Dynamics of the Memristor FitzHugh-Nagumo Equation and the Memristor Chua Circuit Equation

Makoto Itoh [1]
1-19-20-203, Arae, Jonan-ku,
Fukuoka, 814-0101 JAPAN
Email: itoh-makoto@jcom.home.ne.jp

Abstract
In this paper, we show that a voltage-controlled nonlinear resistor can be regarded as a voltage-controlled memristor when its behavior is observed using the time derivatives of state variables. Next, we derive the memristor FitzHugh-Nagumo equation by differentiating the original FitzHgh-Nagumo equation with respect to time. We also show that the trajectory of the memristor FitzHugh-Nagumo equation is highly dependent on the initial flux condition. This trajectory can be changed by applying a small single pulse. Furthermore, the square wave forcing creates many complex behaviors such as chaotic attractors and limit cycles. These behaviors also depend on the initial flux condition. Therefore, it is impossible to predict the trajectory without knowing the initial flux condition. Next, we observe the trajectory's evolution by adding a small DC bias current to the square wave forcing. We show that as time progresses, the trajectory changes shape and eventually shrinks. This resembles the refractory period of neurons. Finally, we study the signal transmission between the two memristor FitzHugh-Nagumo circuits which are connected by a unity-gain buffer. We show that, for the signal to be transmitted almost exactly, the initial flux condition must be identical. Additionally, we derive the memristor Chua circuit equation by differentiating the original Chua circuit equation with respect to time. Our findings are nearly identical to those for the memristor FitzHugh-Nagumo equation. We also study the signal transmission between two memristor Chua circuits that are connected by a unity-gain buffer. We show that nearly perfect transmission requires the identical initial flux condition. It should be noted that requiring the identical initial flux conditions results in more secure communication with the memristor Chua circuits than with the original Chua circuits. Finally, we would like to emphasize the following points: The memristor FitzHugh-Nagumo equation and the memristor Chua circuit equation both exhibit rich dynamics. The identification of the circuit elements depends on the coordinate system used to observe them.

Keywords
nonlinear resistor; memristor; FitzHugh-Nagumo equation; memristor FitzHugh-Nagumo equation; Chua circuit; memristor Chua circuit; initial flux condition; coordinate system; synchronization; evolution of trajectory; square wave forcing; refractory period; excitation mode; pulse signal; signal transmission; DC bias current; secure communication.

1 Introduction

The FitzHugh-Nagumo equation is a two-dimensional simplification of the Hodgkin-Huxley model of spike generation in squid giant axons. It can exhibit the qualitative behavior of neurons [1, 2]. The circuit model

[1] He has no affiliation since his retirement.

of the FitzHugh-Nagumo equation consists of six elements: a linear resistor, a voltage-controlled nonlinear resistor (tunnel diode), a capacitor, an inductor, a battery (DC voltage source), and a stimulus current source [1, 2].

In this paper, we first show that, when observed using the time derivatives of state variables, a voltage-controlled nonlinear resistor can be considered a voltage-controlled memristor. We also show that identifying the circuit elements depends on the coordinate system used to observe them.

Next, we derive the memristor FitzHugh-Nagumo equation by taking the time derivative of the original FitzHugh-Nagumo equation. We also present a circuit realization of the memristor FitzHugh-Nagumo equation. This realization requires fewer elements: a linear resistor, a voltage-controlled memristor, a capacitor, an inductor, and a stimulus current source. In other words, it does not require a battery.

We then study the trajectory of the memristor FitzHugh-Nagumo equation and show that its behavior is highly dependent on the initial flux condition, that is, the total amount of flux change in the past. Furthermore, this trajectory can be changed by applying a small single pulse. Additionally, we observe the behavior of the memristor FitzHugh-Nagumo equation under square wave forcing, and show that it exhibits complex behaviors, such as chaotic attractors and limit cycles. These behaviors also depend on the initial flux condition. Therefore, it is impossible to predict the trajectory without knowing the initial flux condition.

We also observe how the trajectory evolves when a small DC bias current is added to the square wave forcing. We show that the trajectory changes shape and eventually shrinks over time. This resembles the refractory period of neurons, during which they return to a resting state after a strong reaction. In order to transition from a reduced trajectory to a larger one again, a small DC bias current must be applied to the square wave.

