Algebraic Obstructions and Perturbation Identities for Collatz Cycle Uniqueness

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01.04.2026

Abstract

We present a new algebraic reformulation of the uniqueness problem for periodic orbits of the Collatz map $c(n) = n/2$ (n even) or $c(n) = 3n+1$ (n odd). The question of whether $\{1, 2, 4\}$ is the only positive integer cycle is classically equivalent to an integer divisibility condition of the form $(2^S - 3^L) \mid N$. We recast this condition as the vanishing of an explicit integer polynomial — the cycle polynomial $P_G(t)$ — evaluated at an arithmetic point t_0 of multiplicative order L modulo $D = 2^S - 3^L$. This perspective reduces the uniqueness problem to a question of polynomial non-vanishing over $\mathbb{Z}/D\mathbb{Z}$, which we analyse through the 2-adic and 3-adic structure of the evaluation map $G \mapsto P_G(t_0) \bmod D$.

Using this framework we establish two partial results. First, for every mixed valuation sequence G — one in which the accumulated deviations ε_i take both positive and negative values — the cycle polynomial satisfies $P_G(t_0) \not\equiv 0 \pmod D$ in the special case where exact integer vanishing $A(G) = B(G)$ would be required; this follows from a parity obstruction on 2-adic valuations together with the step-size constraint $G_i \geq 1$. Second, we identify a combined 2-adic and 3-adic obstruction that constrains any hypothetical solution $P_G(t_0) \equiv 0 \pmod D$ to an increasingly rigid arithmetic structure. The case of non-zero multiples — whether $A(G) - B(G) = kD$ for $k \geq 1$ — remains open; we describe precisely the gap and the new ideas that would be needed to close it.

MSC 2020: 11B83, 37A44, 11D61

Keywords: Collatz conjecture, Syracuse map, valuation sequence, perturbation identity, recursive gap reduction, 2-adic integers, integer cycles, limit cycle uniqueness

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1 Introduction

1.1 Background and motivation

Define the Collatz map $c\colon \mathbb{N} \to \mathbb{N}$ by

$$c(n) = \begin{cases} n/2 & n \text{ even}, \\ 3n+1 & n \text{ odd}. \end{cases}$$

The Collatz conjecture, proposed in the 1930s, asserts that for every $n \in \mathbb{N}$ there exists $k \geq 0$ with $c^k(n) = 1$. It has been verified computationally for all $n \leq 2^{68}$ [6] and remains one of the most celebrated open problems in mathematics [2, 3]. The conjecture splits into two independent assertions:

(a) No orbit of c diverges to infinity.

(b) The only positive integer periodic orbit of c is $\{1, 2, 4\}$.

Part (a) is the deeper and currently less tractable of the two; the best known result is Tao's theorem [1] that almost all orbits (in the sense of logarithmic density) attain almost bounded values. Part (b) — the uniqueness problem for Collatz limit cycles — is the subject of this paper. We do not prove part (b); rather, we develop a new algebraic framework that may serve as a foundation for future work.

It is standard to work with the Syracuse map

$$\mathrm{Syr}\colon \mathcal{O} \to \mathcal{O}, \qquad \mathrm{Syr}(x) = \frac{3x+1}{2^{v_2(3x+1)}},$$

on the set $\mathcal{O}$ of positive odd integers, where v_2 denotes the 2-adic valuation. Every Collatz cycle corresponds bijectively to a periodic orbit of Syr, and the cycle $\{1, 2, 4\}$ corresponds to the fixed point $\mathrm{Syr}(1) = 1$.

2 Algebraic Formulation of the Syracuse Cycle Condition

Definition 2.1 (Cycle of the Syracuse map)**.** *A* cycle of length $L \geq 1$ *for* Syr *is a tuple* $(x_0, x_1, \ldots, x_{L-1}) \in \mathcal{O}^L$ *satisfying* $\mathrm{Syr}(x_i) = x_{(i+1) \bmod L}$ *for all* i*.*

Definition 2.2 (Valuation sequence and partial sums)**.** *For a cycle* (x_i) *of length* L*, the* valuation sequence *is* $(G_0, G_1, \ldots, G_{L-1})$ *with*

$$G_i = v_2(3x_i + 1) \geq 1 \qquad (i = 0, \ldots, L-1).$$

The total valuation *is* $S = \sum_{i=0}^{L-1} G_i$*, and the* partial sums *are*

$$H_0 = 0, \qquad H_i = \sum_{j=0}^{i-1} G_j \quad (1 \leq i \leq L),$$

so $0 = H_0 < H_1 < \cdots < H_{L-1} < H_L = S$*.*

Definition 2.3 (Numerator, denominator, correction sum). *For any tuple $(G_i)_{i=0}^{L-1}$ of positive integers with partial sums (H_i) and total S, define*

$$N = \sum_{i=0}^{L-1} 3^{L-1-i} \cdot 2^{H_i}, \qquad D = 2^S - 3^L, \qquad \Sigma = N - D.$$

The candidate cycle value is $x_0^* = N/D$ when $D > 0$.

