On physical aspects of the '(st)ringy'

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Abstract

We try to let the '(st)ringy' (viXra:1912.0532 [v1]) explain massive graviton, leptonic spin-$\frac{3}{2}$ particle, and so on. Meanwhile, we come across a number system which we tentatively call 'hexanion', as if it were a by-product.

1 Glossary

$a \in A$: a is an element of the set A .

BH: black hole .

$\mathbb{C}$: the set of complex numbers .

DE: dark energy .

DM: dark matter .

DSR: doubly special relativity .

GC: great circle .

GIMP: GNU Image Manipulation Program .

ι: imaginary unit .

ℓ_{p}: Planck length .

MT: multiplication table .

$\mathbb{O}$: the set of octonions .

q: quaternion .

$\mathbb{R}$: the set of real numbers .

resp.: respectively .

RHS: right-hand side .

$\mathbb{R}^n$: n-dimensional real space .

SM: standard model .

ST: string theory .

$||\cdot||$: norm .

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2 Introduction

Most theories containing gravitons suffer from severe problems . As for ST, it also seems to suffer some problem, having the graviton as a massless state of a fundamental string . So we emphasise the role of massiveness to present some ideas encompassing massive graviton, leptonic spin-$\frac{3}{2}$ particle, etc.

3 Dealing with the graviton [1]

We start from photon sphere (PS) [2, Fig.'s 1, 2, etc.], which is regarded as composed of two rings and decomposed as follows.

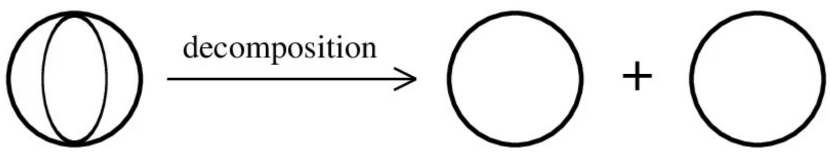


Fig. 1. Decomposition of PS into rings

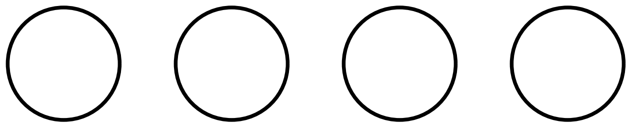
Spin of photon being 1 , we assign $\frac{1}{2}$-spin to each ring. With regard to mass, according to a model in which loss of a self-intersection in PS leads to mass acquisition [2, Fig. 5], each ring is endowed with > 0 mass. That said, unfortunately, 'halving' PS gives rise to $\frac{\ell_{\mathrm{p}}}{2}$, since it comes from ℓ_{p}. Each ring embodying the subplanckian, such a decomposition seems pointless *physically* [1]. So we will have to tackle our task mathematically rather than physically for a while. Specifically, we undertake 'ring combinatorics' in what follows, using such rings as building blocks for some particles.

3.1 Four types of gravitons

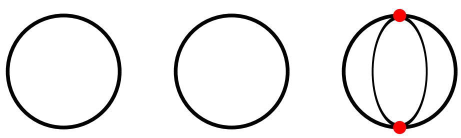
We take up the graviton, whose spin is 2 . Considering $2/\frac{1}{2} = 4$ rings, we enumerate thinkable combinations of them to make a rough classification according to the presence/absence of 'ring coalescence' and resultant formation of self-intersections, which are indicated by red dots:

[1] Actually, we can circumvent this 'physical pointlessness', if we wish. See **7.1**.

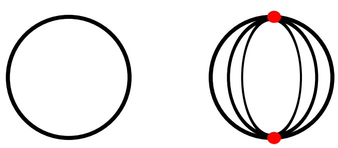
Type 1



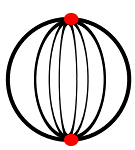
Type 2



Type 3



Type 4



Explanation by words is

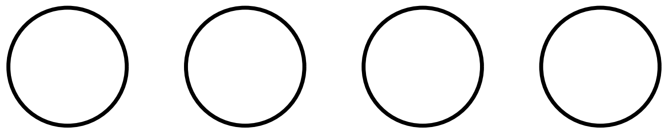
- type 1: no coalescence/no self-intersection;
- type 2: two rings coalesce to form two self-intersections;
- type 3: three rings coalesce to form two self-intersections;
- type 4: all rings coalesce to form two self-intersections.

3.2 Further classification

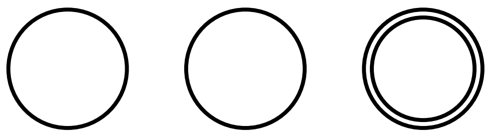
Taking into consideration thinkable overlaps and > 2 self-intersections of these rings, we further classify the above types into some subtypes:

Type 1

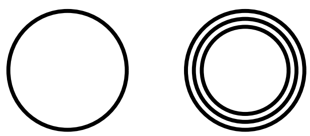
1A



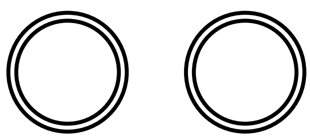
1B



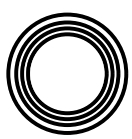
1C



1D

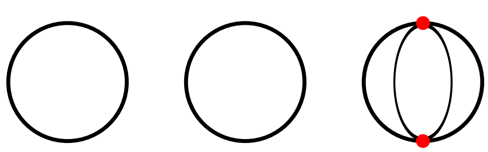


1E

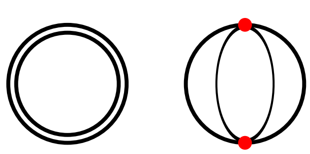


Type 2

2A

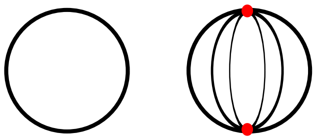


2B

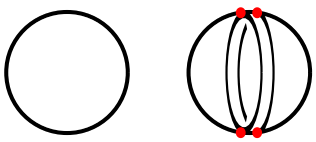


Type 3

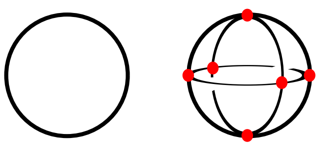
3A



3B

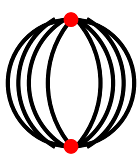


3C

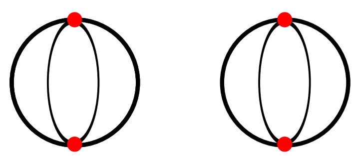


Type 4

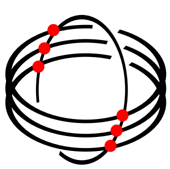
4A



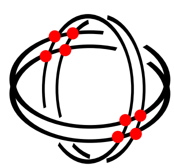
4B



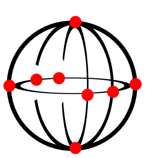
4C



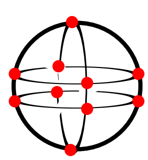
4D



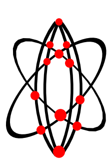
4E



4F



4G



Prior to explaining the above, we make some preparations:

Definition 3.2.1. Overlaps of two, three, and four rings are doublet, triplet, and quadruplet, respectively. A single ring is sometimes called singlet.

Example 3.2.2. 1B consists of two rings and a doublet.

Example 3.2.3. 1C consists of a singlet and a triplet.

Notation 3.2.4. We write the numbers of self-intersections as $\mathbb{2}$, $\mathbb{4}$, etc [2] .

Example 3.2.5. There are two red dots, or self-intersections, in 2A, which is denoted by $\mathbb{2}$ in **Table 1**.

Now we are ready to explain subtypes 1A$-$4G briefly:

- 1A: see the caption of type 1;
- 1B: composed of two rings and a doublet;
- 1C: composed of a ring and a triplet;
- 1D: two doublets;
- 1E: one quadruplet;
- 2A: see the caption of type 2;
- 2B: composed of a doublet and a PS;
- 3A: see the caption of type 3;
- 3B: composed of two rings, one of which contains a doublet [3] ;
- 3C: composed of a ring and a ring-containing PS;
- 4A: see the caption of type 4;
- 4B: two PS's;
- 4C: combination of a ring and a triplet;
- 4D: combination of two doublets;
- 4E: a PS plus two rings;

[2]This kind of notation might be reminiscent of notations used in a certain field of physics, which are *e.g.*, **10**+**5** , **64**=**8**+**56** , and so on.

[3]Inspired by "alpha" curve [3, Figure 2], which has one self-intersection and reflects the root $x=0$ of $y=x^2(x+1)$, a root of multiplicity 2 , we think of 3B as having $2\times 2=4$ self-intersections. *Cf.* $y^2=x^2(x+1)$. Similar reasoning will be applied to 4C, 4D, and so on.

- 4F: combination of a PS and a doublet;
- 4G: combination of a PS and two singlets.

We return to physics. In [2, Fig. 5], we postulated the neutrino gets mass by the decrease in the number of self-intersections in PS, that is, from 2 to 1, and we hit upon the following.

Interpretation 3.2.6. If the number of red dots, or self-intersections, of subtypes is 2, 4, 6, etc., such subtypes are interpreted as massless. Otherwise, massive.

Example 3.2.7. 1A is regarded as massive, since there are no red dots.

Example 3.2.8. 2A is regarded as massive, since it is the sum of two rings without red dots, which are massive, and a ring with two red dots, which is massless [4] .

Example 3.2.9. Likewise, 3A is regarded as massive.

Example 3.2.10. 4C is regarded as massless, since it has six red dots, or self-intersections.

Again mathematically, inspired by auxiliary constructs in plane geometry , we imagine spheres that are associated with rings in each subtype and expected to be helpful.

Definition 3.2.11. Such spheres are called 'auxiliary spheres' (AS's).

Example 3.2.12. In 1A, we imagine that the leftmost ring is a GC of a certain sphere. So there are four AS's in total.

Example 3.2.13. In 1B, overlap of rings on the right is regarded as a GC of a sphere, and thus there are three AS's in total.

Example 3.2.14. In 2A, two rings on the right are regarded as GC's of a sphere, and thus there are three AS's in total.

Example 3.2.15. In 3A, three rings on the right are regarded as GC's of a sphere, and thus there are two AS's in total.

