The error in Schrödinger's equation

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Abstract: The article points to the basic error in the wavefunction concept. The solution for the Schrödinger equation for the hydrogen atom is not an orbital, it is an orbit, especially, it is an elliptic orbit. The error is that the Quantum Mechanical solution ignores the angular momentum conservation law which in quantum physics is expressed as quantization of angular momentum. The Schrödinger equation must obey this rule as it is basically the same as de Broglie's wavelength, which is the stated motivation for making the momentum substitution.

Keywords: Schrödinger equation, de Broglie wavelength, angular momentum, quantization rules.

1. Quantization of angular momentum and the Schrödinger equation

The Schrödinger equation for a hydrogen atom is

$$\hat{H}\Psi = E\Psi \tag{1}$$

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0 r}\right)\Psi = E\Psi \tag{2}$$

where μ is the reduced mass of an electron. We can write $\mu = m_e$ with sufficient precision.

Schrödinger's equation (1) comes from the non-relativistic Hamiltonian describing a hydrogen atom

$$E = \frac{p^2}{2m} - \frac{g}{r} \tag{3}$$

where $g = e^2/4\pi\varepsilon_0$ and $p^2 = p_r^2 + p_\theta^2$ with $p_r = m\dot{r}$ and $p_\theta = mr\dot{\theta}$. We assume here that $p_\phi = 0$. It is allowed as the Quantum Mechanical solution has quantum number m, and if $m = 0$, then $p_\phi = 0$.

Schrödinger's substitution is

$$p^2 \quad \rightarrow \quad -\hbar^2\nabla^2. \tag{4}$$

This substitution is motivated by de Broglie's wavelength

$$\lambda = \frac{h}{p} \tag{5}$$

which for p_θ is the same as Bohr's quantization rule for angular momentum in the first Bohr radius ($n = 1$)

$$p_\theta\lambda = h \quad \rightarrow \quad p_\theta 2\pi r = h \quad \rightarrow \quad L = p_\theta r = \hbar. \tag{6}$$

Bohr's quantization rule is

$$L = p_\theta r = n\hbar. \tag{7}$$

This rule both assures that the angular momentum is conserved because the rule sets it to a constant, and makes L quantized. Also in Sommerfeld's atomic model there is an angular momentum quantization rule, see [2] or [3]. In Sommerfeld's model the solution is not an ellipse, it is a precessing ellipse.

The rule $L = \hbar$ is constant allows replacing p_θ by $p_\theta = L/r$. The Hamiltonian takes the form

$$E = \frac{p_r^2}{2m} + \frac{L^2}{2m}\frac{1}{r^2} - \frac{g}{r} \tag{8}$$

$$\frac{p_r^2}{2m} = E - \frac{g}{r} - \frac{L^2}{2m}\frac{1}{r^2}. \tag{9}$$

Now we do Schrödinger's substitution, the equation only has the r variable

$$-\hbar^2 \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\Psi = \left(E - \frac{g}{r} - \frac{L^2}{2m}\frac{1}{r^2}\right)\Psi. \tag{10}$$

The solution is $\Psi = Ar^l e^{\alpha r}$ as in the Quantum Machanical solution to the wavefunction of the hydrogen atom. Inserting this solution gives the equations

$$\alpha^2 = -\frac{2mE}{\hbar^2} \tag{11}$$

$$2\alpha(l+1) = \frac{2mg}{\hbar^2} \tag{12}$$

$$l(l+1) = \frac{L^2}{\hbar^2}. \tag{13}$$

Solving them gives

$$\alpha = i\frac{\sqrt{2mE}}{\hbar} \qquad l+1 = i\sqrt{\frac{m}{2E}}\frac{g}{\hbar} \qquad l = -i\frac{L^2}{\hbar g}\sqrt{\frac{2E}{m}}. \tag{14}$$

These values are impossible: l should be an integer, it is a quantum number in Quantum Mechanics.

The Quantum Mechanical solution is given in [1]. It is a slightly modified solution to Laplace's equation. The equation has a separating solution

$$\Phi = R(r)Y_l^m(\theta,\phi) \tag{15}$$

where $Y_l^m(\theta,\phi)$ are spherical harmonics

$$Y_l^m(\theta,\phi) = Ne^{im\phi}P_l^m(cos\theta) \tag{16}$$

and

$$R(r) = e^{-\frac{1}{2}ar}(ar)^l L(ar). \tag{17}$$

In this solution we can set $m = 0$ and then $p_\phi = 0$. Therefore it is possible to have $p^2 = p_r^2 + p_\theta^2$ in (8).

