Some variations of the secretary problem

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Abstract

We consider two variations of the classical secretary problem.

- A variation of the returning secretary problem where each interviewee may appear a second time with a fixed probability p. The decision-maker observes interviewees sequentially and must choose whether to accept or reject each appearance. We characterize the optimal threshold rule and examine its dependence on the reappearance probability p, highlighting how additional information from repeated appearances improves selection performance.
- A variation of the secretary problem in which success is defined as selecting any one of the top three interviewees rather than the single best. Interviewees are observed sequentially in random order, and decisions are irreversible. We estimated the success probability under this relaxed success criterion using the threshold strategy of the classical secretary problem. The results show that allowing selection among the top three significantly increases the success probability and shifts the optimal stopping threshold earlier than in the classical problem. This model provides insight into realistic decision-making scenarios where top interviewees are more or less similar.

1 Introduction

The secretary problem is a problem in optimal stopping theory[1, 2] and also a classic example of the random-arrival model[3]. In this problem, there are n interviewees for the secretary position. These interviewees arrive in random order and are interviewed sequentially. We know (or can remember) the best interviewee interviewed so far. However, the employer is unaware of the interviewees he has not yet seen. Moreover, immediately after the interview, the employer either has to select or reject the interviewee. An interviewee once rejected cannot be reconsidered. We are to determine a strategy such that the best interviewee is selected with maximum probability[2].

This problem has been studied extensively in the fields of applied probability, statistics, and decision theory. This problem has been extensively studied and the best shortest proof so far is by the odds algorithm[4]. We basically reject approximately the first n/e (cutoff value) interviewees and select the first one who is better than those already seen[5].

This classical problem has many variants, such as:

- K-choice secretary problem[6].
- The knapsack secretary problem[7].
- The matroid secretary problem[8].
- Returning secretary problem[9, 10, 11].
- The best or worst problem[12]

1.1 Problem Statement

Let N distinct interviewees be totally ordered by a strict ranking, with ranks $\{1, 2, \ldots, N\}$, where rank 1 is the best. Interviewees arrive randomly over time according to a uniform distribution.

Each interviewee arrives at least once. At each appearance, the decision-maker observes only the relative rank of the appearing interviewee among all interviewees observed so far (across all appearances), but not the interviewee's absolute rank nor whether the interviewee will reappear again in the future.

After observing an appearance, the decision-maker must immediately either *accept* the interviewee (terminating the process) or *reject* the appearance. Rejection of an interviewee at his/her last appearance permanently eliminates that interviewee from future consideration. The goal is to maximize the probability of selecting the rank one (the best) interviewee.

A stopping rule is a (possibly randomized) policy generated by the observed relative ranks. The problem is to characterize an optimal stopping rule and the corresponding maximum success probability.

1.2 Assumptions and terminologies

Let us define some terminology,

- An interviewee who is better than all the interviewees who have been interviewed so far is called *leading*.
- Interviewee selected following the proposed strategy is called *chosen*.

2 Technical results

Here, we summarize some technical results that we have used for analysis throughout the paper. These results are mainly from Ribas [11].

Theorem 2.1. *Ribas[Proposition-1][11]: Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of real functions with F_n defined by integer values in $\{0, ..., n\}$ and let $M(n)$ be the value for which the function F_n reaches its maximum. Assume that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined by $f_n(x) := F_n(\lfloor nx \rfloor)$, converges uniformly on $[0, 1]$ to f continuous in $[0, 1]$ and that θ is the only global maximum of f in $[0, 1]$. Then,*

1. $\lim\limits_{n} \frac{M(n)}{n} = \theta$
2. $\lim\limits_{n} F_n(M(n)) = f(\theta)$
3. *If* $\mathfrak{M} \sim M(n)$ *then* $\lim\limits_{n} F_n(\mathfrak{M}(n)) = f(\theta)$

Theorem 2.2. *Ribas[Theorem-1][11]: Consider the sequences of functions $\{F_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ and $\{H_n\}_{n\in\mathbb{N}}$ with $F_n, G_n, H_n : [0, n] \cap \mathbb{Z} \to \mathbb{R}$ and defined recursively by the conditions:*

$$F_n(k) = G_n(k) + H_n(k)F_n(k-1) \ \ and \ \ F_n(0) = \mu$$

Also for all x, $0 \leq x \leq 1$, Let $f_n(x) := F_n(\lfloor nx \rfloor)$, $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$. If the following conditions hold:

- *Both $h_n(x)$ and $g_n(x)$ converges on $(0, 1)$ and uniformly on a $[\epsilon, \epsilon']$ for all $0 < \epsilon < \epsilon' < 1$ to continuous functions in $(0, 1)$, $h(x)$ and $g(x)$, respectively.*
- *$f_n(x)$ converges uniformly in $[0, 1]$ to a continuous function $f(x)$.*

Then, $f(0) = \mu$ and f satisfy in $(0, 1)$,

$$f'(x) = -f(x)h(x) + g(x)$$

Theorem 2.3. *Ribas[Theorem-2][11]: Consider the sequences of functions $\{F_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ and $\{H_n\}_{n\in\mathbb{N}}$ with $F_n, G_n, H_n : [0, n] \cap \mathbb{Z} \to \mathbb{R}$ and defined recursively by the conditions:*

$$F_n(k) = G_n(k) + H_n(k)F_n(k+1) \ \ and \ \ F_n(n) = \mu$$

Also for all x, $0 \leq x \leq 1$, Let $f_n(x) := F_n(\lfloor nx \rfloor)$, $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$. If the following conditions hold:

- *Both $h_n(x)$ and $g_n(x)$ converges on $(0, 1)$ and uniformly on a $[\epsilon, \epsilon']$ for all $0 < \epsilon < \epsilon' < 1$ to continuous functions in $(0, 1)$, $h(x)$ and $g(x)$, respectively.*
- *$f_n(x)$ converges uniformly in $[0, 1]$ to a continuous function $f(x)$.*

Then, $f(1) = \mu$ and f satisfy in $(0, 1)$,

$$f'(x) = f(x)h(x) - g(x)$$

Here $f'(x)$ is the derivative of $f(x)$.

3 Secretary Problem with probabilistic second arrival

In this case, each interviewee arrives at least once. After his/her first appearance, interviewee i may appear a second time with probability $p \in [0, 1]$. No interviewee arrives more than twice.

Note that:

1. If k distinct interviewees are interviewed, and the employer has to select the leading interviewee from them, then the probability that the chosen interviewee is the overall best is $\frac{k}{n}$, as in the secretary problem[11].
2. If the interviewee is guaranteed to appear a second time, it is always preferable for an employer not to accept an interviewee on his first arrival, as other better interviewees might appear before selection, which will increase the probability of success.
3. When an interviewee who is inferior to the best interviewee interviewed so far arrives for the interview, it is irrelevant for the employer to interview him. Thus, he/she can be rejected directly [11].
4. The only relevant information to select the overall best interviewee with maximum probability is the number of distinct interviewees has been interviewed so far and the number of times the interviewer has interviewed the leading interviewee.
5. Therefore, after each interview, the employer can directly reject all interviewees who have already been interviewed and are inferior to the best interviewee so far [11].

Remark. Instead of rejecting inferior interviewees, we can model this by saying that all his/her appearances have already been interviewed. For example, suppose an employer has interviewed k distinct interviewees, in which the leading interviewee has appeared only once. In that case, we can consider that all non-leading interviewees in k, i.e., $(k - 1)$, have exhausted all their appearances. So now the employer has taken approximately $(p + 1)(k - 1) + 1$ interviews from $(1 + p)n$ interviews.

3.1 Proposed strategy

In Observation Phase, we interview k different interviewees, reject everyone but note down the leading among them, and the number of times he/she has appeared for the interview.

Then comes the Selection Phase. After interviewing k distinct interviewees, select the first interviewee that satisfies any of the following criteria in sequential order:

1. S/he is the leading interviewee and s/he arrives again.
2. A better (better interviewee than all the previously seen interviewees) interviewee arrives for the first time, select him/her with probability $(1 - p)$. Note that he/she can appear again with probability p.
3. A better (better interviewee than all the previously seen interviewees) interviewee arrives a second time, select him/her.

3.2 Probability of success for probabilistic second arrival

The probability of success following the proposed strategy is the probability that the chosen interviewee is the overall best, assuming we have rejected k distinct interviewees.
The probability of success can be calculated by combining the probability that the chosen interviewee is overall best, with respect to two mutually exclusive and exhaustive events:

- χ_1: Chosen interviewee is the overall best, assuming we have rejected k distinct interviewees and the leading interviewee has appeared only once.
- χ_2: Chosen interviewee is the overall best, assuming we have rejected k distinct interviewees and the leading interviewee has appeared twice.

Let us now introduce some notations. This is similar to that in [11]. Let

- $\Phi_n(k)$: Probability that the chosen interviewee is overall best, assuming we have rejected k distinct interviewees and the leading interviewee has appeared only once. This is the probability of event χ_1.
- $\Psi_n(k)$: Probability that the chosen interviewee is overall best, assuming we have rejected k distinct interviewees and the leading interviewee has appeared twice. This is the probability of event χ_2.