Finally, we examine the signal transmission between two memristor FitzHugh-Nagumo circuits that are connected by a unity-gain buffer. We show that nearly perfect transmission requires an identical initial flux condition.

Additionally, we derive the memristor Chua circuit equation by differentiating the original Chua circuit equation with respect to time. As a result, our findings were nearly identical to those for the memristor FitzHugh-Nagumo equation. That is, its behavior is highly dependent on the initial flux condition. We also examine the signal transmission between two memristor Chua circuits that are connected by a unity-gain buffer. We show that nearly perfect signal transmission requires an identical initial flux condition. If this requirement is satisfied, then the two memristor Chua circuits exhibit nearly identical behavior. Thus, they are in a special relationship as a pair. It should be noted that requiring identical initial flux conditions results in more secure communication with the memristor Chua circuits than with the original Chua circuits.

In conclusion, both the memristor FitzHugh-Nagumo and the memristor Chua circuit equations exhibit rich dynamics.

2 Memristors

The Memristor is a 2-terminal electronic device, which was postulated by Chua [3, 4, 5] and found by Strukov et al. [6]. An ideal memristor can be described by a constitutive relation between the charge q and the flux φ,

$$q = g(\varphi) \quad \text{or} \quad \varphi = f(q), \tag{1}$$

where $g(\cdot)$ and $f(\cdot)$ are differentiable scalar-valued functions. Its terminal voltage v and terminal current i are described by (see Fig. 1)

$$i = G(\varphi)v \quad \text{or} \quad v = R(q)i, \tag{2}$$

where

$$v = \frac{d\varphi}{dt} \quad \text{and} \quad i = \frac{dq}{dt}, \tag{3}$$

which represent Faraday's law of induction and its dual law, respectively. Note that the flux $\varphi(t)$, the voltage $v(t)$, the charge $q(t)$, and the current $i(t)$ satisfy the following universal relationships:

$$\varphi(t) = \int_{-\infty}^{t} v(\tau)d\tau \quad \text{and} \quad q(t) = \int_{-\infty}^{t} i(\tau)d\tau. \qquad (4)$$

The nonlinear functions $G(\varphi)$ and $R(\varphi)$, called memductance and memristance, respectively, are defined by

$$G(\varphi) \triangleq \frac{dg(\varphi)}{d\varphi}, \quad \text{and} \quad R(q) \triangleq \frac{df(q)}{dq}, \qquad (5)$$

where we assume that they are the continuous scalar-valued functions. They represent the slope of the scalar function $q = g(\varphi)$ and $\varphi = f(q)$, respectively (called the memristor constitutive relation).

Thus, the voltage-controlled ideal memristor is defined by Eq. (1) or the state-dependent Ohm's law and its associated state equation given by

voltage-controlled ideal memristor

$$\left.\begin{aligned} i &= G(\varphi)v. \\ \frac{d\varphi}{dt} &= v. \end{aligned}\right\} \qquad (6)$$

Similarly, those for the current-controlled ideal memristor are given by

current-controlled ideal memristor

$$\left.\begin{aligned} v &= R(q)i, \\ \frac{dq}{dt} &= i, \end{aligned}\right\} \qquad (7)$$

(see also Fig. 1). The classification of the more generalized memristors is shown in [7, 8, 9].

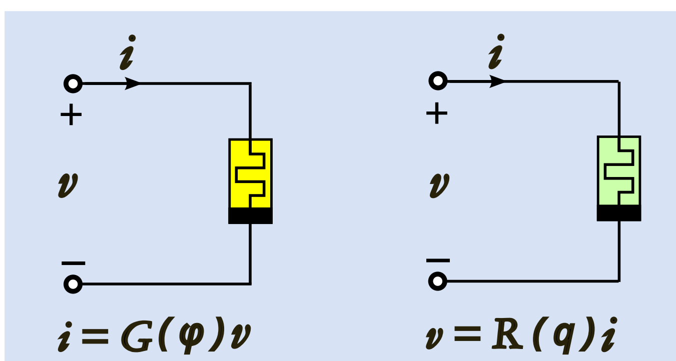


Figure 1: Flux-controlled memristor with the terminal current $i = G(\varphi)\, v$ (left). Charge-controlled memristor with the terminal voltage $v = R(q)\, i$ (right). Here v and i are the voltage across and the current through the ideal memristors, respectively, φ and q are the flux and the charge of the ideal memristors, respectively, and $G(\varphi)$ and $R(\varphi)$ are called the memductance and the memristance, respectively.