Example 3.2.16. In 4A, all rings are regarded as GC's of one sphere, and thus there is one AS.

3.3 Wrap-up

Based on the above interpretation and definition, we tabulate the results we have so far obtained:

[4] Imagine *e.g.,* $2+0=2>0$.

Table 1 Classification of gravitons

Type	Subtype	Number of self-intersections	Mass	Number of AS's
1	A	0	>0	4
	B	0	>0	3
	C	0	>0	2
	D	0	>0	2
	E	0	>0	1
2	A	2	>0	3
	B	2	>0	2
3	A	2	>0	2
	B	4	>0	2
	C	6	>0	2
4	A	2	0	1
	B	4	0	2
	C	6	0	1
	D	8	0	1
	E	8	0	1
	F	10	0	1
	G	12	0	1

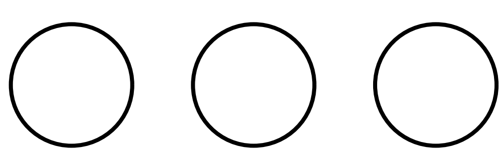
4 What if we let the number of rings decrease to 3?

In this case, we necessarily consider combinations of $4-1=3$ rings and the resultant spin-$\frac{3}{2}$ $(=\frac{1}{2}\times 3)$ particles as follows.

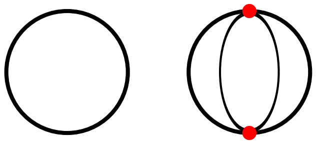
4.1 Three types of spin-$\frac{3}{2}$ particles

Like **3.1**, we enumerate thinkable types.

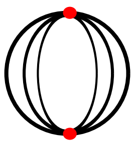
Type I



Type II



Type III



In words.

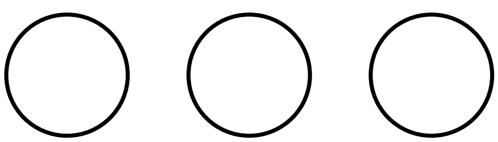
- type I: no coalescence/no self-intersection;
- type II: two rings coalesce to form two self-intersections;
- type III: all rings coalesce to form two self-intersections.

4.2 Further classification

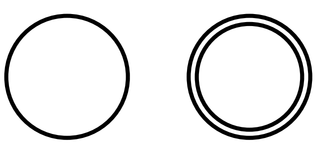
Like **3.2**, we further classify these types into some subtypes.

Type I

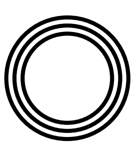
IA



IB

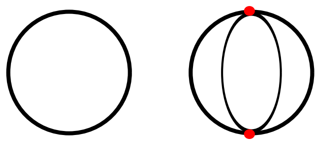


IC

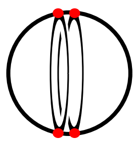


Type II

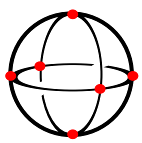
IIA



IIB

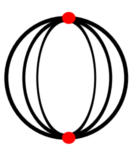


IIC



Type III

No further classification.



We explain IA−III succinctly:

- IA: see the caption of type I;
- IB: composed of a ring and a doublet;
- IC: a triplet;
- IIA: see the caption of type II;
- IIB: combination of a ring and a doublet;
- IIC: combination of a ring and a PS;
- III: see the caption of type III.

4.3 Wrap-up

Like **Table 1**, we tabulate the findings we have got:

Table 2 Classification of spin-$\frac{3}{2}$ particles

Type	Subtype	Number of self-intersections	Mass	Number of AS's
I	A	0	>0	3
	B	0	>0	2
	C	0	>0	1
II	A	2	>0	2
	B	4	0	1
	C	6	0	1
III	None	2	0	1

5 Deriving 'hexanion' from q

5.1 Some questions

Regarding each ring as $2+0$D stuff [5] , we compute $(2+0)\times 4 = 8+0$, which characterises the graviton as $8+0$D entity mathematically. That said, it is natural that one should raise the following question.

Question 5.1.1. How is mathematical $8+0$D connected with our (conventional) $3+1$D?

We substitute the following figure for our answer.

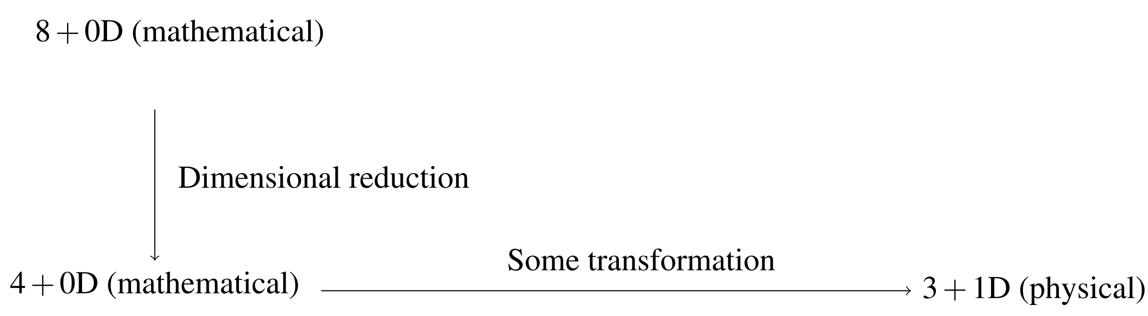


Fig. 2. Schematic diagram for 'realisation' of the graviton in our space-time

[5] By $m+n$D, we mean spatial and temporal dimensions are m and n, respectively. For example, $2+1$D means that spatial dimension is two, and temporal dimension is one. *Cf.* here .