- $\Upsilon_n(k)$: Probability that the leading interviewee has appeared only once when the k^{th} distinct interviewee makes his/her initial appearance.
- $F_n(k)$: Probability that the chosen interviewee is overall best, assuming we have rejected k distinct interviewees.

3.2.1 Calculation of $\Phi_n(k)$

We calculate the value of $\Phi_n(k)$ as follows:

Proposition 3.1. *For all natural numbers n and k where $k < n$, we have,*

$$\Phi_n(k) = \frac{pak + (1-p)(1-pa)}{n} + \frac{(p+k)(1-pa)}{k+1}\Phi_n(k+1),$$
$$where \;\; a = \frac{1}{(1+p)(n-k)+1}$$

and $\Phi_n(n) = p$.

Proof. By hypothesis, when all n distinct interviewees have been interviewed, then leading is overall best, and he/she has only appeared once, so he/she may appear again with probability p. So $\Phi_n(n) = p$,
Now, for the case when $k < n$, the remaining interviews can be categorized into two mutually exclusive events:

1. *E: The leading interviewee will appear again*
   As each interviewee can appear a second time with probability p, the probability that the leading interviewee will appear again is
   $$P(E) = p$$
   Let $P(S_1)$ be $\Phi_n(k)|E$ i.e, probability obtained with respect to event E.
   For the next interviewee that arrives, the following scenarios are possible:

   Case-1: *X_1: Next interviewee is the second appearance of the leading interviewee.*
   - As we have observed k distinct interviewees out of n distinct interviewees, the number of remaining distinct interviewees is $n - k$. Since each of these interviewees can appear $(1 + p)$ times, and the second appearance of the current leading interviewee is also yet to occur, the total number of interviewees yet to be seen is $(1 + p)(n - k) + 1$. So, the probability that the next interviewee is the second appearance of the leading interviewee is
   $$P(X_1) = \frac{1}{(1+p)(n-k)+1}$$
   - And according to the proposed strategy, this interviewee will be our chosen interviewee, so the probability that the chosen interviewee is overall best in this case is $\frac{k}{n}$ [Page-3, Point-1][3]. Therefore,
   $$P(S_1|X_1) = \frac{k}{n}$$

   Case-2: *X_2: Next interviewee is a new leading interviewee.*
   - For the next interviewee to be a new leading interviewee, the following two events should happen together:
     (a) Next interviewee should not be the second appearance of the leading interviewee, which happens with probability $(1 - P(X_1))$.
     (b) Next interviewee should be the new leading interviewee. As the leading interviewee will be only one among the $k + 1$ distinct interviewees interviewed so far, the probability of this happening is $(\frac{1}{k+1})$

     The total probability is the product of the probabilities of both events, i.e.
   $$P(X_2) = \left(1 - \frac{1}{(1+p)(n-k)+1}\right)\frac{1}{k+1}$$
   - And according to the proposed strategy, we will accept this interviewee with probability $1 - p$ and reject him/her with probability p. Once rejected, the interview process will continue(but we have seen $k + 1$ different candidates). So the probability $P(S_1)$ with respect to event X_2 is.
   $$P(S_1|X_2) = (1-p)\left(\frac{k+1}{n}\right) + p\left(\Phi_n(k+1)\right)$$

Case-3: X_3*: Next interviewee is a new non-leading interviewee.*

- For the next interviewee to be a new non-leading interviewee, the following two events should happen together:
  (a) Next interviewee should not be the second appearance of the leading interviewee, which happens with probability $(1 - P(X_1))$.
  (b) Next interviewee should be a new non-leading interviewee. As there are k non-leading interviewees among the $k+1$ distinct interviewees interviewed so far. Therefore the probability is $\left(\frac{k}{k+1}\right)$

  The total probability is the product of the probabilities of the above two events together, i.e.

$$P(X_3) = \left(1 - \frac{1}{(1+p)(n-k)+1}\right) \frac{k}{k+1}$$

- And according to the proposed strategy, we will always reject this interviewee. So the success probability remains the same, but the number of distinct interviewees interviewed has been increased by one. Thus, the probability is.

$$P(S_1|X_3) = \Phi_n(k+1)$$

By the rule of total probability,

$$P(S_1) = P(S_1|X_1)P(X_1) + P(S_1|X_2)P(X_2) + P(S_1|X_3)P(X_3)$$

$$\begin{aligned} P(S_1) &= \left(\frac{k}{n}a\right) + \left((1-p)\left(\frac{k+1}{n}\right) + p\left(\Phi_n(k+1)\right)\right)\left((1-a)\frac{1}{k+1}\right) \\ &+ \Phi_n(k+1)\left((1-a)\frac{k}{k+1}\right) \\ &= \frac{ka + (1-a)(1-p)}{n} + \frac{(1-a)(p+k)}{k+1}\Phi_n(k+1) \end{aligned}$$

where $a = \frac{1}{(1+p)(n-k)+1}$.

2. $\bar{E}$: *The leading interviewee will not appear again*
   As each interviewee may appear a second time with probability p, the probability that the leading interviewee will not appear again is

$$P(\bar{E}) = 1 - p$$

Let $P(S_2)$ be $\Phi_n(k)|\bar{E}$ i.e, probability obtained with respect to event $\bar{E}$.
For the next interviewee that arrives following scenarios are possible:

Case-1: Y_1*: Next interviewee is a new leading interviewee.*

- Since the current leading interviewee cannot appear again, the next interviewee must necessarily be distinct. Among $k+1$ distinct interviewees observed so far, only one can be the leading interviewee. Therefore, the probability that the next interviewee is a new leading interviewee is

$$P(Y_1) = \frac{1}{k+1}$$

- And according to the proposed strategy, we will accept this interviewee with probability $1-p$ and reject him/her with probability p. Once rejected, the interview process will continue for the next interviewee. Therefore, the probability $P(S_2)$ with respect to event Y_1 is

$$P(S_2|Y_1) = (1-p)\left(\frac{k+1}{n}\right) + p\Phi_n(k+1)$$

  This is similar to $P(S_1|X_2)$

Case-2: Y_2*: Next interviewee is a new non-leading interviewee.*

- Since the current leading interviewee cannot appear again, the next interviewee must necessarily be distinct. Among $k+1$ distinct interviewees observed so far, k distinct interviewees will be non-leading interviewees. Therefore, the probability that the next interviewee is a new non-leading interviewee is

$$P(Y_2) = \frac{k}{k+1}$$

- And according to the proposed strategy, we will always reject this interviewee. As now the interviewer has interviewed $k+1$ distinct interviewees. Therefore, the probability is

$$P(S_2|Y_2) = \Phi_n(k+1)$$

By applying the rule of total conditional probability,

$$P(S_2) = P(S_2|Y_1)P(Y_1) + P(S_2|Y_2)P(Y_2)$$

$$\begin{aligned} P(S_2) &= \left((1-p)\left(\frac{k+1}{n}\right) + p\Phi_n(k+1)\right)\left(\frac{1}{k+1}\right) + \Phi_n(k+1)\left(\frac{k}{k+1}\right) \\ &= \frac{1-p}{n} + \frac{p+k}{k+1}\Phi_n(k+1) \end{aligned}$$

By combining the probabilities from both event E and $\bar{E}$ using the rule of total probability,

$$\Phi_n(k) = P(E)P(S_1) + P(\bar{E})P(S_2)$$

$$\begin{aligned} \Phi_n(k) &= p\left(\frac{ka + (1-a)(1-p)}{n} + \frac{(1-a)(p+k)}{k+1}\Phi_n(k+1)\right) \\ &+ (1-p)\left(\frac{1-p}{n} + \frac{p+k}{k+1}\Phi_n(k+1)\right) \\ &= \frac{pak + (1-p)(1-pa)}{n} + \frac{(p+k)(1-pa)}{k+1}\Phi_n(k+1) \end{aligned}$$

where $a = \frac{1}{(1+p)(n-k)+1}$. □

Proposition 3.2. *The sequence of functions* $\hat{\Phi}_n(x) := \Phi_n(\lfloor nx \rfloor)$ *converges uniformly on* $[0,1]$ *for* $\Phi(1) = p$ *to*

$$\Phi'(x) = \left(\frac{1-p}{x} + \frac{p}{(1+p)(1-x)}\right)\Phi(x) - \left(\frac{px}{(1+p)(1-x)} + (1-p)\right)$$

Proof. From Proposition 3.1 we know that,

$$\Phi_n(k) = G_n(k) + H_n(k)\Phi_n(k+1)$$

where $G_n(k) = \frac{pak+(1-p)(1-pa)}{n}$ and $H_n(k) = \frac{(p+k)(1-pa)}{k+1}$ and $a = \frac{1}{(1+p)(n-k)+1}$.
From Theorem 2.3, where $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$ we have,

$$g_n(x) = n\left(\frac{pa\lfloor nx \rfloor + (1-p)(1-pa)}{n}\right) = pa\lfloor nx \rfloor + (1-p)(1-pa)$$

$$h_n(x) = n\left(1 - \left(\frac{(p+\lfloor nx \rfloor)(1-pa)}{\lfloor nx \rfloor + 1}\right)\right)$$

As $n \to \infty$, $g_n(x) \to g(x)$, $h_n(x) \to h(x)$ and $\lfloor nx \rfloor \approx nx$.