3 Relationship between Nonlinear Resistors and Memristors

Consider the nonlinear resistor with a characteristic curve defined by

$$i_R = f(v_R), \tag{8}$$

where i_R and v_R are the current through and voltage across the nonlinear resistor, respectively, and $f(\cdot)$ is assumed to be a continuously differentiable function. Suppose that the nonlinear resistor is operating within a circuit, such as a Van der Pol oscillator or a Chua circuit. In this case, i_R and v_R are considered to be functions of time.

Differentiating Eq. (8) with respect to time t, we obtain

$$\frac{di_R}{dt} = \frac{\partial f(v_R)}{\partial v_R}\frac{dv_R}{dt} = G(v_R)\frac{dv_R}{dt}, \tag{9}$$

where

$$G(v_R) \triangleq \frac{\partial f(v_R)}{\partial v_R}, \tag{10}$$

which is a nonlinear conductance function. We assume that $G(v_R)$ is a continuous bounded scalar-valued function of v_R.

Define the new state variables i_M, v_M, q_M and φ_M by

$$\left.\begin{array}{rcl} i_M &\triangleq& \dfrac{di_R}{dt}, \\ v_M &\triangleq& \dfrac{dv_R}{dt}, \\ q_M &\triangleq& i_R, \\ \varphi_M &\triangleq& v_R. \end{array}\right\} \tag{11}$$

Then, from Eq. (9), we obtain the state-dependent Ohm's law and its associated state equation

$$\left.\begin{array}{rcl} i_M &=& G(\varphi_M)\, v_M, \\ \dfrac{d\varphi_M}{dt} &=& v_M, \end{array}\right\} \tag{12}$$

where $G(\varphi_M)$ is considered to be a memductance function. Note that i_M, v_M, and φ_M are considered the new current, voltage, and flux variables, respectively, for the new coordinate system.

The relationship between Eqs. (8) and (12) is described as follows:

Eq. (8) for the nonlinear resistor $\xrightarrow{\text{differentiation}}$ Eq. (12) for the memristor

Since we assumed that the memductance $G(\varphi_M)$ is a continuous bounded scalar-valued function, we conclude that $i_M = 0$ when $v_M = 0$ from Eq. 12. Therefore, if we connect a periodic input voltage $v_s(t)$ (for example, $v_s(t) = A\sin(\omega t)$) across the memristor (12), and if we plot the voltage $v_M(t)$ and current $i_M(t)$ of the memristor on the two-dimensional (v_M, i_M)-plane, then the trajectory passes through the origin whenever $v_s(t) = v_M(t) = 0$. That is, $i_M(t) = 0$ whenever $v_M(t) = 0$. In this case, A and ω are some constants. It is important to note that the trajectory may intersect itself or with the horizontal v_M-axis, but not with the vertical i_M-axis [7]. Therefore, the voltage-controlled memristor defined by Eq. (12) satisfies the basic properties of a memristor.

Therefore, we conclude as follows:

- Equation (8) is transformed into Eq. (12) by differentiating with respect to time t.
- The (v_R, i_R)-characteristic of the voltage-controlled nonlinear resistor is transformed into the state-dependent Ohm's law and its associated state equation of the voltage-controlled memristor.
- The voltage-controlled nonlinear resistor is regarded as the voltage-controlled memristor on the new coordinate system, i.e., (v_M, i_M)-plane.

In summary, the above discussion can be summarized as follows:

We can consider the nonlinear resistor to be a memristor when we observe its behavior on the $\left(\frac{dv_R}{dt}, \frac{di_R}{dt}\right)$ plane.