Let us explain this more detailedly. We start from mathematical $8+0$D, which is subjected to dimensional reduction to become mathematical $4+0$D. Next, this $4+0$D is subjected to some spatio-temporal transformation to reach physical $3+1$D, our (conventional) space-time. By the way, talking of mathematical $8+0$D, it is known that norm of $\mathbb{O}$ agrees with the standard 8-dimensional Euclidean norm on $\mathbb{R}^8$. This inclines us to regard the graviton as something octonionic—— 'octonionic gravitons' [6] .

As for spin-$\frac{3}{2}$ particle, we can think of the following in a similar way.

$$
\begin{array}{ccc}
6+0\text{D (mathematical)} & & \\
\Big\downarrow \text{Dimensional reduction} & & \\
3+0\text{D (mathematical)} & \xrightarrow{\text{Some transformation}} & 2+1\text{D (physical)}
\end{array}
$$

Fig. 3. Schematic diagram for 'realisation' of spin-$\frac{3}{2}$ particle in $2+1$D

Explanation is as follows. Because spin-$\frac{3}{2}$ particle consists of three rings, we start from $(2+0\text{D})\times 3=6+0$D mathematically. Next, it is subjected to dimensional reduction to become mathematical $3+0$D. And this $3+0$D reaches physical $2+1$D via some spatio-temporal transformation. Because we have encountered mathematical $6+0$D, the second question is

Question 5.1.2. Is it worth taking pains to devise 'hexanion' (*hex*) to characterise spin-$\frac{3}{2}$ particles as 'hexanionic' stuff like 'octonionic gravitons'?

To address this question, we try to 'extend' q's, while keeping them as intact as possible. Specifically, we make use of

$$
\begin{cases} ijk=-1 \ , \\ kji=1. \end{cases}
$$

'Extending' the above with care [7], we set

$$
\begin{cases} ijk\ell m=-1, \\ m\ell kji=1. \end{cases}
$$

[6] Actually, we have already related the graviton with $\mathbb{O}$. See [2, **Table 1**].

[7] In other words, writing two letters following k after 'ijk' or before 'kji' in reverse order,. . .

We thus get the relations

$$\ell m = m\ell = 1. \tag{1}$$

In analogy with q's, we set $i^2 = j^2 = k^2 = \ell^2 = m^2 = -1$. Using (1) and the relations such as

$$\begin{cases} i = jk\ , \\ j = ki\ , \end{cases}$$

and so on, we obtain

Table 3 MT of *hex*'s under way

$\times$	1	i	j	k	ℓ	m
1	1	i	j	k	ℓ	m
i	i	-1	k	$-j$	$*$	$*$
j	j	$-k$	-1	i	$*$	$*$
k	k	j	$-i$	-1	$*$	$*$
ℓ	ℓ	$*$	$*$	$*$	-1	1
m	m	$*$	$*$	$*$	1	-1

N.B. In the above MT, '$*$''s signify entries which are pending.

5.2 Computing square of the norm of *hex*

We make a definition.

Definition 5.2.1. *hex* (resp. $\overline{hex}$) $= a+bi+cj+dk+e\ell+fm$ (resp. $a-bi-cj-dk-e\ell-fm$), where $a,b,c,d,e,f \in \mathbb{R}$.

And we try computing the product of *hex* and $\overline{hex}$.

$$\begin{aligned} hex\cdot\overline{hex} &= (a+bi+cj+dk+e\ell+fm)\cdot(a-bi-cj-dk-e\ell-fm) \\ &= a\cdot(a-bi-cj-dk-e\ell-fm)+bi\cdot(a-bi-cj-dk-e\ell-fm) \\ &\quad +cj\cdot(a-bi-cj-dk-e\ell-fm)+dk\cdot(a-bi-cj-dk-e\ell-fm) \\ &\quad +e\ell\cdot(a-bi-cj-dk-e\ell-fm)+fm\cdot(a-bi-cj-dk-e\ell-fm) \end{aligned}$$