- Let us estimate the value of $g(x)$ where $a = \frac{1}{(1+p)(n-nx)+1}$,

$$\begin{aligned} g(x) &= \lim_{n\to\infty} (panx + (1-p)(1-pa)) \\ &= \lim_{n\to\infty} (panx) + (1-p)\lim_{n\to\infty}(1-pa) \end{aligned}$$

Now let us consider each part separately as $\lim\limits_{n\to\infty}$.

$$\begin{aligned} panx &= \frac{pnx}{(1+p)(n-nx)+1} = \frac{pnx}{n(1+p)(1-x)+1} \\ &= \frac{pnx}{(1+p)n(1-x)}\left(\frac{1}{1+\frac{1}{(1+p)n(1-x)}}\right) \end{aligned}$$

Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{1}{(1+p)n(1-x)}$. We have,

$$\begin{aligned} panx &= \frac{px}{(1+p)(1-x)}\left(1 - \frac{1}{(1+p)n(1-x)} + O(n^{-2})\right) \\ &= \frac{px}{(1+p)(1-x)} + O(n^{-1}) \end{aligned} \tag{1}$$

Let's now estimate $1 - pa$,

$$\begin{aligned} pa &= \frac{p}{(1+p)(n-nx)+1} = \frac{p}{n(1+p)(1-x)+1} \\ &= \frac{p}{(1+p)n(1-x)}\left(\frac{1}{1+\frac{1}{(1+p)n(1-x)}}\right) \end{aligned}$$

Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{1}{(1+p)n(1-x)}$. We have,

$$\begin{aligned} pa &= \frac{p}{(1+p)n(1-x)}\left(1 - \frac{1}{(1+p)n(1-x)} + O(n^{-2})\right) \\ &= \frac{p}{(1+p)n(1-x)} + O(n^{-2}) \\ 1 - pa &= 1 - \frac{p}{(1+p)n(1-x)} + O(n^{-2}) \end{aligned} \tag{2}$$

Substituting values for $(panx)$ and $(1-pa)$ from equation 1 and 2. We get, as $n \to \infty$

$$g(x) = \frac{px}{(1+p)(1-x)} + O(n^{-1}) + (1-p)\left(1 - \frac{p}{(1+p)n(1-x)} + O(n^{-2})\right)$$

As $n \to \infty$ then $O(n^{-1})$ and higher order term vanishes, So,

$$g(x) = \frac{px}{(1+p)(1-x)} + (1-p) \tag{3}$$

- Let us estimate the value of $h(x)$:

$$h(x) = \lim_{n\to\infty} n\left(1 - \left(\frac{p+nx}{nx+1}(1-pa)\right)\right)$$

Let us consider $\lim_{n\to\infty}\left(\frac{p+nx}{nx+1}\right)$.

$$\lim_{n\to\infty}\left(\frac{p+nx}{nx+1}\right) = \left(\frac{p+nx}{nx}\right)\left(\frac{1}{1+\frac{1}{nx}}\right)$$

Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{1}{nx}$. We have,

$$\begin{aligned} \lim_{n\to\infty}\left(\frac{p+nx}{nx+1}\right) &= \left(\frac{p+nx}{nx}\right)\left(1 - \frac{1}{nx} + O(n^{-2})\right) \\ &= \left(\frac{p}{nx}+1\right)\left(1 - \frac{1}{nx} + O(n^{-2})\right) \\ &= \frac{p}{nx} + 1 - \frac{1}{nx} + O(n^{-2}) \\ &= 1 - \frac{1-p}{nx} + O(n^{-2}) \end{aligned} \tag{4}$$

Substituting values from equations 2 and 4. We get,

$$\begin{aligned} h(x) &= n\left(1 - \left(1 - \frac{1-p}{nx} + O(n^{-2})\right)\left(1 - \frac{p}{(1+p)n(1-x)} + O(n^{-2})\right)\right) \\ &= n\left(1 - \left(1 - \frac{1-p}{nx} - \frac{p}{(1+p)n(1-x)} + O(n^{-2})\right)\right) \\ &= n\left(\frac{1-p}{nx} + \frac{p}{(1+p)n(1-x)} + O(n^{-2})\right) \\ &= \left(\frac{1-p}{x} + \frac{p}{(1+p)(1-x)} + O(n^{-1})\right) \end{aligned}$$

As $n \to \infty$ then $O(n^{-1})$ and higher order term vanishes, So,

$$h(x) = \frac{1-p}{x} + \frac{p}{(1+p)(1-x)} \tag{5}$$

Using Theorem 2.3 and Substituting $g(x)$ and $h(x)$ from above equations 3, 5, we get,

$$\Phi'(x) = \left(\frac{1-p}{x} + \frac{p}{(1+p)(1-x)}\right)\Phi(x) - \left(\frac{px}{(1+p)(1-x)} + (1-p)\right)$$

□

3.2.2 Calculation of $\Psi_n(k)$

We calculate the value of $\Psi_n(k)$ as follows:

Proposition 3.3. *For all natural numbers n and k where $k < n$, we have,*

$$\Psi_n(k) = \frac{1-p}{n} + \frac{p}{k+1}\Phi_n(k+1) + \frac{k}{k+1}\Psi_n(k+1)$$

and $\Phi_n(n) = p, \Psi_n(n) = 0$

Proof. By hypothesis, when all n distinct interviewees have been interviewed, then the leading interviewee is the overall best, and he/she has already appeared twice, so he/she will not appear again. Thus, $\Psi_n(n) = 0$,
Now for the general case k, since the leading interviewee has already appeared twice, he/she will not appear again. So the next interviewee interviewed can only be one of the following:

Case-1: Z_1*: Next interviewee is a new leading interviewee.*

- Since the current leading interviewee cannot appear again, the next interviewee must necessarily be distinct. Among $k+1$ distinct interviewees observed so far, only one can be the leading interviewee. Therefore, the probability that the next interviewee is a new leading interviewee is

$$P(Z_1) = \frac{1}{k+1}$$

- And according to the proposed strategy, we will accept this interviewee with probability $1-p$ and reject him/her with probability p. Therefore, the probability $\Psi_n(k)$ with respect to event Z_1 is

$$\Psi_n(k)|Z_1 = (1-p)\left(\frac{k+1}{n}\right) + p\left(\Phi_n(k+1)\right)$$

Case-2: Z_2*: Next interviewee is a new non-leading interviewee.*

- Since the current leading interviewee cannot appear again, the next interviewee must necessarily be distinct. Among $k+1$ distinct interviewees observed so far, k distinct interviewees will be non-leading interviewees. Therefore, the probability that the next interviewee is a new non-leading interviewee is

$$P(Z_2) = \frac{k}{k+1}$$

- And according to the proposed strategy, we will always reject this interviewee. Therefore, the probability is

$$\Psi_n(k)|Z_2 = \Psi_n(k+1)$$

By the rule of total probability,

$$\Psi_n(k) = (\Psi_n(k)|Z_1)P(Z_1) + (\Psi_n(k)|Z_2)P(Z_2)$$

$$\begin{aligned}
\Psi_n(k) &= \left((1-p)\left(\frac{k+1}{n}\right) + p\Phi_n(k+1)\right)\frac{1}{k+1} + \Psi_n(k+1)\frac{k}{k+1} \\
&= (1-p)\frac{k+1}{n}\frac{1}{k+1} + p\Phi_n(k+1)\frac{1}{k+1} + \Psi_n(k+1)\frac{k}{k+1} \\
&= \frac{1-p}{n} + \frac{p}{k+1}\Phi_n(k+1) + \frac{k}{k+1}\Psi_n(k+1)
\end{aligned}$$

□

Proposition 3.4. *The sequence of functions* $\hat{\Psi}_n(x) := \Psi_n(\lfloor nx \rfloor)$ *converges uniformly on* $[0,1]$ *for* $\Psi(1) = 0$ *to*

$$\Psi'(x) = \frac{1}{x}\Psi(x) - \Big(1 - p + \frac{p}{x}\Phi(x)\Big)$$

Proof. As

$$\Psi_n(k) = G_n(k) + H_n(k)\Psi_n(k+1)$$

where $G_n(k) = \frac{1-p}{n} + \frac{p}{k+1}\Phi_n(k+1)$ and $H_n(k) = \frac{k}{k+1}$.
To use Theorem 2.3 we let $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$, now

$$g_n(x) = n\left(\frac{1-p}{n} + \frac{p}{\lfloor nx \rfloor + 1}\Phi_n(\lfloor nx \rfloor + 1)\right) = 1 - p + \frac{pn}{\lfloor nx \rfloor + 1}\Phi_n(\lfloor nx \rfloor + 1)$$

$$h_n(x) = n\left(1 - \left(\frac{\lfloor nx \rfloor}{\lfloor nx \rfloor + 1}\right)\right) = \left(\frac{n}{\lfloor nx \rfloor + 1}\right)$$

As $n \to \infty$, $g_n(x) \to g(x)$, $h_n(x) \to h(x)$ and $\lfloor nx \rfloor \approx nx$.