The states $\frac{dv_R}{dt}$ and $\frac{di_R}{dt}$ can be observed theoretically by using an op-am-based differentiator and an op-amp-based current-to-voltage converter.[2]

In Appendix, we show that similar results hold for the voltage-controlled capacitors, current controlled inductors. and linear resistors. Based on these results, we can draw the following conclusion:

The behavior of circuit elements can appear different depending on the coordinate system used to observe them.

4 Memristor FitzHugh-Nagumo Equation

4.1 Deviation of the memristor FitzHugh-Nagumo equation

(a) FitzHugh-Nagumo equation

The FitzHugh-Nagumo equation is a two-dimensional simplification of the Hodgkin-Huxley model, which is defined by

FitzHugh-Nagumo equation

$$\left.\begin{aligned} \frac{dV}{dt} &= V - \frac{V^3}{3} - W + I, \\ \frac{dW}{dt} &= 0.08(V + 0.7 - 0.8W), \end{aligned}\right\} \qquad (13)$$

where V is the membrane potential, W is a recovery variable, I is the magnitude of stimulus current [1, 2]. Equation (13) is realized by the circuit shown in Fig. 2, as described in [1].

[2]An op-amp-based differentiator produces an output voltage, which is equal to the differential of input voltage. An op-amp-based current-to-voltage converter (transimpedance amplifier) converts an input current into a proportional output voltage.

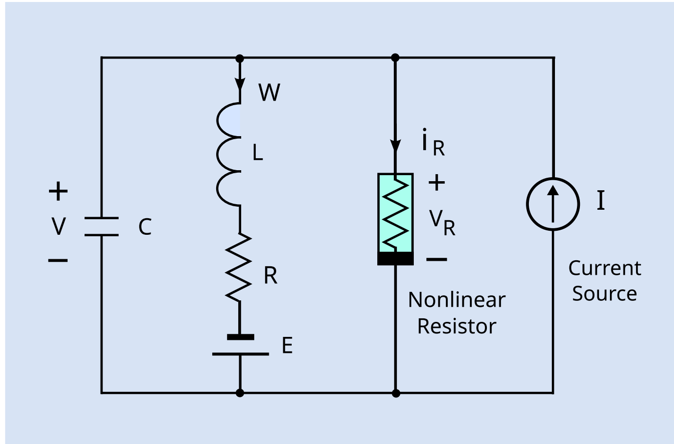


Figure 2: FitzHugh-Nagumo circuit. Parameters: $C = 1$, $L = \dfrac{1}{0.08}$, $R = 0.8$, $E = 0.7$. The characteristic of the nonlinear resistor is given by $i_R = \dfrac{{v_R}^3}{3} - v_R$. The symbol I denotes a current source.

(b) Memristor FitzHugh-Nagumo equation

Differentiating Eq. (13) with respect to time t, we obtain

$$\left.\begin{aligned} \frac{d^2V}{dt^2} &= \left(1 - V^2\right)\frac{dV}{dt} - \frac{dW}{dt} + \frac{dI}{dt}, \\ \frac{d^2W}{dt^2} &= 0.08\left(\frac{dV}{dt} - 0.8\frac{dW}{dt}\right). \end{aligned}\right\} \tag{14}$$

Define next the new state variables i, v, q and φ by

$$\left.\begin{aligned} v &\triangleq \frac{dV}{dt}, \\ i &\triangleq \frac{dW}{dt}, \\ \varphi &\triangleq V, \\ q &\triangleq W, \\ J &\triangleq \frac{dI}{dt}, \end{aligned}\right\} \tag{15}$$

Then, from Eq. (14), we obtain the **memristor FitzHugh-Nagumo equation**

Memristor FitzHugh-Nagumo equation

$$\left.\begin{aligned} \frac{dv}{dt} &= -\left(\varphi^2 - 1\right)v - i + J, \\ \frac{di}{dt} &= 0.08\left(v - 0.8i\right), \\ \frac{d\varphi}{dt} &= v, \end{aligned}\right\} \tag{16}$$

which is defined the new coordinate (v, i, φ)-space. The memristor FitzHugh-Nagumo equation (16) can be realized by using the circuit shown in Fig. 3.[3] Note that the flux φ can be observed theoretically by using the op-amp-based integrator[4], since $\varphi(t)$ is given by