$$
\begin{aligned}
&= a^2 - abi - acj - adk - ae\ell - afm + bia - bibi - bicj - bidk - bie\ell - bifm \\
&\quad + cja - cjbi - cjcj - cjdk - cje\ell - cjfm + dka - dkbi - dkcj - dkdk - dke\ell \\
&\quad - dkfm + e\ell a - e\ell bi - e\ell cj - e\ell dk - e\ell e\ell - e\ell fm + fma - fmbi - fmcj - fmdk \\
&\quad - fme\ell - fmfm \\
&= a^2 - abi - acj - adk - ae\ell - afm + abi - b^2 i^2 - bcij - bdik - bei\ell - bfim \\
&\quad + acj - bcji - c^2 j^2 - cdjk - cej\ell - cfjm + adk - bdki - cdkj - d^2 k^2 - dek\ell \\
&\quad - dfkm + ae\ell - be\ell i - ce\ell j - de\ell k - e^2 \ell^2 - ef\ell m + afm - bfmi - cfmj - dfmk \\
&\quad - efm\ell - f^2 m^2 \\
&= a^2 - b^2 i^2 - c^2 j^2 - d^2 k^2 - e^2 \ell^2 - f^2 m^2 - bc(ij + ji) - bd(ik + ki) - be(i\ell + \ell i) \\
&\quad - bf(im + mi) - cd(jk + kj) - ce(j\ell + \ell j) - cf(jm + mj) - de(k\ell + \ell k) \\
&\quad - df(km + mk) - ef(\ell m + m\ell). \qquad (2)
\end{aligned}
$$

By the way, let $z \in \mathbb{C}$ and write $z = a + b\imath$, $\bar{z} = a - b\imath$, where $a, b \in \mathbb{R}$. Then, we have

$$||z||^2 = z\bar{z} = \bar{z}z = a^2 + b^2. \qquad (3)$$

As for q's, writing $q = a + bi + cj + dk$, $\bar{q} = a - bi - cj - dk$, where $a, b, c, d \in \mathbb{R}$, one has

$$||q||^2 = q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2 \text{ [8]}. \qquad (4)$$

So our question is as follows.

Question 5.2.2. $hex \cdot \overline{hex} = \overline{hex} \cdot hex$ (like (3) and/or (4))?

To answer this question, we simply compute $\overline{hex} \cdot hex$.

$$
\begin{aligned}
\overline{hex} \cdot hex &= (a - bi - cj - dk - e\ell - fm) \cdot (a + bi + cj + dk + e\ell + fm) \\
&= a \cdot (a + bi + cj + dk + e\ell + fm) - bi \cdot (a + bi + cj + dk + e\ell + fm) \\
&\quad - cj \cdot (a + bi + cj + dk + e\ell + fm) - dk \cdot (a + bi + cj + dk + e\ell + fm) \\
&\quad - e\ell \cdot (a + bi + cj + dk + e\ell + fm) - fm \cdot (a + bi + cj + dk + e\ell + fm) \\
&= a^2 + abi + acj + adk + ae\ell + afm - bia - bibi - bicj - bidk - bie\ell - bifm \\
&\quad - cja - cjbi - cjcj - cjdk - cje\ell - cjfm - dka - dkbi - dkcj - dkdk - dke\ell \\
&\quad - dkfm - e\ell a - e\ell bi - e\ell cj - e\ell dk - e\ell e\ell - e\ell fm - fma - fmbi - fmcj - fmdk \\
&\quad - fme\ell - fmfm
\end{aligned}
$$

[8] *Cf.* [4].

$$
\begin{aligned}
&= a^2+abi+acj+adk+ae\ell+afm-abi-b^2i^2-bcij-bdik-bei\ell-bfim \\
&\quad -acj-bcji-c^2j^2-cdjk-cej\ell-cfjm-adk-bdki-cdkj-d^2k^2-dek\ell \\
&\quad -dfkm-ae\ell-be\ell i-ce\ell j-de\ell k-e^2\ell^2-ef\ell m-afm-bfmi-cfmj-dfmk \\
&\quad -efm\ell-f^2m^2 \\
&= a^2-b^2i^2-c^2j^2-d^2k^2-e^2\ell^2-f^2m^2-bc(ij+ji)-bd(ik+ki)-be(i\ell+\ell i) \\
&\quad -bf(im+mi)-cd(jk+kj)-ce(j\ell+\ell j)-cf(jm+mj)-de(k\ell+\ell k) \\
&\quad -df(km+mk)-ef(\ell m+m\ell). \qquad (5)
\end{aligned}
$$

Because $(2)=(5)$, square of the norm of *hex* has proven to be independent of the order of multiplication like (3) and/or (4).

We slightly simplify (2) (or (5)) using established entries in **Table 3** as follows.

$$
\begin{aligned}
&a^2-b^2\cdot(-1)-c^2\cdot(-1)-d^2\cdot(-1)-e^2\cdot(-1)-f^2\cdot(-1)-bc\{k+(-k)\}-bd\{(-j)+j\} \\
&-be(i\ell+\ell i)-bf(im+mi)-cd\{i+(-i)\}-ce(j\ell+\ell j)-cf(jm+mj)-de(k\ell+\ell k) \\
&-df(km+mk)-ef(1+1) \\
&= a^2+b^2+c^2+d^2+e^2+f^2-2ef-be(i\ell+\ell i)-bf(im+mi)-ce(j\ell+\ell j)-cf(jm+mj) \\
&-de(k\ell+\ell k)-df(km+mk).
\end{aligned}
$$

For the sake of clarification of '$*$''s in **Table 3**, we try setting

$$
\begin{cases}
i\ell+\ell i=0, & (6)\\
im+mi=0, & (7)\\
j\ell+\ell j=0, & (8)\\
jm+mj=0, & (9)\\
k\ell+\ell k=0, & (10)\\
km+mk=0. & (11)
\end{cases}
$$

As a result, we obtain

$$
||hex||^2 = hex\overline{hex} = \overline{hex}hex = a^2+b^2+c^2+d^2+(e-f)^2. \qquad (12)
$$

And applying $(6)-(11)$ to **Table 3**, we get

Table 4 MT of *hex*'s

$\times$	1	i	j	k	ℓ	m
1	1	i	j	k	ℓ	m
i	i	-1	k	$-j$	$-\ell i$	$-mi$
j	j	$-k$	-1	i	$-\ell j$	$-mj$
k	k	j	$-i$	-1	$-\ell k$	$-mk$
ℓ	ℓ	$-i\ell$	$-j\ell$	$-k\ell$	-1	1
m	m	$-im$	$-jm$	$-km$	1	-1

(12) leads us to consider the following three cases.