- Let us estimate the value of $g(x)$, as $n \to \infty$:

  $$g(x) = 1 - p + \lim_{n\to\infty}\left(\frac{pn}{nx+1}\Phi_n(nx+1)\right)$$

  Now let us consider $\lim_{n\to\infty}\frac{pn}{nx+1}$.

  $$\frac{pn}{nx+1} = \frac{pn}{nx}\left(\frac{1}{1+\frac{1}{nx}}\right)$$

  Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{1}{nx}$. So,

  $$\begin{aligned}\frac{pn}{nx+1} &= \frac{p}{x}\left(1 - \frac{1}{nx} + O(n^{-2})\right)\\ &= \frac{p}{x} + O(n^{-1})\end{aligned} \tag{6}$$

  From Proposition 3.1, and as $n \to \infty$, $\Phi_n(nx+1) = \Phi_n(nx)$

  $$\lim_{n\to\infty}\Phi_n(nx+1) = \Phi(x) \tag{7}$$

  Substituting values for $(\frac{pn}{nx+1})$ and $\Phi_n(nx+1)$ from equations 6, 7, We get,

  $$g(x) = 1 - p + \left(\frac{p}{x} + O(n^{-1})\right)\Phi(x)$$

  As $n \to \infty$ then $O(n^{-1})$ and higher order term vanishes, So,

  $$g(x) = 1 - p + \frac{p}{x}\Phi(x) \tag{8}$$

- Let us estimate the value of $h(x)$:

  $$h(x) = \lim_{n\to\infty}\frac{n}{nx+1} = \lim_{n\to\infty}\frac{n}{nx}\left(\frac{1}{1+\frac{1}{nx}}\right)$$

  Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{1}{nx}$. So,

  $$\begin{aligned}h(x) &= \lim_{n\to\infty}\frac{1}{x}\left(1 - \frac{1}{nx} + O(n^{-2})\right)\\ &= \frac{1}{x} + O(n^{-1})\end{aligned}$$

  As $n \to \infty$ then $O(n^{-1})$ and higher order term vanishes, So,

  $$h(x) = \frac{1}{x} \tag{9}$$

Using Theorem 2.3 and Substituting $g(x)$ and $h(x)$ from above equations 8, 9, we get,

$$\Psi'(x) = \frac{1}{x}\Psi(x) - \Big(1 - p + \frac{p}{x}\Phi(x)\Big)$$

□

3.2.3 Calculation of $\Upsilon_n(k)$

We calculate the value of $\Upsilon_n(k)$ as follows:

Proposition 3.5. *For all natural numbers n and k where $1 < k \leq n$, we have,*

$$\Upsilon_n(k) = \frac{1}{k} + \left(1 - \frac{p}{(1+p)(n-k+1)+1}\right)\left(1 - \frac{1}{k}\right)\Upsilon_n(k-1)$$

and $\Upsilon_n(1) = 1$

Proof. By hypothesis, when the first (distinct) interviewee makes his/her initial appearance, that interviewee is necessarily the leading interviewee. Therefore, the probability that the leading interviewee appears only once when the first distinct interviewee is interviewed, i.e., $\Upsilon_n(1)$, is equal to 1.
Now, for the general case of k distinct interviews, let's consider two mutually exclusive events.

1. *E: The leading interviewee has appeared only once when the $(k-1)^{th}$ distinct interviewee makes his/her initial appearance.*
   Let $P(Q_1)$ be $\Upsilon_n(k)|E$ i.e. probability conditioned on event E.
   This case can be categorized further into two mutually exclusive events:

   Case-1: *R: When the future appearance of the leading interviewee observed in $(k-1)$ distinct interviewee interviews is possible*
   As each interviewee may appear a second time with probability p. Therefore,

   $$P(R) = p$$

   Now, for the leading interviewee to be observed only once in k distinct interviewee interviews, one of the following two events should happen.

   - *T_1: The leading interviewee appears again before the k^{th} distinct interviewee is interviewed and the k^{th} distinct interviewee is a new leading interviewee.*
     Since only one remaining appearance of the current leading interviewee is among the unobserved interviewees, the probability that he/she arrives again after $(k-1)^{th}$ distinct interviewee has been interviewed is $\frac{1}{(1+p)(n-(k-1))+1}$. Furthermore, the probability that the k^{th} distinct interviewee is a new leading interviewee is $\frac{1}{k}$. Hence,

     $$P(T_1) = \frac{1}{k}\frac{1}{(1+p)(n-(k-1))+1}$$

   - *T_2: The leading interviewee does not appear again before k^{th} distinct interviewee is interviewed.*
     The probability that the leading interviewee does not appear again before the k^{th} distinct interviewee is one minus the probability of the second appearance of the leading interviewee. Therefore,

     $$P(T_2) = 1 - \frac{1}{(1+p)(n-(k-1))+1}$$

     .

   By using the rule of total probability,

   $$P(Q_1|R) = P(T_1) + P(T_2)$$

   $$\begin{aligned} P(Q_1|R) &= \frac{1}{k}\frac{1}{(1+p)(n-(k-1))+1} + 1 - \frac{1}{(1+p)(n-(k-1))+1} \\ &= 1 - \left(1 - \frac{1}{k}\right)\left(\frac{1}{(1+p)(n-k+1)+1}\right) \end{aligned}$$

   Case-2: *$\bar{R}$: When the future appearance is not possible for the leading interviewee observed in $(k-1)$ distinct interviewee interviews.*
   As each interviewee may appear a second time with probability p. Therefore,

   $$P(\bar{R}) = 1 - p$$

   Since the future appearance is not possible for the leading interviewee, the next interviewee can only be a new distinct interviewee, so in either case, when this new distinct interviewee is a new leading or not, we will always have the case when the leading interviewee is observed once in k distinct interviews. Therefore,

   $$P(Q_1|\bar{R}) = 1$$

By the rule of total probability,

$$P(Q_1) = P(Q_1|R)P(R) + P(Q_1|\bar{R})P(\bar{R})$$

$$\begin{aligned} P(Q_1) &= \left(1 - \left(1 - \frac{1}{k}\right)\left(\frac{1}{(1+p)(n-k+1)+1}\right)\right)p + (1-p) \\ &= p - p\left(1 - \frac{1}{k}\right)\left(\frac{1}{(1+p)(n-k+1)+1}\right) + 1 - p \\ &= 1 - p\left(1 - \frac{1}{k}\right)\left(\frac{1}{(1+p)(n-k+1)+1}\right) \end{aligned}$$

2. *$\bar{E}$: The leading interviewee has appeared twice when the $(k-1)^{th}$ distinct interviewee makes his/her initial appearance.*
   Let $P(Q_2)$ be $\Upsilon_n(k)|\bar{E}$ i.e. probability conditioned on $\bar{E}$.
   As the current leading interviewee has already appeared twice, for the single appearance of the leading interview to be present in k distinct interviewees, the k^{th} distinct interviewee must be a new leading interviewee. Therefore,

$$P(Q_2) = \frac{1}{k}$$

As the probability of event E is $\Upsilon_n(k-1)$ and event $\bar{E}$ is $1 - \Upsilon_n(k)$. Therefore, by combining the probabilities from both events E and $\bar{E}$ using the rule of total probability, we get,

$$\Upsilon_n(k) = P(E)P(Q_1) + P(\bar{E})P(Q_2)$$

$$\begin{aligned} \Upsilon_n(k) &= \Upsilon_n(k-1)\left(1 - p\left(1 - \frac{1}{k}\right)\left(\frac{1}{(1+p)(n-k+1)+1}\right)\right) + (1 - \Upsilon_n(k-1))\left(\frac{1}{k}\right) \\ &= \frac{1}{k} + \left(1 - \frac{1}{k}\right)\left(1 - \frac{p}{(1+p)(n-k+1)+1}\right)\Upsilon_n(k-1) \end{aligned}$$

□

Proposition 3.6. *The sequence of functions $\hat{\Upsilon}_n(x) := \Upsilon_n(\lfloor nx \rfloor)$ converges uniformly on $[0,1]$ for $\Upsilon(0) = 1$ to*

$$\Upsilon'(x) = -\left(\frac{1}{x} + \frac{p}{(1+p)(1-x)}\right)\Upsilon(x) + \frac{1}{x}$$

Proof. From Proposition 3.5 we know that,

$$\Upsilon_n(k) = G_n(k) + H_n(k)\Upsilon_n(k-1)$$

where $G_n(k) = \frac{1}{k}$ and $H_n(k) = \left(1 - \frac{p}{(1+p)(n-k+1)+1}\right)\left(1 - \frac{1}{k}\right)$.
We use Theorem 2.2 with $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$. Now

$$g_n(x) = n\frac{1}{\lfloor nx \rfloor}$$

$$\begin{aligned} h_n(x) &= n\left(1 - \left(1 - \frac{p}{(1+p)(n - \lfloor nx \rfloor + 1) + 1}\right)\left(1 - \frac{1}{\lfloor nx \rfloor}\right)\right) \\ &= n\left(\frac{1}{\lfloor nx \rfloor} + \left(\frac{p}{(1+p)(n - \lfloor nx \rfloor + 1) + 1}\right)\left(1 - \frac{1}{\lfloor nx \rfloor}\right)\right) \end{aligned}$$

As $n \to \infty$, $g_n(x) \to g(x)$, $h_n(x) \to h(x)$ and $\lfloor nx \rfloor \approx nx$.