The error in the derivation in [1] (i.e., the derivation in Quantum Mechanics) is that the variables are separated in the equation but the quantization rule for the angular momentum is not used. The variables cannot be separated. When the angular momentum quantization rule is inserted in an early stage, there is no acceptable solution to Schrödinger's equation for the hydrogen atom.

In order to understand this error better, let us see how the Hamiltonian (3) can be correctly solved. It is a very old problem, long ago solved.

2. How to solve the Keplerian equation

The elegant way to solve (3) in Sommerfeld's solution (which is not the way Sommerfeld used, I found it in [2]. It is an old solution, I did check it in [3]) is to write

$$\dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} = \dot{\theta}r' \tag{18}$$

and

$$s = \frac{1}{r} \qquad s' = \frac{ds}{d\theta} = -\frac{1}{r^2}r'. \tag{19}$$

Then equation (3) is turned into (see [3] or [2] for details)

$$s'' + s - \frac{gm}{L^2} = 0 \tag{20}$$

and the solution is an ellipse. We can solve the equation in many ways, all ways give an ellipse. We can for instance eliminate p_θ with $p_\theta = L/r$ and solve the equation

$$\frac{p_r^2}{2m} = E - \frac{g}{r} - \frac{L^2}{2m}\frac{1}{r^2} \tag{21}$$

$$\dot{r} = \frac{1}{r}\sqrt{\frac{2E}{m}r^2 + \frac{2g}{m}r - \frac{L^2}{m^2}} \tag{22}$$

$$\int \frac{rdr}{\sqrt{\frac{2E}{m}r^2 + \frac{2g}{m}r - \frac{L^2}{m^2}}} = \int dt \tag{23}$$

$$\int \frac{(x+C)dr}{A\sqrt{(x+B)^2+1}} = \int dt \tag{24}$$

for some A, B, C giving $r = r(t)$ and the quantization rule for angular momentum gives $r = r(\theta)$, an ellipse. We can easily integrate the equation, or, like Sommerfeld, solve it with residue calculus by first writing it in complex numbers.

In a similar way, we can eliminate $\dot{r}$ and solve an equation for $\dot{\theta}$ and θ. It also gives an ellipse.

There always remains the relation between r and $\dot{\theta}$ given by $L = \hbar$, or $L = n\hbar$ for other S-states.

3. Conclusions

This error means that in the separating solution to Schrödinger's equation for the hydrogen atom there remains the angular momentum quantization rule, which certainly is intended as Schrödinger's equation is motivated by de Broglie's wavelength, but the rule is not applied.

The solutions to (1) are not orbitals. They are orbits, and especially they are ellipses. In Sommerfeld's solution (1) is changed to the relativistic Hamiltonian, the solutions are precessing ellipses. The relativistic kinetic energy formula is not correct, but with the relatively slow speeds of electrons in a hydrogen atom ($v \sim \alpha c$) the difference between the apparent mass and the relativistic mass is extremely small. It is necessary to use the apparent mass as Lorentz already vey long ago concluded that electrons do have apparent mass.

But quantum physics as it is today know is largely false. The error is in the wavefunction concept. Only old quantum physics of Bohr and Sommerfeld is rather valid. The rest should be revised. Some ideas in quantum physics may be good. Two of those may be Pauli's spinors and the covariant derivative, i.e., derivative that is invariant under a gauge transform of a gauge group, not Lorentz covariant. But too many ideas in quantum physics must be re-evaluated.

4. References

[1] Jormakka J., "Errors in the Quantum Mechanical calculation of the fine structure spectrum of the hydrogen atom", ResearchGate, 2026.

[2] Barley, K., Ruffins, A., Suslov, S.K., Old Quantum Mechanics by Bohr and Sommerfeld from a Modern Perspective, 2025, arxiv.org/html/2506.00408v5.

[3] Jormakka J., "A look at Sommerfeld's fine structure formula", ResearchGate, 2026.