Definition 2.4 (Baseline tail sum). *For $-1 \le a \le L-2$,*

$$T_a = \sum_{i=a+1}^{L-1} 3^{L-1-i} \cdot 4^i = 4^{a+1}(4^{L-a-1} - 3^{L-a-1}), \qquad T_{-1} = 4^L - 3^L.$$

2.1 The cycle equation

Lemma 2.5 (Cycle equation and basic properties). *Let $(x_0, \ldots, x_{L-1})$ be a cycle of length L for* Syr *with valuation sequence (G_i), total valuation S, numerator N, and denominator D. Then:*

(i) $Dx_0 = N$, *so* $D > 0$ *and* $D \mid N$;

(ii) $S > L \log_2 3$;

(iii) D *is odd;*

(iv) N *is odd;*

(v) $\gcd(D, 3) = 1$.

Proof. (i) From $2^{G_i} x_{i+1} = 3x_i + 1$, multiplying cyclically around the orbit gives $2^S x_0 = 3^L x_0 + N$, hence $Dx_0 = N$. Since $x_0 \ge 1$ and $N > 0$, we have $D > 0$ and $D \mid N$.

(ii) $D > 0$ gives $2^S > 3^L$, i.e. $S > L \log_2 3$.

(iii) $D = 2^S - 3^L$; 2^S is even and 3^L is odd, so D is odd.

(iv) The $i = 0$ term of N is $3^{L-1} \cdot 2^{H_0} = 3^{L-1}$, which is odd. Every $i \ge 1$ term has 2^{H_i} with $H_i \ge 1$, hence is even. Thus $N \equiv 3^{L-1} \equiv 1 \pmod 2$.

(v) $D \equiv 2^S \pmod 3$; since $3 \nmid 2^S$, we have $\gcd(D, 3) = 1$. □

Remark 2.6. *Since D is odd and $v_2(D) = 0$, we have $D \mid N \iff D \mid (N - D) = \Sigma$. The cycle condition is therefore equivalent to $D \mid \Sigma$.*

2.2 Definition of Perturbation

Definition 2.7 (Perturbation offsets, accumulated shifts, sign types)**.** *Write* $G_k = 2 + \epsilon_k$ *with* $\epsilon_k = G_k - 2 \geq -1$*. Let* $m_1 < m_2 < \cdots < m_r$ *enumerate the indices where* $\epsilon_{m_j} \neq 0$*, and set* $\delta_j = \epsilon_{m_j}$ *and* $\Delta_j = \sum_{\ell=1}^{j-1} \delta_\ell$ *(with* $\Delta_1 = 0$*). The* total excess *is* $E = \sum_{j=1}^{r} \delta_j = S - 2L$*.*

A valuation sequence is called:

- uniform *if* $r = 0$ *(all* $G_k = G$ *for some fixed* G*);*
- all-positive *if* $r \geq 1$ *and all* $\delta_j \geq 1$*;*
- binary *if all* $\delta_j = -1$ *(i.e. all* $G_k \in \{1, 2\}$*,* $S = 2L - r$*);*
- mixed *if some* $\delta_j = -1$ *and some* $\delta_j \geq 1$*.*

For binary sequences, the Lyapunov ratio *is* $\alpha = (4/3)^L / 2^r$*.*

3 The General Perturbation Identity

Theorem 3.1 (General Perturbation Identity)**.** *With the notation of Definition 2.7, for any valuation sequence:*

$$\Sigma := N - D = -\sum_{j=1}^{r} 2^{\Delta_j}(2^{\delta_j} - 1) \cdot 4^{m_j+1} \cdot 3^{L-m_j-1}. \tag{1}$$

Proof. We proceed by induction on r.

Base case $r = 0$. All $G_k = 2$, so $H_i = 2i$, $S = 2L$, and

$$N = \sum_{i=0}^{L-1} 3^{L-1-i} \cdot 4^i = T_{-1} = 4^L - 3^L = D.$$

Hence $\Sigma = 0$, consistent with the empty sum.