- Let us estimate the value of $g(x)$:

$$g(x) = \lim_{n\to\infty}\left(\frac{1}{x}\right) = \frac{1}{x} \tag{10}$$

- Let us estimate the value of $h(x)$:

$$\begin{aligned}
h(x) &= \lim_{n\to\infty} n\left(\frac{1}{nx} + \left(\frac{p}{(1+p)(n-nx+1)+1}\right)\left(1-\frac{1}{nx}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{n}{nx} + n\left(1-\frac{1}{nx}\right)\left(\frac{p}{(1+p)(n-nx+1)+1}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{x} + n\left(1-\frac{1}{nx}\right)\left(\frac{p}{(1+p)(n-nx)+(p+1)+1}\right)\right) \\
&= \frac{1}{x} + \lim_{n\to\infty}\left(n\left(1-\frac{1}{nx}\right)\left(\frac{p}{(1+p)n(1-x)}\right)\left(\frac{1}{1+\frac{2+p}{(1+p)n(1-x)}}\right)\right)
\end{aligned}$$

Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{2+p}{(1+p)n(1-x)}$. So,

$$\begin{aligned}
h(x) &= \frac{1}{x} + \lim_{n\to\infty}\left(\left(1-\frac{1}{nx}\right)\frac{p}{(1+p)(1-x)}\left(1-\frac{2+p}{(1+p)n(1-x)}+O(n^{-2})\right)\right) \\
&= \frac{1}{x} + \lim_{n\to\infty}\left(\left(1-\frac{1}{nx}\right)\left(\frac{p}{(1+p)(1-x)}+O(n^{-1})\right)\right) \\
&= \frac{1}{x} + \lim_{n\to\infty}\left(\frac{p}{(1+p)(1-x)}+O(n^{-1})\right)
\end{aligned}$$

As $n \to \infty$ then $O(n^{-1})$ and higher order term vanishes, so,

$$h(x) = \frac{1}{x} + \frac{p}{(1+p)(1-x)} \tag{11}$$

Using Theorem 2.2 and substituting $g(x)$ and $h(x)$ from above equations 10 and 11, we get,

$$\Upsilon'(x) = -\left(\frac{1}{x} + \frac{p}{(1+p)(1-x)}\right)\Upsilon(x) + \frac{1}{x}$$

□

3.2.4 Calculation of $F_n(k)$

Overall success probability $F_n(k)$ can be obtained by combining the probabilities of events χ_1 and χ_2 by using the total probability rule.
From Proposition 3.1 we know $\Phi_n(k)$, the probability of event χ_1.
From Proposition 3.3, we know $\Psi_n(k)$, the probability of event χ_2.
From Proposition 3.5, we know the value of $\Upsilon_n(k)$.
Now $\bar{\Upsilon}_n(k)$, the probability that the leading interviewee has appeared twice when the $(k-1)^{th}$ distinct interviewee makes his/her initial appearance is one minus $\Upsilon_n(k)$. Therefore,

$$\bar{\Upsilon}_n(k) = 1 - \Upsilon_n(k)$$

By combining the above probabilities using the law of conditional probability, we can obtain,

$$\begin{aligned}
F_n(k) &= \Upsilon_n(k)\Phi_n(k) + \bar{\Upsilon}_n(k)\Psi_n(k) \\
&= \Upsilon_n(k)\Phi_n(k) + (1-\Upsilon_n(k))\Psi_n(k)
\end{aligned} \tag{12}$$

From Proposition 3.1 we can compute $\phi_n(k)$ from $\phi_n(k+1)$, and from Proposition 3.3 we can compute $\psi_n(k)$ from $\psi_n(k+1)$ and finally from Proposition 3.5 we can compute $\Upsilon_n(k)$ from $\Upsilon_n(k-1)$ in $O(1)$ computation time. Thus, values for all $1 \le k \le n$ can be computed in $O(n)$ time. From above results using Algorithm 1, 2, we can obtain the success probability($F(k_n)$) in $O(n)$ computation time.

Proposition 3.7. *The sequence of functions $\hat{F}_n(x) := F_n(\lfloor nx \rfloor)$ converges uniformly on $[0,1]$ to*

$$F(x) = \Upsilon(x)\Phi(x) + (1-\Upsilon(x))\Psi(x)$$

Proof. From equation 12, we know that,

$$F_n(k) = \Upsilon_n(k)\Phi_n(k) + (1 - \Upsilon_n(k))\Psi_n(k)$$

Substituting $k = \lfloor nx \rfloor$,

$$F_n(k) = F_n(\lfloor nx \rfloor) = \Upsilon_n(\lfloor nx \rfloor)\Phi_n(\lfloor nx \rfloor) + (1 - \Upsilon_n(\lfloor nx \rfloor))\Psi_n(\lfloor nx \rfloor)$$

using the notations introduced above,

$$\hat{F}_n(x) = \hat{\Upsilon}_n(x)\hat{\Phi}_n(x) + (1 - \hat{\Upsilon}_n(x))\hat{\Psi}_n(x)$$

For uniform convergence at $n \to \infty$,

$$\lim_{n\to\infty} \hat{F}_n(x) = \lim_{n\to\infty} \hat{\Upsilon}_n(x)\hat{\Phi}_n(x) + \lim_{n\to\infty} (1 - \hat{\Upsilon}_n(x))\hat{\Psi}_n(x)$$

As per Theorem 2.1, at $n \to \infty$, $F(x) = \hat{F}_n(x), \Upsilon(x) = \hat{\Upsilon}_n(x), \Phi(x) = \hat{\Phi}_n(x)$, $\Psi(x) = \hat{\Psi}_n(x)$, so,

$$F(x) = \Upsilon(x)\Phi(x) + (1 - \Upsilon(x))\Psi(x)$$

□

Algorithm 1: Get Φ, Ψ, Υ arrays

Input: n, p
Output: Φ, Ψ, Υ

Def `get_PW_arrays():`
- $\Phi[n] \longleftarrow p$
- $\Psi[n] \longleftarrow 0$
- **for** $k \leftarrow n-1$ *down* **to** 0 **do**
  - $a = 1/((1+p)*(n-k)+1)$
  - $\Phi[k] = (p*a*k + (1-p)*(1-p*a))/n + (p+k)*(1-p*a)*\Phi[k+1]/(k+1)$
  - $\Psi[k] = (1-p)/n + (p*\Phi[k+1] + k*\Psi[k+1])/(k+1)$
- $\Upsilon[1] \longleftarrow 1$
- **for** $k = 2$ **to** n **do**
  - $\Upsilon[k] = 1/k + (1 - p/((1+p)(n-k+1)+1))*(1-1/k)*\Upsilon[k-1]$
- **return** Φ, Ψ, Υ

Algorithm 2: Get Optimal Probability ($F(k_n)$)

Input: n, Φ, Ψ, Υ
Output: $F[k_n], k_n$

Def `get_optimal_probability():`
- `get_PW_arrays()` `// function from Algorithm-1`
- $F[1,\ldots,n] \longleftarrow 0$
- $F(k_n) \longleftarrow 0$
- $k_n \longleftarrow 0$
- **for** $k = 1$ **to** n **do**
  - $F[k] = \Phi[k]*\Upsilon[k] + (1-\Upsilon[k])*\Psi[k]$
  - **if** $F[k] > F(k_n)$ **then**
    - $F(k_n) \longleftarrow F[k]$
    - $k_n \longleftarrow k$
- **return** $F(k_n), k_n$

3.3 Special cases for probabilistic second arrival

Here we will be showing that the model we have derived in the earlier section reduces to the standard models if $p = 0$ or $p = 1$.