$$\varphi(t) = \int_{-\infty}^{t} v(\tau) d\tau. \tag{17}$$

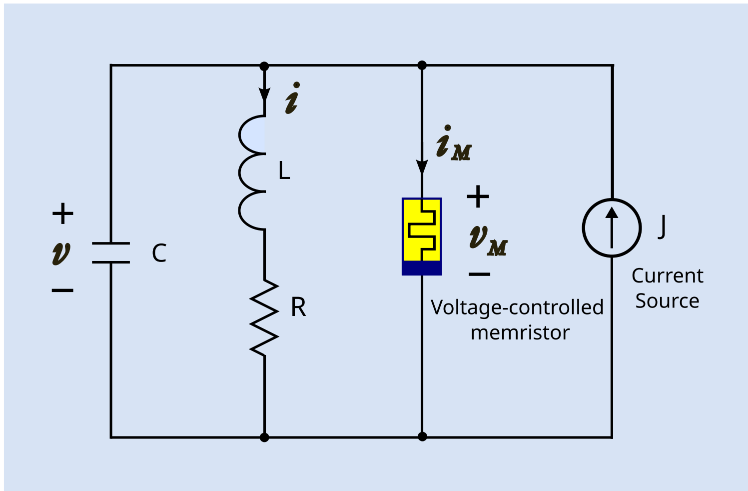


Figure 3: Memristor FitzHugh-Nagumo circuit, where i is the current through the inductor and v is the voltage across the capacitor, and i_M and v_m are the current through and voltage across the voltage-controlled memristor. Parameters: $C = 1$, $L = \dfrac{1}{0.08}$, $R = 0.8$. The voltage-controlled memristor is defined by the state-dependent Ohm's law and its associated state equation, that is, $i_M = ({\varphi_M}^2 - 1)v_M$ and $\dfrac{d\varphi_M}{dt} = v_M$, where φ_M denotes the flux of the memristor. The symbol J denotes a current source.

(b-1) Ohm's law of the voltage-controlled memristor

The state-dependent Ohm's law and its associated state equation for the the voltage-controlled memristor shown in Fig. 3 is given by

$$\left.\begin{aligned} i_M &= G(\varphi_M)\, v_M = ({\varphi_M}^2 - 1)\, v_M, \\ \frac{d\varphi_M}{dt} &= v_M, \end{aligned}\right\} \tag{18}$$

where v_M is the voltage across the memristor, i_M is the current through the memristor, φ_M is the flux of the memristor, and $G(\varphi_M)$ is the memductance which is defined by

$$G(\varphi_M) = {\varphi_M}^2 - 1. \tag{19}$$

Note that in this circuit, $\varphi_M = \varphi = \int_{-\infty}^{t} v(\tau)d\tau$, since $v_M = v$.

[3]The synchronization and chaos in coupled memristor-based FitzHugh-Nagumo circuits is studied in [10, 11]. In these circuits, the memductance of the memristor is defined by a piece-wise function, but it is not a polynomial function.

[4]The op-amp-based integrator is an operational amplifier circuit that performs the mathematical operation of integration.

(b-2) Constitutive relation of the voltage-controlled memristor

The constitutive relation of the memristor in Fig. 3 is obtained by integrating Eq. (18) with respect to time t. This integration yields:

$$\boxed{q_M = \frac{{\varphi_M}^3}{3} - \varphi_M + d_M, \qquad (20)}$$

where d_M is a constant of integration and φ_M and q_M are the flux and charge of the memristor, and they are defined by

$$\varphi_M(t) = \int_{-\infty}^{t} v_M(\tau)d\tau \quad \text{and} \quad q_M(t) = \int_{-\infty}^{t} i_M(\tau)d\tau. \tag{21}$$

Since $\varphi_M = \varphi$, we can derive the following from Eq. (20):