5.2.1 The case $e \neq f$

Because $e-f \neq 0$, replacing $e-f$ in (12) by $g \in \mathbb{R}$, one can write

$$||hex||^2 = a^2+b^2+c^2+d^2+g^2. \tag{13}$$

5.2.2 The case $e = f \neq 0$

In this case, we can rewrite *hex* and $\overline{hex}$ as $a+bi+cj+dk+e(\ell+m)$ and $a-bi-cj-dk-e(\ell+m)$, respectively [9] . And using a 'new' basis element n, one can further rewrite these as

$$\begin{cases} hex = a+bi+cj+dk+en, \\ \overline{hex} = a-bi-cj-dk-en. \end{cases}$$

5.2.3 The case $e = f = 0$

Likewise, we have that

$$\begin{cases} hex = a+bi+cj+dk, & (14) \\ \overline{hex} = a-bi-cj-dk, & (15) \end{cases}$$

which are the same as q and $\bar{q}$, respectively.

[9] See *Def.* 5.2.1.

6 Discussion

At the outset, graviton mass has been estimated to be zero or almost zero . Therefore, that **Table 1** is a mixture of the massive and the massless doesn't seem very unnatural at the time of writing [10], though our primary interest has lain in the massive rather than the massless. Next, we notice that some graviton subtypes contain PS's [11] . This might help to qualitatively explain why light fails to break loose from BH , provided that such inclusion is interpreted as capture or something like that.

What about spin-$\frac{3}{2}$ particle? Since we derived spin-$\frac{3}{2}$ particle from graviton, whose mass is very light at best (perhaps massless), paying attention to the Greek adjective leptós that means thin, light, etc. , we should like to think of spin-$\frac{3}{2}$ particles as leptonic rather than quark -like. And as in **Table 2**, spin-$\frac{3}{2}$ particles are presumed to be a mixture of the massive and the massless like gravitons in **Table 1**. This mixed status reminds us of Dirac/Weyl fermions , though none of the elementary particles in the SM are Weyl fermions . We have sometimes seen the simple fraction $\frac{3}{2}$; we once characterised particle and number system that are related to this value as something unknown [12], failing to mention the spin-$\frac{3}{2}$ Ω^- baryon [6]. Taking this opportunity, we hope introduction of *hex* and leptonic spin-$\frac{3}{2}$ particle will bring about some clarification.

As for 'transformations' in Fig.'s 2 and 3, it might be lofty to try to know whether there are some particles involved in such spatio-temporal transformations.

Incidentally, equating the RHS of (13) with 1 yields

$$a^2+b^2+c^2+d^2+g^2=1.$$

Here one might recall S^4, the four-dimensional unit sphere . That said, it is somewhat odd that we have encountered stuff in five-dimensional space, since our initial purpose was to devise ' *hexa*-nion'. It is unfortunate that we have no explanation about this oddity at present. As to whether the intactness of q's was actually kept during the derivation of *hex* from them, we point out that square of the norm of *hex* was computed like q [13]. Furthermore, if $e=f=0$, *hex*'s coincide with q's [14], which means that *hex*'s can subsume q's. So we believe the desired intactness is acknowledgeable at least partially and *hex*'s can be an 'extension' of q's.

Overall, our arguments have been qualitative rather than quantitative, since we haven't mentioned values such as 125 GeV, 0.511 MeV, $1.616229(38)\times 10^{-35}$ m [7], and so on. In this respect, it is likely that our approach has been quite unsatisfactory in terms of experiments.

Apropos of experiments, unambiguous detection of individual gravitons, though not prohibited by any fundamental law, has been thought impossible with any physically reasonable detector , whereas astronomical observations might point toward massive gravitons . And since it is known that space groups in $3+0$D are relevant to crystal system , physical insight into space groups might lead to crystallography-inspired experiment(s) [15] .

[10] *Cf.* [5, **Exercise 3.3**].

[11] See *e.g.*, 2A, 2B, etc. in **3.2**.

[12] See '?''s on both sides of spin-$\frac{3}{2}$ in [2, **Table 1**].

[13] Compare (12) or (13) with (4).

[14] See (14) and (15).

[15] *Cf.* Fig. 3.

In any event and finally, we wonder if experimental hunting of leptonic spin-$\frac{3}{2}$ particles (either massive or massless) is basically astronomy-oriented and things we have referred to as 'octonionic gravitons' have something to do with the "cosmic significance" of the octonions [8].