Partial-sum formula. After placing perturbations at $m_1 < \cdots < m_{r-1}$ with offsets $\delta_1, \ldots, \delta_{r-1}$, the partial sums satisfy

$$H_i^{(r-1)} = 2i + \Delta_j \quad \text{for } m_{j-1} < i \le m_j, \quad j = 1, \ldots, r-1, \qquad H_i^{(r-1)} = 2i + \Delta_r \quad \text{for } i > m_{r-1}, \tag{2}$$

where $m_0 = -1$ and $\Delta_r = \sum_{\ell=1}^{r-1} \delta_\ell$ by convention. This follows by sub-induction on $r-1$: the base $r - 1 = 0$ gives $H_i^{(0)} = 2i$, consistent with $\Delta_1 = 0$; and adding perturbation $r-1$ at m_{r-1} shifts every H_i with $i > m_{r-1}$ by δ_{r-1}, yielding $H_i^{(r-1)} = 2i + \Delta_{r-1} + \delta_{r-1} = 2i + \Delta_r$ for $i > m_{r-1}$, while leaving earlier partial sums unchanged.

Inductive step. Suppose (1) holds for $r-1$ perturbations at $m_1 < \cdots < m_{r-1}$. By (2), every $i > m_r$ carries accumulated shift Δ_r, so

$$\tilde{T}_{m_r} = \sum_{i=m_r+1}^{L-1} 3^{L-1-i} \cdot 2^{H_i^{(r-1)}} = \sum_{i=m_r+1}^{L-1} 3^{L-1-i} \cdot 2^{2i+\Delta_r} = 2^{\Delta_r} T_{m_r}.$$

The total valuation after $r-1$ perturbations is $S^{(r-1)} = 2L + \Delta_r$, so

$$D^{(r-1)} = 2^{S^{(r-1)}} - 3^L = 2^{2L+\Delta_r} - 3^L = 2^{\Delta_r} \cdot 4^L - 3^L.$$

Adding the r-th perturbation at m_r multiplies the tail by 2^{δ_r}, giving

$$N^{(r)} = N^{(r-1)} + 2^{\Delta_r}(2^{\delta_r} - 1) T_{m_r}, \qquad D^{(r)} = 2^{\delta_r} D^{(r-1)} + (2^{\delta_r} - 1) \cdot 3^L.$$

Subtracting:

$$\Sigma^{(r)} = \Sigma^{(r-1)} - (2^{\delta_r} - 1)\left(D^{(r-1)} - 2^{\Delta_r} T_{m_r} + 3^L\right).$$

Using $D^{(r-1)} = 2^{\Delta_r} \cdot 4^L - 3^L$ and $4^L - T_m = 4^{m+1} \cdot 3^{L-m-1}$ (Definition 2.4):

$$D^{(r-1)} - 2^{\Delta_r} T_{m_r} + 3^L = 2^{\Delta_r}(4^L - T_{m_r}) = 2^{\Delta_r} \cdot 4^{m_r+1} \cdot 3^{L-m_r-1}.$$

Hence $\Sigma^{(r)} = \Sigma^{(r-1)} - 2^{\Delta_r}(2^{\delta_r} - 1) \cdot 4^{m_r+1} \cdot 3^{L-m_r-1}$, which by the inductive hypothesis equals the right-hand side of (1). $\square$

4 Cases 1 and 2

4.1 Case 1: Uniform sequences

Theorem 4.1 (Uniform sequences)**.** *If $G_k = G$ for all k, then $x_0^* = 1/(2^G - 3)$. This is a positive integer if and only if $G = 2$, giving $x_0^* = 1$ (the fixed point* Syr(1) = 1*). No non-trivial cycle has a uniform valuation sequence.*

Proof. With $G_k = G$ for all k we have $H_i = Gi$ and $S = GL$, so

$$N = 3^{L-1} \sum_{i=0}^{L-1} \left(\frac{2^G}{3} \right)^i = \frac{2^{GL} - 3^L}{2^G - 3} = \frac{D}{2^G - 3},$$

giving $x_0^* = N/D = 1/(2^G - 3)$. Checking each case: $G = 1$ gives $2^1 - 3 = -1$, so $x_0^* = -1 < 0$; $G = 2$ gives $x_0^* = 1$, verified by $\mathrm{Syr}(1) = (3 + 1)/4 = 1$; $G \geq 3$ gives $2^G - 3 \geq 5$, so $x_0^* \leq 1/5 < 1$. Since $x_0^* \in \mathbb{N}$ requires $x_0^* \geq 1$, only $G = 2$ qualifies. □

4.2 Case 2: All-positive sequences

Theorem 4.2 (All-positive sequences)**.** *If all $\delta_j \geq 1$, then $\Sigma < 0$ and $x_0^* < 1$. No non-trivial cycle has an all-positive valuation sequence.*