3.3.1 Case $p = 0$ (Classical Secretary Problem)

When $p = 0$, no interviewee arrives a second time, and each interviewee is observed exactly once. On substituting $p = 0$ in the equation from Proposition 3.1, we get,

$$\Phi_n(k) = \frac{1}{n} + \frac{k}{k+1}\Phi_n(k+1)$$

Since each interviewee only arrives once, $\Upsilon_n(k)$ will always be 1. Therefore,

$$\begin{aligned} F_n(k) &= \Upsilon_n(k)\Phi_n(k) + \bar{\Upsilon}_n(k)\Psi_n(k) \\ &= \Phi_n(k) + (1-1)\Psi_n(k) \\ &= \Phi_n(k) \end{aligned}$$

Therefore,

$$F_n(k) = \frac{1}{n} + \frac{k}{k+1}F_n(k+1) \;\; for \;\; 1 \le k \le n$$

This is the same recursive function obtained by Ribas[11] for the Classical Secretary Problem, which gives Optimal Probability as $1/e$ at optimal threshold n/e, see Ribas[11] for details.

3.3.2 Case $p = 1$ (Guaranteed Reappearance)

When $p = 1$, every interviewee arrives exactly twice.

- *Value for $\Phi_n(k)$ at $p = 1$:*
  On substituting $p = 1$ in the equation from Proposition 3.1, we get,

  $$\Phi_n(k) = \frac{ak}{n} + \frac{(1+k)(1-a)}{k+1}\Phi_n(k+1)$$

  where $a = \frac{1}{2n-2k+1}$. Therefore,

  $$\begin{aligned} \Phi_n(k) &= \frac{k}{n}\frac{1}{2n-2k+1} + \left(1 - \frac{1}{2n-2k+1}\right)\Phi_n(k+1) \\ &= \frac{k}{n}\frac{1}{2n-2k+1} + \frac{2n-2k}{2n-2k+1}\Phi_n(k+1) \end{aligned}$$

- On substituting $p = 1$ in the equation from Proposition 3.3, we get,

  $$\Psi_n(k) = \frac{1}{k+1}\Phi_n(k+1) + \frac{k}{k+1}\Psi_n(k+1)$$

- On substituting $p = 1$ in the equation from Proposition 3.5, we get,

  $$\Upsilon_n(k) = \frac{1}{k} + \left(1 - \frac{1}{2n-2k+3}\right)\left(1 - \frac{1}{k}\right)\Upsilon_n(k-1)$$

  Ribas[11] has obtained $\bar{\Upsilon}_n(k)$, the probability that the leading interviewee has appeared twice when the $(k-1)^{th}$ distinct interviewee makes his/her initial appearance. As $\bar{\Upsilon}_n(k) = 1 - \Upsilon_n(k)$,

$$\begin{aligned} 1 - \bar{\Upsilon}_n(k) &= \frac{1}{k} + \left(1 - \frac{1}{2n-2k+3}\right)\left(1 - \frac{1}{k}\right)(1 - \bar{\Upsilon}_n(k-1)) \\ \bar{\Upsilon}_n(k) &= 1 - \frac{1}{k} - \left(1 - \frac{1}{2n-2k+3}\right)\left(1 - \frac{1}{k}\right)(1 - \bar{\Upsilon}_n(k-1)) \\ &= \left(\frac{k-1}{k(2n-2k+3)}\right) + \left(\frac{2(k-1)(n-k+1)}{k(2n-2k+3)}\right)\bar{\Upsilon}_n(k-1) \end{aligned}$$

These equations (for case $p = 1$) are the same recursive equation obtained by Ribas[11]. So we can say that the optimal probability of success is the same as the one obtained by Ribas[11]. Therefore, the optimal probability is 0.76 at an optimal threshold of 0.47.

3.4 Numerical Simulation

Table 1 shows the optimal threshold k_n together with the corresponding optimal success probability $P(k_n)$ obtained using Algorithm 2 for various values of p when $n = 100$. The table also reports the normalized threshold k_n/n. The results demonstrate how allowing reappearances (increasing p) affects both the optimal stopping rule and the probability of successfully selecting the best candidate.

$\mathbf{p}$	0	0.001	0.1	0.25	0.5	0.75	0.9	0.999	1
$\mathbf{k_n}$	37	37	47	55	57	54	51	48	48
$\mathbf{\frac{k_n}{n}}$	0.37	0.37	0.47	0.55	0.57	0.54	0.51	0.48	0.48
$\mathbf{P(k_n)}$	0.371	0.372	0.484	0.597	0.6874	0.7328	0.7546	0.7695	0.7697

Table 1: Optimal values k_n, k_n/n, and $P(k_n)$ for different values of p when $n = 100$

This figure illustrates the success probability $P(k)$ as a function of the stopping threshold k for $n = 100$ under different values of the reappearance probability p. Each curve corresponds to a different p, and the dashed vertical lines indicate the respective optimal thresholds. The results show that as p increases, both the optimal threshold and the maximum success probability increase, demonstrating the positive effect of candidate reappearance on selection success.

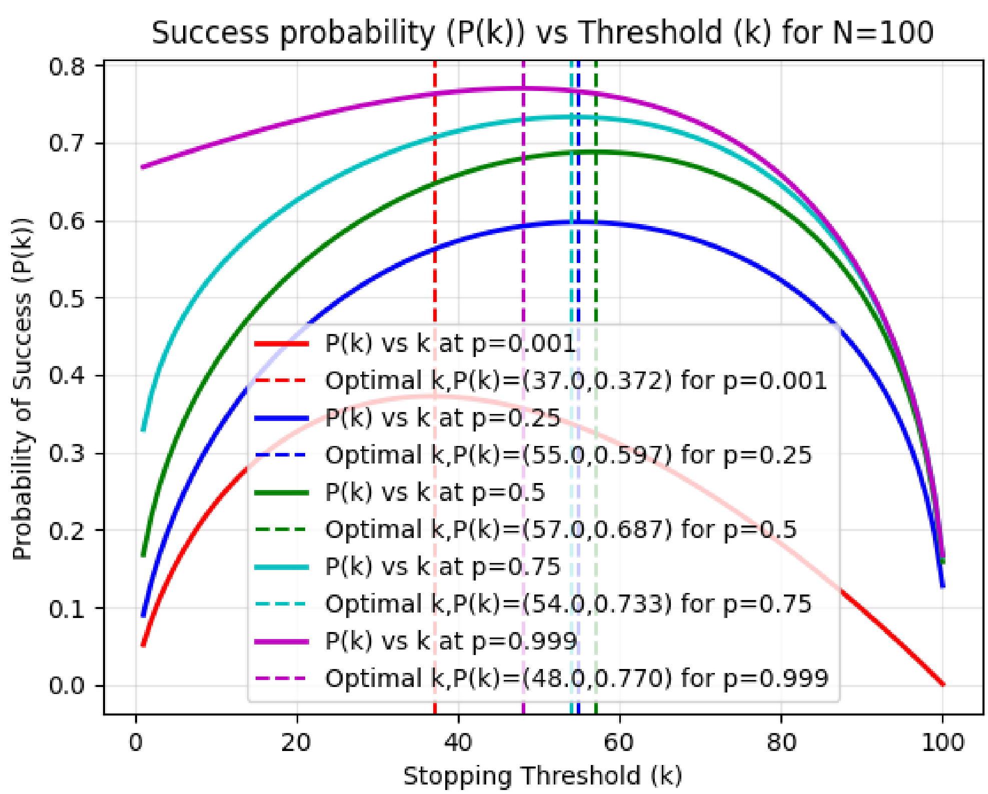

Figure 1: Success probability graph w.r.t. k at $n = 100$

4 Secretary problem with top-3 success

In this case, each interview arrives exactly once. But, we will be satisfied if the chosen candidate is amongst the best (top) three.

We use the optimal strategy of the classical secretary problem, i.e. interview k interviewees, reject all of them but note down the leading one among them.

After interviewing k interviewees, select the first interviewee that is better than all the previously seen interviewees.

4.1 Success probability for top-3 secretary problem

Let,
$P_n(k)$: Probability that the chosen interviewee is one of top-3 assuming we have rejected the first k interviewees.

4.1.1 Calculation of $P_n(k)$

For $P_n(k)$, we have the below proposition,

Proposition 4.1. *For all natural numbers n and k where $k < n$, we have,*

$$P_n(k) = \frac{1}{k+1}\left(1 - \frac{\binom{n-3}{k+1}}{\binom{n}{k+1}}\right) + \frac{k}{k+1}P_n(k+1)$$

and $P_n(n) = 0$.