Acknowledgment. We should like to thank the developers of GIMP, Okular , PostScript , and Ti*k*Z for their indirect help, which enabled us to prepare figures and pictures for submission.

References

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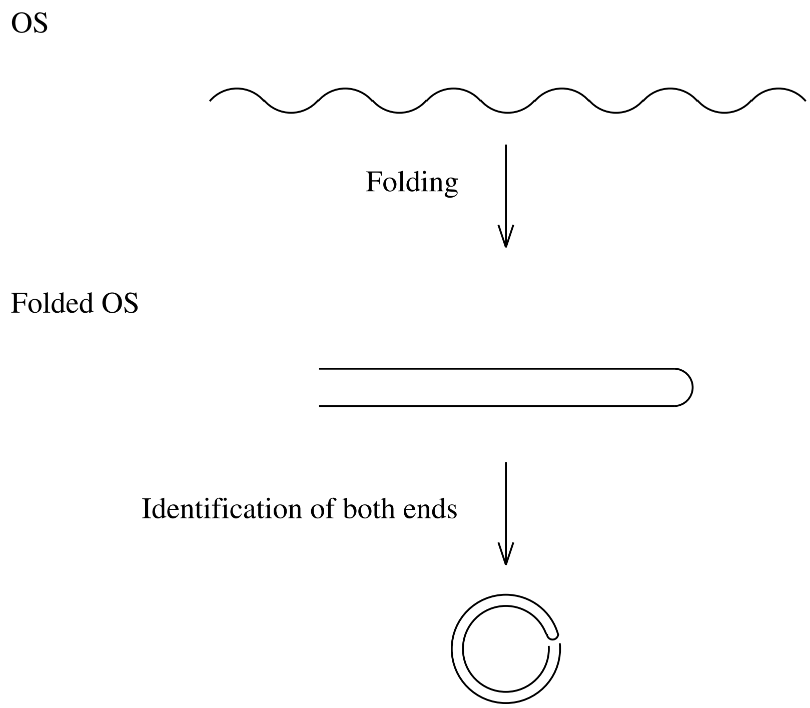
7 Appendix

7.1 If physical violation of the planckian is *too* annoying...

We fold an open string (OS), whose length is $\ell_{\rm p}$, identify one end with another, and prepare a ‘ring’ for ‘ring combinatorics’:

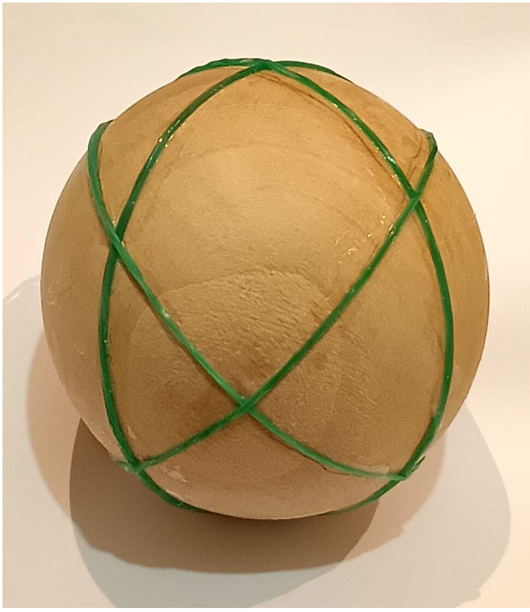
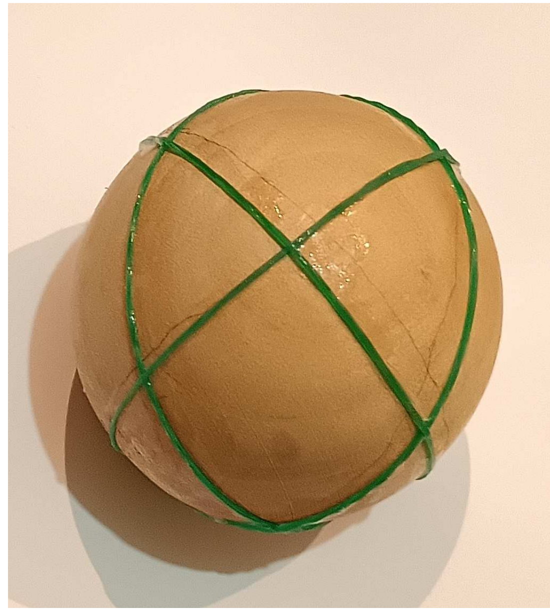


Unlike the process in Fig. 1, the string isn’t subjected to decomposition. So $\ell_{\rm p}$ is kept intact, and the folded string yields a circle whose circumference is $\frac{\ell_{\rm p}}{2}$. By substituting a pair of two such circles for the rings in Fig. 1, one circumvents the violation of the planckian somehow, though we aren’t sure whether this is a wrinkle.