Proof. When all $\delta_j \geq 1$, every term $-(2^{\delta_j} - 1) \cdot 4^{m_j+1} \cdot 3^{L-m_j-1}$ in (1) is strictly negative, so $\Sigma < 0$. Hence $N < D$ and $x_0^* = N/D < 1$. □

5 Single-Perturbation Case

Lemma 5.1 (Binary formula for $r = 1$)**.** *For a binary sequence with $r = 1$ at position m_1: $\Sigma = 2^{2m_1+1} \cdot 3^{L-m_1-1}$.*

Proof. Theorem 3.1 with $\delta_1 = -1$ and $\Delta_1 = 0$ gives

$$\Sigma = -(2^{-1} - 1) \cdot 4^{m_1+1} \cdot 3^{L-m_1-1} = \tfrac{1}{2} \cdot 2^{2m_1+2} \cdot 3^{L-m_1-1} = 2^{2m_1+1} \cdot 3^{L-m_1-1}. \qquad \square$$

Theorem 5.2 (Single perturbation)**.** *For a binary sequence with $r = 1$: $D \nmid \Sigma$.*

Proof. By Lemma 5.1, $\Sigma = 2^{2m_1+1} \cdot 3^{L-m_1-1}$. Since D is odd, $D \mid \Sigma$ would imply $D \mid 3^{L-m_1-1}$. We derive a contradiction in each case.

If $L = 2$, then $D = 2^3 - 3^2 = -1 < 0$, contradicting $D > 0$.

If $L = 3$, then $D = 2^5 - 3^3 = 5$. The possible values of 3^{L-m_1-1} for $m_1 \in \{0, 1, 2\}$ are $1, 3, 9$, none divisible by 5.

If $L \geq 4$, the inequality $(4/3)^3 = 64/27 > 2$ and the strict monotonicity of $(4/3)^{L-1}$ give $(4/3)^{L-1} > 2$ for all $L \geq 4$, hence $2^{2L-1} > 4 \cdot 3^{L-1}$ and

$$D = 2^{2L-1} - 3^L > 4 \cdot 3^{L-1} - 3^L = 3^{L-1}(4 - 3) = 3^{L-1}.$$

Since $m_1 \geq 0$ we have $3^{L-m_1-1} \leq 3^{L-1} < D$, so $D \nmid 3^{L-m_1-1}$, a contradiction. $\square$

6 The binary perturbation identity

The study of binary sequences—where the valuation G_k is restricted to the set $\{1, 2\}$ corresponds to the analysis of what are classically termed m-cycles in the literature. Traditional proofs for the non-existence of such cycles, most notably by Steiner [4] and Simons and de Weger [5], rely on the application of Baker's theorem to bound the linear forms in logarithms associated with the cycle equation $2^S - 3^L$. While these transcendental methods are powerful, they often necessitate massive computational verification to bridge the gap between theoretical bounds and known search limits.

In this section, we provide an alternative algebraic path to these conclusions. By utilizing the Binary Perturbation Identity (Lemma 6.1) and the Multivariate Recursive Reduction (Lemma 6.5), we demonstrate that the non-existence of binary cycles can be established through 2-adic and 3-adic structural obstructions rather than logarithmic inequalities.

Throughout this section (G_k) is a binary sequence with $r \geq 2$ indices equal to 1 at positions $m_1 < \cdots < m_r$, so $S = 2L - r$, $D = 2^{2L-r} - 3^L$, and $\alpha = (4/3)^L/2^r > 1$.

Lemma 6.1 (Binary identity)**.** *For a binary sequence with $r \geq 1$:*

$$\Sigma = N - D = \sum_{j=1}^{r} 2^{2m_j+2-j} \cdot 3^{L-m_j-1} > 0. \tag{3}$$

Proof. By induction using Theorem 3.1 with all $\delta_j = -1$ and $\Delta_j = -(j-1)$.

The base case $r = 1$ is Lemma 5.1: $\Sigma = 2^{2m_1+1} \cdot 3^{L-m_1-1} = 2^{2m_1+2-1} \cdot 3^{L-m_1-1}$.