Proof. By hypothesis, when an employer has already interviewed all n interviewees, then the leading interviewee is the overall best, and he/she has already appeared and was rejected, so the probability of selecting the best is 0, or $P_n(n) = 0$.
Now for the general case of k interviews, let's look at time instant $(k+1)$:
At $(k+1)^{th}$ time instant, either of the following events occurs:

- *E: The $(k+1)^{th}$ interviewee is a new leading interviewee.*
  According to the proposed strategy, this interviewee will be accepted immediately. Therefore, the success probability with respect to event E is the probability of choosing one of the top 3. Therefore,

  $$P(\text{success}|E) = P(\text{choosen in top 3}|E)$$

  The complement of this event is the event that none of the first $k+1$ interviewees is among top three. The probability of this is:

  $$P(\text{not in top 3}) = \frac{\binom{n-3}{k+1}}{\binom{n}{k+1}}$$

  Here $\binom{n}{r}$ is the number of ways to choose r items from n items.

  $$P(\text{in top 3}|E) = 1 - \frac{\binom{n-3}{k+1}}{\binom{n}{k+1}} \quad (13)$$

  Now, the probability that the leading interviewee appears at $k+1$ is

  $$P(E) = \frac{1}{k+1} \quad (14)$$

- *$\bar{E}$: The $(k+1)^{th}$ interviewee is a non-leading interviewee.*
  According to the proposed strategy, this interviewee will not be chosen (we will always reject him/her and will continue the process). Therefore,

  $$P_n(k)|\bar{E} = P_n(k+1) \quad (15)$$

  And the probability that a non-leading interviewee appears at $(k+1)^{th}$ interview is one minus the probability that a leading interviewee does not appear at $(k+1)^{th}$ interview. Therefore,

  $$P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{k+1} = \frac{k}{k+1} \quad (16)$$

By using the rule of total probability,

$$P_n(k) = P(E)(P_n(k)|E) + P(\bar{E})(P_n(k)|\bar{E})$$

from equations 14, 13, 16 and 15, we get,

$$P_n(k) = \frac{1}{k+1}\left(1 - \frac{\binom{n-3}{k+1}}{\binom{n}{k+1}}\right) + \frac{k}{k+1}P_n(k+1)$$

□

The above recursion for the probability of success can be solved in $O(n)$ computation time.

Algorithm 3: Get Optimal Probability ($P(k_n)$)

Input: n
Output: $P(k_n), k_n$

Def `get_optimal_probability():`
    $P[n] \longleftarrow 0$
    $P(k_n) \longleftarrow 0$
    $k_n \longleftarrow 0$
    for $k \leftarrow n-1$ *down* **to** 0 **do**
        $r = ((n-k-1)*(n-k-2)*(n-k-3))/(n*(n-1)*(n-2))$
        $P[k] = (1/(k+1))*(1-r) + (k/(k+1))*P_n[k+1]$
        if $P[k] > P(k_n)$ **then**
            $P(k_n) \longleftarrow P[k]$
            $k_n \longleftarrow k$
    return $P(k_n), k_n$

4.2 Asymptotic analysis of P_n

For asymptotic analysis of P_n, we have the following,

Proposition 4.2. *The sequence of functions $\hat{P}_n(x) := P_n(\lfloor nx \rfloor)$ converges uniformly on $[0,1]$ with $P(1) = 0$ to*

$$P(x) = -3x\ln(x) + 3x^2 - \frac{x^3}{2} - \frac{5x}{2}$$

Proof. From Proposition 4.1 we know that,

$$P_n(k) = G_n(k) + H_n(k)P_n(k+1)$$

where $G_n(k) = \frac{1}{k+1}\left(1 - \frac{\binom{n-3}{k+1}}{\binom{n}{k+1}}\right)$ and $H_n(k) = \frac{k}{k+1}$.
Using expansion of $\binom{n}{r}$, we have

$$\frac{\binom{n-3}{k+1}}{\binom{n}{k+1}} = \frac{(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)}$$

From Theorem 2.3 with $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$ we have,

$$g_n(x) = n\left(\frac{1}{\lfloor nx \rfloor + 1}\left(1 - \frac{(n-\lfloor nx \rfloor - 1)(n-\lfloor nx \rfloor - 2)(n-\lfloor nx \rfloor - 3)}{n(n-1)(n-2)}\right)\right)$$

and

$$h_n(x) = n\left(1 - \left(\frac{\lfloor nx \rfloor}{\lfloor nx \rfloor + 1}\right)\right) = \left(\frac{n}{\lfloor nx \rfloor + 1}\right)$$

As $n \to \infty$, $g_n(x) \to g(x)$, $h_n(x) \to h(x)$ and $\lfloor nx \rfloor \approx nx$.

- Let us estimate the value of $g(x)$:

$$g(x) = \lim_{n\to\infty}\left(\frac{n}{nx+1}\left(1 - \frac{(n-nx-1)(n-nx-2)(n-nx-3)}{n(n-1)(n-2)}\right)\right)$$

Now let us consider $\lim\limits_{n\to\infty} \frac{n}{nx+1}$.

$$\frac{n}{nx+1} = \frac{n}{nx\left(1+\frac{1}{nx}\right)}$$

Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{1}{nx}$. So,

$$\frac{n}{nx+1} = \frac{n}{nx}\left(1 - \frac{1}{nx} + O(n^{-2})\right)$$

$$\frac{n}{nx+1} = \frac{1}{x} + O(n^{-1}) \tag{17}$$

Now for $\frac{(n-nx-1)(n-nx-2)(n-nx-3)}{n(n-1)(n-2)}$,

$$\begin{aligned}\frac{(n-nx-1)(n-nx-2)(n-nx-3)}{n(n-1)(n-2)} &= \frac{n^3(1-x-\frac{1}{n})(1-x-\frac{2}{n})(1-x-\frac{3}{n})}{n^3(1-\frac{1}{n})(1-\frac{2}{n})} \\ &= \frac{1-x-\frac{1}{n}}{1-\frac{1}{n}}\frac{1-x-\frac{2}{n}}{1-\frac{2}{n}}\left(1-x-\frac{3}{n}\right)\end{aligned} \tag{18}$$

Using expansion for $\frac{1}{1+\epsilon}$ where $\epsilon = \frac{1}{n}$. So,

$$\frac{1-x-\frac{1}{n}}{1-\frac{1}{n}} = \left(1-x-\frac{1}{n}\right)\left(1+\frac{1}{n}+O(n^{-2})\right) = \left(1-x+O(n^{-1})\right)$$

similarly,

$$\frac{1-x-\frac{2}{n}}{1-\frac{2}{n}} = \left(1-x-\frac{2}{n}\right)\left(1+\frac{2}{n}+O(n^{-2})\right) = \left(1-x+O(n^{-1})\right)$$

Substituting the above results in equation 18, we get,

$$\begin{aligned}\frac{(n-nx-1)(n-nx-2)(n-nx-3)}{n(n-1)(n-2)} &= \left(1-x+O(n^{-1})\right)\left(1-x+O(n^{-1})\right)\left(1-x-\frac{3}{n}\right) \\ &= (1-x)^3 + O(n^{-1})\end{aligned} \tag{19}$$

Finally combining above results from equations 19, 17 and $g(x)$, we get,

$$\begin{aligned}g(x) &= \lim_{n\to\infty}\left(\frac{1}{x}+O(n^{-1})\right)\left(1-\left((1-x)^3+O(n^{-1})\right)\right) \\ &= \lim_{n\to\infty}\frac{1}{x}\left(1-(1-x)^3\right)+O(n^{-1})\end{aligned}$$

As $n\to\infty$ then $O(n^{-1})$ and higher order term vanishes, So,

$$g(x) = \frac{1}{x}\left(1-(1-x)^3\right) \tag{20}$$

- Let us estimate the value of $h(x)$:

$$h(x) = \lim_{n\to\infty}\left(\frac{n}{nx+1}\right)$$

from equation 17, we know that,

$$h(x) = \lim_{n\to\infty}\left(\frac{1}{x}+O(n^{-1})\right)$$

As $n\to\infty$ then $O(n^{-1})$ and higher order term vanishes, So,

$$h(x) = \frac{1}{x} \tag{21}$$

Using Theorem 2.3 and substituting $g(x)$ and $h(x)$ from above equations 20, 21. We get,

$$P'(x) = \frac{1}{x}P(x) - \frac{1}{x}\left(1-(1-x)^3\right) \tag{22}$$

Using the general solution of a first-order linear differential equation. We get,

$$P(x) = \mu(x)^{-1}\left[\mu(1)P(1) + \int_x^1 \mu(t)g(t)dt\right],$$
$$\mu(x) = \exp\left(-\int h(x)dx\right)$$

As $P(1) = 0$, So

$$P(x) = \mu(x)^{-1}\left[\int_x^1 \mu(t)g(t)dt\right],$$
$$\mu(x) = \exp\left(-\int h(x)dx\right), h(x) = \frac{1}{x}, g(x) = \frac{1}{x}\left(1-(1-x)^3\right)$$

Lets calculate the value for $\mu(x)$:

$$\mu(x) = \exp\left(-\int h(x)dx\right) = e^{\left(-\int \frac{1}{x}dx\right)} = e^{-\ln(x)} = x^{-1}$$

Substituting value of $\mu(x)$ is above equation for $P(x)$,

$$P(x) = x\left[\int_x^1 \frac{1}{t}\left(\frac{1}{t}\left(1-(1-t)^3\right)\right)dt\right]$$

