) = 2\,. \tag{18}$$

Since the right side of the latter equation is not a multiple of 4, at least one of α, β has to be equal to 1. For the choice $\alpha = \beta = 1$ the equation (18) reduces to

$$v(2A-1) - t(2B-1) = 1\,, \tag{19}$$

which is impossible since the difference of two odd numbers has to be even. Therefore, either $u \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 4$, or $u \equiv 0 \pmod 4$ and $w \equiv 0 \pmod 2$. In both cases, equation (16) implies

$$\kappa \equiv 0 \pmod 8. \tag{20}$$

The remaining question is whether every multiple of 8 does occur as a curvature. To answer this question we consider the images $ST^2ST^{m_1}(\mathcal{L})$ for $m_1 \geq 0$. The x-coordinates of the left and right endpoints of the semi-circles $ST^2ST^{m_1}(\mathcal{L})$ are

$$\frac{t}{u} = \frac{-(2m_1+1)}{4m_1} \tag{21}$$

and

$$\frac{v}{w} = \frac{-1}{2} \tag{22}$$

respectively. That is $t = -(2m_1+1)$, $u = 4m_1$, $v = -1$ and $w = 2$. The curvature of $ST^2ST^{m_1}(\mathcal{L})$ therefore is

$$\kappa\left(ST^2ST^{m_1}(\mathcal{L})\right) = 8m_1\,. \tag{23}$$

The latter implies that every multiple of 8 does occur as a curvature.

5 Remarks

Remark 1: If we allow the value 0 for u and w we can also denote infinity as a fraction: $\infty = \frac{1}{0}$ and $-\infty = \frac{-1}{0}$. If we regard a half-line to run from a point $\frac{2m_1+1}{2}$ on the real axis to ∞, then $M = \begin{pmatrix} 1 & 2m_1+1 \\ 0 & 2 \end{pmatrix}$ and $\det(M) = 2$. If we regard a half-line to run from $-\infty$ to a point $\frac{2m_1+1}{2}$ on the real axis, then $M = \begin{pmatrix} 2m_1+1 & -1 \\ 2 & 0 \end{pmatrix}$ and $\det(M) = 2$. That is, no matter whether we take $-\infty$ or ∞ as an endpoint of a half-line, the determinant satisfies equation (13). Accepting $-\infty$ or ∞ as an endpoint of a half-line would make the restriction to orbits not to pass through a half-line redundant, although it requires to consider separately the action of S for the cases $0 = t < v$, $t < v = 0$ and $\frac{t}{u} = \frac{-1}{1} < \frac{v}{w} = \frac{1}{1}$.

Remark 2: There is some similarity between the $SL(2,\mathbb{Z})$ images of the standard fundamental domain and the Farey diagrams of Farey sequences F_n. The Farey diagrams are $SL(2,\mathbb{Z})$ images of the vertical half-lines with one endpoint at $x = 1$ and the other endpoint at infinity. As a consequence, two adjacent fractions $\frac{v}{w} > \frac{t}{u}$ in a Farey sequence satisfy the relation $vu - tw = 1$ and a semi-circle connecting the fractions $\frac{v}{w}$ and $\frac{t}{u}$ has curvature $2uw$. These values differ by a factor 2 with the values for the images of the half-line $\mathcal{L}$.

6 Summary

The curvatures of the boundaries of images of the fundamental domain of the modular group, satisfy

$$\kappa \equiv 0, 1, 3, 5, 7 \pmod 8). \tag{24}$$

Furthermore all odd numbers and all multiples of 8 can occur as a curvature. In other words, if $\mathbb{N}_0$ denotes the set of positive integers including 0, then the set of odd curvatures is equal to $2\mathbb{N}_0 + 1$ and the set of even curvatures is equal to $8\mathbb{N}_0$.

Explicitly, the ordered set of curvatures of the boundaries of tiles in the $SL(2, \mathbb{Z})$ tessellation of the upper half-plane is given by:
{0, 1, 3, 5, 7, 8, 9, 11, 13, 15, 16, 17, 19, 21, 23, 24, 25, 27, 29, 31, 32, 33, 35, 37, 39, 40, 41, 43, 45, 47, 48, 49, 51, 53, 55, 56, 57, 59, 61, 63, 64, 65, 67, 69, 71, 72, 73, 75, ...}.

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