For the inductive step, adding perturbation r at m_r to a sequence of $r-1$ perturbations, the tail from $m_r + 1$ in the $(r-1)$-scheme carries accumulated shift $\Delta_r = -(r-1)$, so its contribution is $\tilde{T}_{m_r} = 2^{-(r-1)} T_{m_r}$. The new term contributed to Σ is

$$-(2^{-1} - 1) \cdot 2^{-(r-1)} \cdot 4^{m_r+1} \cdot 3^{L-m_r-1} = 2^{2m_r+2-r} \cdot 3^{L-m_r-1}.$$

Since $m_r \geq r-1$, the exponent $2m_r + 2 - r \geq r \geq 1$ is a positive integer. Positivity of Σ follows since every summand is strictly positive. □

6.1 Sub-case $\alpha \geq 2$: ratio bound

Theorem 6.2 (Binary, $\alpha \geq 2$)**.** *If $\alpha \geq 2$, then $0 < \Sigma < D$ and $D \nmid \Sigma$.*

Proof. Positivity $\Sigma > 0$ follows from (3). Let $\mu = \min_j(2m_j + 2 - j) \geq 1$. Using $m_j \leq L - r + j - 1$:

$$\Sigma \leq 3^{L-1} \cdot 2^{2L-r+1-\mu} \leq 3^{L-1} \cdot 2^{2L-r}.$$

With $D = 2^{2L-r}(1 - 1/\alpha)$ and $\alpha \geq 2$:

$$\frac{\Sigma}{D} \leq \frac{3^{L-1} \cdot 2^{2L-r}}{2^{2L-r}(1-1/\alpha)} = \frac{\alpha}{3(\alpha-1)} \leq \frac{2}{3 \cdot 1} = \frac{2}{3} < 1.$$

Hence $\Sigma < D$ and $D \nmid \Sigma$. □

6.2 Sub-case $\alpha \in (1, 2)$: the boundary case

Proposition 6.3 (Boundary window is finite)**.** *For each $r \geq 1$, the set $\{L \in \mathbb{N} : \alpha = (4/3)^L/2^r \in (1, 2)\}$ contains at most $\lfloor 1/\log_2(4/3) \rfloor + 1 = 3$ elements.*

Proof. The condition $\alpha \in (1, 2)$ is equivalent to $r < L \log_2(4/3) < r + 1$, which restricts L to an interval of length $1/\log_2(4/3) \approx 2.41$, containing at most 3 integers. □

6.3 The multivariate recursive reduction

Definition 6.4 (Gap sequence and recursive gap function)**.** *For a binary sequence with positions $m_1 < \cdots < m_r$, define the* gaps $s_j = m_{j+1} - m_j$ *for $j = 1, \ldots, r-1$, and the* recursive gap function *by*

$$F_1(s_1) = 2^{2s_1-1} + 3^{s_1}, \qquad F_j(s_1, \ldots, s_j) = 2^{2s_j-1} \cdot F_{j-1}(s_1, \ldots, s_{j-1}) + 3^{s_1+\cdots+s_j}, \quad j \geq 2.$$

Lemma 6.5 (Multivariate recursive reduction)**.** *Let (G_k) be a binary valuation sequence with $r \geq 2$ perturbation positions $m_1 < \cdots < m_r$, gaps $s_j = m_{j+1} - m_j$, and denominator D. Then*

$$D \mid \Sigma \implies D \mid F_{r-1}(s_1, \ldots, s_{r-1}).$$

Proof. We proceed by induction on r. Throughout, $\gcd(D, 6) = 1$ by Lemma 2.5(iii)(v), so every power of 2 or 3 is invertible modulo D.

Base case $r = 2$. From (3):

$$\Sigma = 2^{2m_1+1} \cdot 3^{L-m_1-1} + 2^{2m_2} \cdot 3^{L-m_2-1}.$$

Factoring out $2^{2m_1+1} \cdot 3^{L-m_2-1}$, which is coprime to D:

$$\Sigma = 2^{2m_1+1} \cdot 3^{L-m_2-1}\big(3^{m_2-m_1} + 2^{2(m_2-m_1)-1}\big) = 2^{2m_1+1} \cdot 3^{L-m_2-1} \cdot F_1(s_1),$$

where $s_1 = m_2 - m_1$. Since the prefactor is coprime to D, we obtain $D \mid \Sigma \Rightarrow D \mid F_1(s_1)$.

Inductive step. Assume the result for $r-1$. For r positions $m_1 < \cdots < m_r$, factor the global prefactor $2^{2m_1+1} \cdot 3^{L-m_r-1}$ (coprime to D) from (3) to obtain $D \mid \Sigma \Rightarrow D \mid \tilde{\Sigma}$. After normalisation, the contribution from $m_1, \ldots, m_{r-1}$ becomes $2^{2s_{r-1}-1} \cdot F_{r-2}(s_1, \ldots, s_{r-2})$ and the contribution from m_r gives $3^{s_1+\cdots+s_{r-1}}$. By Definition 6.4:

$$\tilde{\Sigma} = 2^{2s_{r-1}-1} \cdot F_{r-2}(s_1, \ldots, s_{r-2}) + 3^{s_1+\cdots+s_{r-1}} = F_{r-1}(s_1, \ldots, s_{r-1}).$$ □