As $1-(1-t)^3 = 1-(1-t^3-3t+3t^2) = t^3+3t-3t^2$,

$$\begin{aligned}
P(x) &= x\left[\int_x^1 \frac{1}{t^2}\left(t^3+3t-3t^2\right)dt\right] \\
&= x\left[\int_x^1 \left(t-3+\frac{3}{t}\right)dt\right] \\
&= x\left[\frac{t^2}{2} - 3t + 3\ln(t)\right]_x^1 \\
&= x\left[\left(\frac{1^2}{2} - 3(1) + 3\ln(1)\right) - \left(\frac{x^2}{2} - 3x + 3\ln(x)\right)\right] \\
&= x\left[\frac{-5}{2} - \left(\frac{x^2}{2} - 3x + 3\ln(x)\right)\right] \\
&= x\left[-3\ln(x) + 3x - \frac{x^2}{2} - \frac{5}{2}\right]
\end{aligned}$$

$$-3x\ln(x) + 3x^2 - \frac{x^3}{2} - \frac{5x}{2}$$

□

4.2.1 Optimal Threshold

The optimal threshold is the time instant at which the probability of success, $P(x)$, is maximized. We can calculate this by taking the derivative of $P(x)$ and equating it to zero. By substituting the value of $P(x)$ from Propositions 4.2 in equation 22, We get,

$$
\begin{aligned}
P'(x) &= \frac{1}{x}\left(-3x\ln(x) + 3x^2 - \frac{x^3}{2} - \frac{5x}{2}\right) - \frac{1}{x}\left(1-(1-x)^3\right) \\
&= \left(-3\ln(x) + 3x - \frac{x^2}{2} - \frac{5}{2}\right) - \frac{1}{x}\left(x^3 + 3x - 3x^2\right) \\
&= \left(-3\ln(x) + 3x - \frac{x^2}{2} - \frac{5}{2}\right) - \left(x^2 + 3 - 3x\right) \\
&= -3\ln(x) + 3x - \frac{x^2}{2} - \frac{5}{2} - x^2 - 3 + 3x \\
&= -3\ln(x) + 6x - \frac{3}{2}x^2 - \frac{11}{2}
\end{aligned}
$$

Equating it to 0, we get,

$$
-3\ln(x) + 6x - \frac{3}{2}x^2 - \frac{11}{2} = 0 \tag{23}
$$

Root of the above equation 23 is $x^* \approx 0.259$.
At $x^* \approx 0.259$,

$$
P(x^*) = -3(0.259)\ln(0.259) + 3(0.259)^2 - \frac{(0.259)^2}{2} - \frac{5(0.259)}{2} = 0.59
$$

So the optimal stopping threshold is $x^* \approx 0.259$ with maximum success probability as $P(x^*) \approx 0.59$.

4.3 Numerical Simulation

Table 2 shows the optimal threshold k_n, the normalized threshold k_n/n, and the corresponding optimal success probability $P(k_n)$ obtained using Algorithm 3 for various values of n. The results illustrate the convergence of the optimal stopping proportion and success probability as n increases.

$\mathbf{n}$	10	100	1000	10000	100000	1000000	10000000
$\mathbf{k_n}$	2	26	260	2599	25997	259971	2599716
$\frac{\mathbf{k_n}}{\mathbf{n}}$	0.2	0.26	0.26	0.2599	0.25997	0.259971	0.2599716
$\mathbf{P(k_n)}$	0.6640	0.6008	0.5953	0.59479	0.59473	0.59473	0.59472

Table 2: Optimal values k_n, k_n/n and $P(k_n)$ for different values of n

Figure 2 shows the theoretical success probability $P(k)$ as a function of the stopping threshold k for $n = 100$ in the top-3 secretary problem. The probability increases initially, reaches its maximum value $P(k) \approx 0.6008$ at the optimal threshold $k = 26$, and then gradually decreases, illustrating the existence of an optimal stopping rule.

Figure 3 presents the simulated success probability based on $10,000$ trials for $n = 100$ in the top-3 secretary problem. The simulation closely matches the theoretical behavior, with the maximum success probability (≈ 0.6124) occurring near the optimal threshold $k = 26$, validating the analytical results.

5 Conclusion

Here, we have studied two variations of the secretary problem. Firstly, we studied a variation where each candidate may reappear a second time with probability p. We proposed a threshold-based strategy and derived a recursive formula to calculate the probability of success at a general threshold k. We also provided an $O(n)$ time dynamic programming algorithm to calculate the probability of success at a general threshold k. We analyzed the extreme cases $p = 0$ and $p = 1$, showing that the model reduces respectively to the classical and returning secretary problem. The asymptotic analysis of the proposed strategy is formulated in terms of differential equations, which can be further studied to obtain optimal bounds.

Finally, we analyzed a relaxed variation of the classical secretary problem, where selecting a candidate ranked among the top three globally is considered a success. Using the optimal threshold structure of the classical secretary problem,

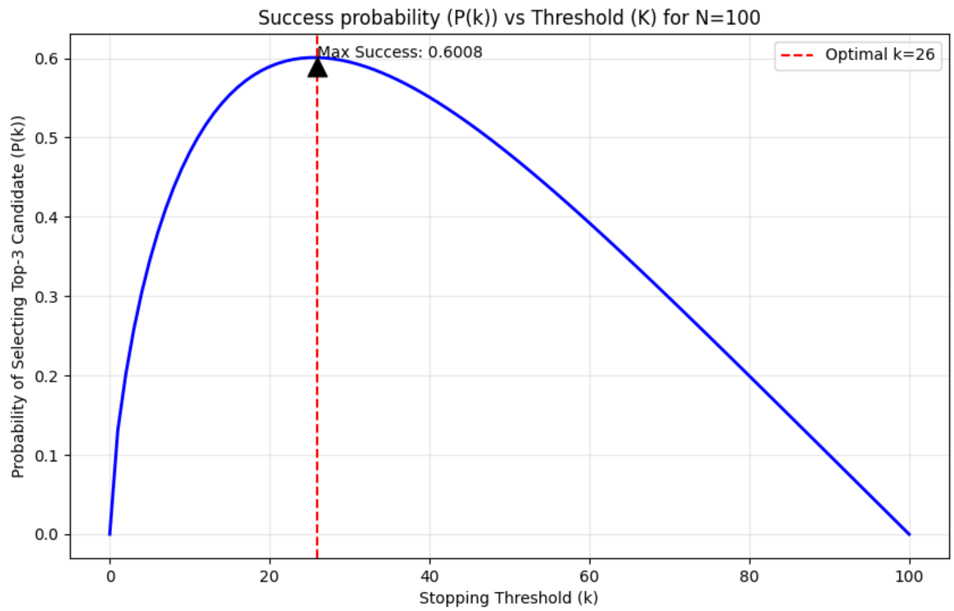

Figure 2: Success probability graph w.r.t k at $n = 100$

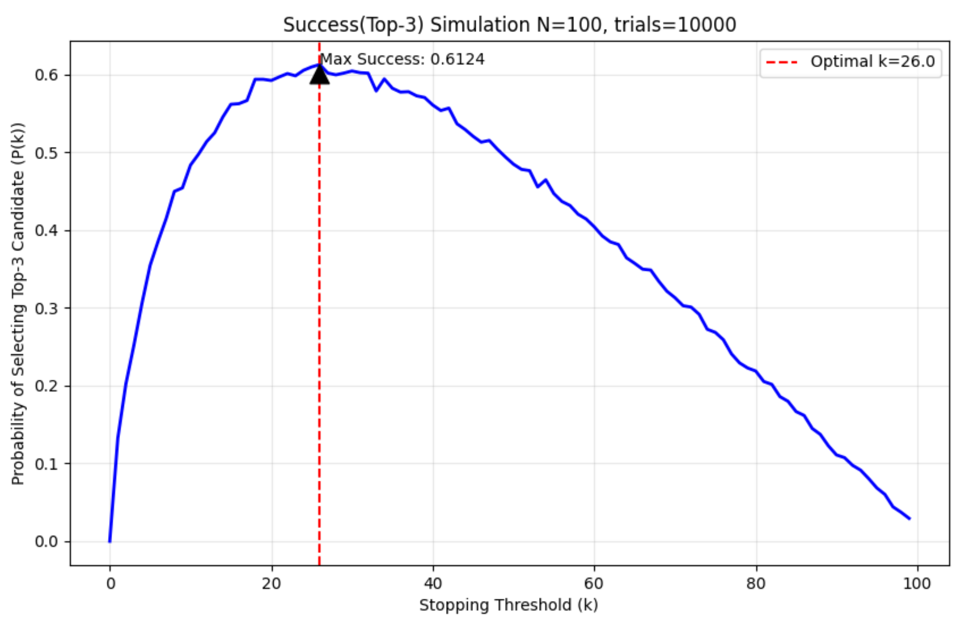

Figure 3: Simulation Success top-3 graph w.r.t k at $n = 100$, $trials = 10000$

we derived a recursive formula and an $O(n)$ time dynamic programming algorithm to calculate the probability of success at a general threshold k. We further studied the asymptotic behavior of the model and the optimal threshold and corresponding optimal probability of success.

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