7.2 Since 4G has seemed least accessible among graviton subtypes...

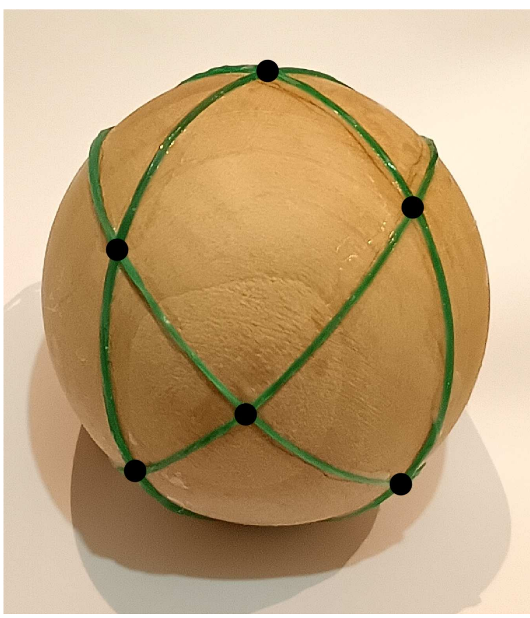
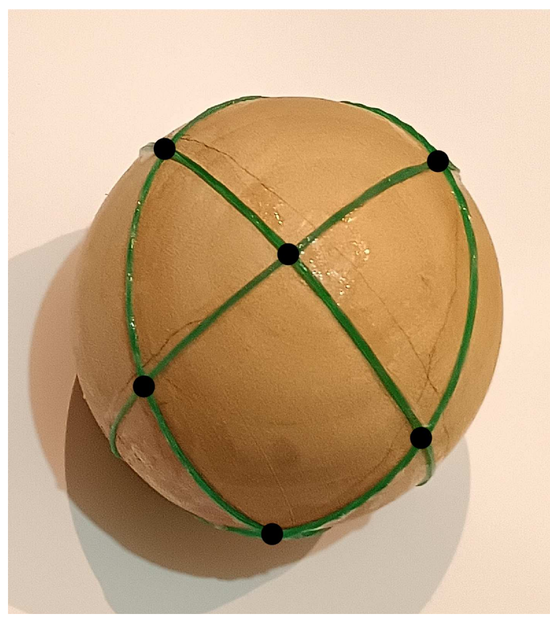
We liken rubber bands [16] and a wooden ball to the rings and AS of 4G, respectively as shown below:

[16]Green rubber bands were fixed by a kind of glue.

Front view   Back view

For the sake of emphasis, we overlap black dots on the self-intersections of the rubber bands in the above pictures [17] .

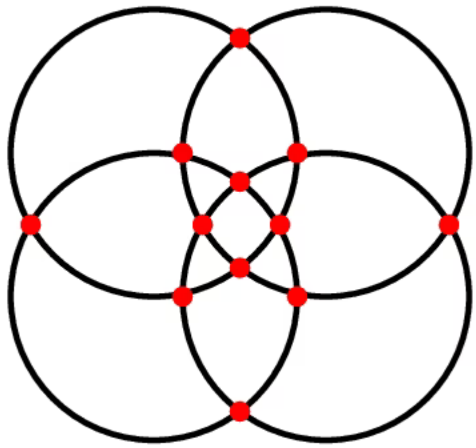
 

We see that there are $6+6=12$ dots, which will make $\mathbb{12}$, or the number of self-intersections of 4G, in **Table 1** more accessible (hopefully).

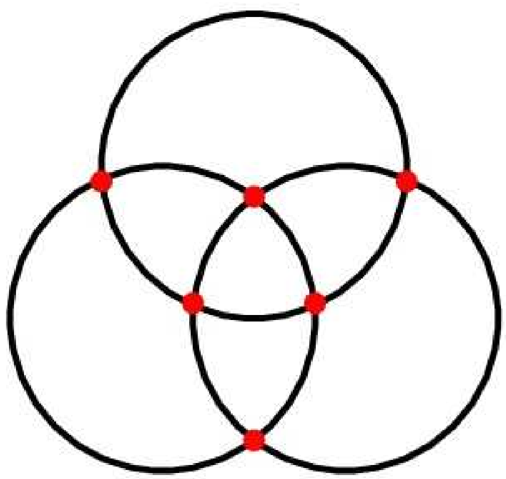
[17] GIMP ver.2.10.36 was used.

7.3 'Rough' understanding of some subtypes

At the sacrifice of the so-called mathematical rigour, we identify 4G and IIC with the following figures [18].



Remark 7.3.1. Like 4G, this is composed of four rings and has twelve self-intersections (indicated by red dots).



Remark 7.3.2. Like IIC, this consists of three rings [19] and has six self-intersections (indicated like the above figure) [20] .

[18] They might be of some practical use, however. See footnote 19.

[19] If somebody dares to associate these rings with 1987A supernova remnant , we might be able to say that the above 'three-ring-figure' has played some practical role. See footnote 18.

[20] Some might recall the Borromean rings [9, Figure 9.3(c)].

These ‘rough’ identifications might make 4G and IIC more accessible.

7.4 Will something in Table 1 or Table 2 turn out to be relevant to DM?

It is believed that DM hardly interacts with any of the SM particles through strong or electroweak forces, but interacts only through gravity [10]. So **Table 1**, which resulted from graviton classification, doesn’t seem irrelevant to DM, but....

7.5 What about DE?

Time will tell.

7.6 Can one say that Table 1 intimates DSR just because 4B consists of two PS’s?

Again, time will tell.