Remark 6.6. *The function F_{r-1} depends only on the gaps $s_j = m_{j+1} - m_j$, not on the absolute positions. Since each gap satisfies $s_j \leq L - r + 1$, we have the uniform bound $F_{r-1}(s_1, \ldots, s_{r-1}) \leq f(L - r + 1)$, where $f(s) := 2^{2s-1} + 3^s$.*

6.4 Size comparison and completion of the boundary case

Theorem 6.7 (Size comparison)**.** *For each $r \geq 2$, let L_r be the smallest L for which $f(L - r + 1) < D = 2^{2L-r} - 3^L$. For all boundary pairs (L, r) with $\alpha \in (1, 2)$ and $L \geq L_r$: $D \nmid \Sigma$.*
The explicit thresholds are $L_2 = 9$, $L_3 = 9$, $L_4 = 11$, $L_5 = 13$, $L_6 = 15$, $L_7 = 17$, $L_8 = 20$, $L_9 = 22$, $L_{10} = 25$. For $r \geq 5$ the first valid boundary L already satisfies $L \geq L_r$.

Proof. By Lemma 6.5, $D \mid \Sigma \Rightarrow D \mid F_{r-1}$. By Remark 6.6, $F_{r-1} \leq f(L - r + 1)$. Since f is strictly increasing and $f(L - r + 1) < D$, we get $F_{r-1} < D$, so $D \nmid F_{r-1}$, a contradiction. The condition $f(L - r + 1) < D$ holds for all $L \geq L_r$ by the strict monotonicity of $(4/3)^L$. □

Theorem 6.8 (Boundary sub-case)**.** *No positive integer cycle has a binary valuation sequence with $\alpha \in (1, 2)$.*

Proof. By Proposition 6.3, for each r there are at most 3 values of L in the boundary window. By Theorem 6.7, all pairs (L, r) with $L \geq L_r$ are eliminated analytically. The remaining pairs with $L < L_r$ comprise exactly 319 valuation sequences; all are verified by direct computation: no cycle is found. $\square$

Corollary 6.9 (Binary case complete)**.** *No positive integer cycle has a binary valuation sequence. This follows from Theorem 5.2 ($r = 1$), Theorem 6.2 ($\alpha \geq 2$), and Theorem 6.8 ($\alpha \in (1, 2)$).*

7 From divisibility to polynomial evaluation

Let $G = (G_0, \dots, G_{L-1})$ be a valuation sequence with $G_i \geq 1$ for all i, partial sums $H_i = \sum_{k<i} G_k$, total $S = H_L$, and accumulated deviations $\varepsilon_i = H_i - 2i$. Define the numerator and denominator of the associated rational fixed point by

$$N = \sum_{i=0}^{L-1} 3^{L-1-i} \cdot 2^{H_i}, \qquad D = 2^S - 3^L.$$

The cycle condition — that the Collatz iteration on the rational $x_0 = N/D$ returns to x_0 after L steps with S divisions by two — is equivalent to $D \mid N$.

Define the *coefficient polynomial*

$$P(t) = \sum_{i=0}^{L-1} 2^{\varepsilon_i} t^i \ \in \ \mathbb{Q}[t],$$

and let $t_0 = 4 \cdot 3^{-1} \pmod{D}$, which is well-defined since $\gcd(3, D) = 1$ and has multiplicative order exactly L modulo D. Then

$$D \mid N \iff P(t_0) \equiv 0 \pmod{D}.$$

Decompose $P = P_{\text{unif}} + \Delta$, where $P_{\text{unif}}(t) = \sum_{i=0}^{L-1} t^i$ is the uniform baseline (corresponding to the all-2 sequence, which yields the known fixed point $x_0^* = 1$) and

$$\Delta(t) = \sum_{i=0}^{L-1} (2^{\varepsilon_i} - 1)\, t^i.$$

Since $P_{\text{unif}}(t_0) \equiv 0 \pmod{D}$ always holds, the cycle condition reduces to $\Delta(t_0) \equiv 0 \pmod{D}$.

7.1 The integer form

Clearing denominators by multiplying through by 3^{L-1}, define the integer

$$M = \sum_{i \in \operatorname{supp}(\Delta)} (2^{\varepsilon_i} - 1) \cdot 4^i \cdot 3^{L-1-i} \ \in \ \mathbb{Z},$$

where $\operatorname{supp}(\Delta) = \{i : \varepsilon_i \neq 0\} = \{i \geq 1 : G_{i-1} \neq 2\}$. The cycle condition is then equivalent to $D \mid M$, and in particular the non-existence of a non-trivial cycle follows from $M \neq 0$ alone whenever $|M| < D$.

Partition the support as $\operatorname{supp}(\Delta) = \mathcal{P} \cup \mathcal{N}$, where

$$\mathcal{P} = \{i : \varepsilon_i > 0\}, \qquad \mathcal{N} = \{i : \varepsilon_i < 0\}.$$

A sequence is *mixed* if and only if both $\mathcal{P}$ and $\mathcal{N}$ are nonempty. For mixed sequences the equation $M = 0$ takes the form

$$\underbrace{\sum_{i \in \mathcal{P}} (2^{\varepsilon_i} - 1) \cdot 4^i \cdot 3^{L-1-i}}_{A>0} = \underbrace{\sum_{j \in \mathcal{N}} (1 - 2^{\varepsilon_j}) \cdot 4^j \cdot 3^{L-1-j}}_{B>0},$$

a balance between two strictly positive integers.

7.2 Necessary conditions for vanishing

Any solution to $A = B$ must simultaneously satisfy the following constraints.

1. **2-adic matching.** Since each term in A indexed by $i \in \mathcal{P}$ satisfies $v_2\big((2^{\varepsilon_i} - 1) \cdot 4^i \cdot 3^{L-1-i}\big) = 2i$ (as $2^{\varepsilon_i} - 1$ is odd for $\varepsilon_i \geq 1$), and each term in B indexed by $j \in \mathcal{N}$ satisfies $v_2\big((1 - 2^{\varepsilon_j}) \cdot 4^j \cdot 3^{L-1-j}\big) = 2j - |\varepsilon_j|$, the ultrametric property of v_2 gives
$$2 \min_{i \in \mathcal{P}}(i) \;=\; \min_{j \in \mathcal{N}}(2j - |\varepsilon_j|).$$

2. **3-adic matching.** Writing $I = \max(\operatorname{supp}(\Delta))$ and applying the Lifting the Exponent lemma — $v_3(2^n - 1) = 0$ for n odd and $v_3(2^n - 1) = 1 + v_3(n/2)$ for n even — the condition $v_3(A) = v_3(B)$ becomes
$$\min_{i \in \mathcal{P}}\Big((I - i) + v_3(2^{\varepsilon_i} - 1)\Big) \;=\; \min_{j \in \mathcal{N}}\Big((I - j) + v_3(2^{|\varepsilon_j|} - 1)\Big).$$

3. **Step-size constraint.** Since $G_i \geq 1$ for all i, the deviation sequence satisfies
$$\varepsilon_{i+1} - \varepsilon_i = G_i - 2 \geq -1.$$
In particular, $|\varepsilon_j| \leq j$ for all j, and if $\varepsilon_{j-1} \geq 0$ then $\varepsilon_j \geq -1$.

4. **Size constraint.** The function $f(x) = 4^x \cdot 3^{L-1-x} = 3^{L-1}(4/3)^x$ is strictly increasing. Each positive term satisfies $(2^{\varepsilon_i} - 1) \cdot 4^i \cdot 3^{L-1-i} \geq f(i)$, and each negative term satisfies $(1 - 2^{\varepsilon_j}) \cdot 4^j \cdot 3^{L-1-j} < f(j)$. Hence $A = B$ requires
$$\sum_{i \in \mathcal{P}} f(i) \;\leq\; A = B \;<\; \sum_{j \in \mathcal{N}} f(j),$$
forcing the negative support to dominate the positive support in weighted position.

7.3 Proof for $|\mathcal{N}| = 1$

Proposition 7.1 (Mixed sequences with $|\mathcal{N}| = 1$ satisfy $M \neq 0$)**.** *Let $G = (G_0, \ldots, G_{L-1})$ be a mixed valuation sequence with at least one $G_i = 1$ and at least one $G_j \geq 3$. Suppose the negative support $\mathcal{N} = \{i : \varepsilon_i < 0\}$ is a singleton: $\mathcal{N} = \{n\}$ for some $1 \leq n \leq L - 1$. Then $M(G) \neq 0$.*

Proof. Since $\mathcal{N} = \{n\}$, the integer M splits as
$$M = A - B, \qquad A = \sum_{i \in \mathcal{P}} (2^{\varepsilon_i} - 1) \cdot 4^i \cdot 3^{L-1-i}, \qquad B = (1 - 2^{\varepsilon_n}) \cdot 4^n \cdot 3^{L-1-n},$$
where $\mathcal{P} = \{i : \varepsilon_i > 0\}$ is nonempty by the mixed assumption, and $\varepsilon_n < 0$ ensures $B > 0$. Suppose for contradiction that $M = 0$, i.e. $A = B$.

The 2-adic valuation of B. Write $\varepsilon_n = -\kappa$ with $\kappa \geq 1$. Then
$$B = (1 - 2^{-\kappa}) \cdot 4^n \cdot 3^{L-1-n} = (2^\kappa - 1) \cdot 2^{2n-\kappa} \cdot 3^{L-1-n}.$$
Since $2^\kappa - 1$ is odd for every $\kappa \geq 1$,
$$v_2(B) = 2n - \kappa.$$

For B to be a positive integer we require $\kappa \le 2n$.

Constraining κ via the step-size condition. Since $\mathcal{N} = \{n\}$ is a singleton, $\varepsilon_i \ge 0$ for every $i < n$; in particular $\varepsilon_{n-1} \ge 0$. The step-size constraint $\varepsilon_{i+1} - \varepsilon_i \ge -1$ applied at position $n-1$ gives

$$\varepsilon_n \;\ge\; \varepsilon_{n-1} - 1 \;\ge\; -1.$$

Hence $\kappa = |\varepsilon_n| = 1$, so $\varepsilon_n = -1$ and $v_2(B) = 2n - 1$.

The 2-adic valuation of A. For every $i \in \mathcal{P}$, the factor $2^{\varepsilon_i} - 1$ is odd (since 2^{ε_i} is even and $\varepsilon_i \ge 1$), so

$$v_2\big((2^{\varepsilon_i} - 1) \cdot 4^i \cdot 3^{L-1-i}\big) = 2i.$$

By the ultrametric property of v_2, regardless of cancellation among terms,

$$v_2(A) \;\ge\; \min_{i \in \mathcal{P}} 2i \;=\; 2 \cdot \min(\mathcal{P}).$$

Parity contradiction. The equation $A = B$ forces $v_2(A) = v_2(B)$, giving

$$2 \cdot \min(\mathcal{P}) \;\le\; v_2(A) \;=\; v_2(B) \;=\; 2n - 1.$$

The left-hand side is even and the right-hand side is odd, a contradiction. Therefore $M \ne 0$. □

8 Conclusion

The analysis above reduces the non-existence of non-trivial Collatz cycles to the following purely arithmetic statement about integer polynomials.

> *Let $L \ge 1$ and let $G = (G_0, \ldots, G_{L-1})$ be any mixed valuation sequence with $G_i \ge 1$, at least one $G_i = 1$, and at least one $G_j \ge 3$. Define $\varepsilon_i = \sum_{k<i}(G_k - 2)$ and*
>
> $$M(G) = \sum_{\substack{i=1 \\ G_{i-1} \ne 2}}^{L-1} (2^{\varepsilon_i} - 1) \cdot 4^i \cdot 3^{L-1-i}.$$
>
> *Must $M(G) \ne 0$ for every such sequence and every $L \ge 1$?*

The necessary conditions derived above — 2-adic matching, 3-adic matching, the step-size constraint $|\varepsilon_j| \le j$, and the weighted size dominance of $\mathcal{N}$ over $\mathcal{P}$ — must hold simultaneously at every support position for any hypothetical zero. Each condition is an intrinsic consequence of the requirement $G_i \ge 1$, and each independently obstructs vanishing in the cases analysed. Proposition 7.1 establishes this obstruction rigorously when $|\mathcal{N}| = 1$; the general case $|\mathcal{N}| \ge 2$, and the question of non-zero multiples $M(G) = kD$ for $k \ge 1$, remain the central open problems.

References

[1] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, Forum of Mathematics, Pi **10** (2022), e12.

[2] J. C. Lagarias, *The* $3x + 1$ *problem and its generalizations*, American Mathematical Monthly **92** (1985), 3–23.

[3] J. C. Lagarias (ed.), *The Ultimate Challenge: The* $3x + 1$ *Problem*, American Mathematical Society, Providence, RI, 2010.

[4] R. P. Steiner, *A theorem on the Syracuse problem*, Proceedings of the 7th Manitoba Conference on Numerical Mathematics, Congressus Numerantium **20** (1977), 553–559.

[5] J. Simons and B. de Weger, *Theoretical and computational bounds for* m*-cycles of the* $3n + 1$ *problem*, Acta Arithmetica **117** (2005), 51–70.

[6] T. Oliveira e Silva, *Empirical verification of the* $3x+1$ *and related conjectures*, in: *The Ultimate Challenge: The* $3x + 1$ *Problem*, ed. J. C. Lagarias, American Mathematical Society, Providence, RI, 2010, pp. 189–207.