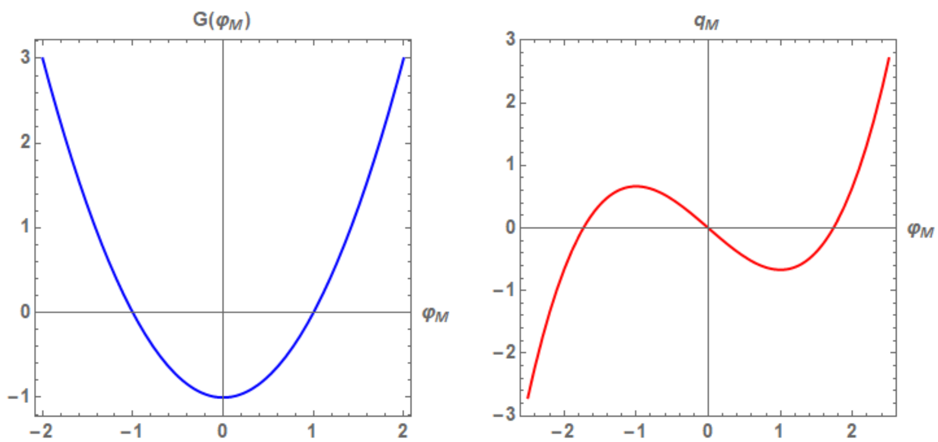
$$q_M = \frac{\varphi^3}{3} - \varphi + d_M. \tag{22}$$

Note that q and q_M are not equivalent, since q is defined by

$$\boxed{q = W = \int_{-\infty}^{t} i(\tau)d\tau, \qquad (23)}$$

where i is the current through the inductor (see Eq. 15).

We plot the memductance $G(\varphi_M) = {\varphi_M}^2 - 1$ and the constitutive relation: $q_M = \dfrac{{\varphi_M}^3}{3} - \varphi_M + d_M$ in Fig. 4, where we set $d_M = 0$ for the sake of simplicity.



(a) Memductance $G(\varphi_M)$ (b) Constitutive relation

Figure 4: Memductance $G(\varphi_M) = {\varphi_M}^2 - 1$ and constitutive relation: $q_M = \dfrac{{\varphi_M}^3}{3} - \varphi_M$ of the voltage-controlled memristor.

(b-3) Ohm's law from the constitutive relation

The Ohm's law is obtained by differentiating the constitutive relation (20) with respect to time t. This differentiation yields:

$$\frac{dq_M}{dt} = ({\varphi_M}^2 - 1)\frac{d\varphi_M}{dt}. \tag{24}$$

It can be recast into the form

$$i_M = G(\varphi_M)v_M. \tag{25}$$

where we set

$$\left.\begin{array}{rcl} \dfrac{dq_M}{dt} & = & i_M, \\ \dfrac{d\varphi_M}{dt} & = & v_M, \\ G(\varphi_M) & = & {\varphi_M}^2 - 1. \end{array}\right\} \tag{26}$$

Thus, we obtained the Ohm's law and its associated state equation (18).

(b-4) Two important relationships

The relationship between the constitutive relation (20) and Ohm's law (18) is described as follows:

$$\text{Constitutive relation (20)} \ \underset{\xleftarrow[\text{integration}]{}}{\xrightarrow{\text{differentiation}}} \ \text{Ohm's law (18)} \tag{27}$$

Furthermore, the relationship between the FitzHugh-Nagumo equation (13) and the memristor FitzHugh-Nagumo equation (16) is described as follows:

$$\text{FitzHugh-Nagumo equation (13)} \xrightarrow{\text{differentiation}} \text{Memristor FitzHugh-Nagumo equation (16)}$$

We conclude as follows:

Equation (13) is transformed into Eq. (16) by differentiating with respect to time t. In other words, this operation transforms the FitzHugh-Nagumo equation into the memristor FitzHugh-Nagumo equation.

Observe that the difference between the circuits in Fig. 2 and Fig. 3 is the nonlinear elements (i.e., a nonlinear resistor and a memristor) and the DC voltage source (i.e., a battery). Thus, they do not have the same dynamics.

4.2 Relationship between the FitzHugh-Nagumo equation and the memristor FitzHugh-Nagumo equation

In this section, we obtain the relationship between the original FitzHugh-Nagumo equation and the memristor FitzHugh-Nagumo equation. By integrating the first two equations of Eq. (16) with respect to time t, we obtain

$$\left.\begin{aligned} \frac{d\varphi}{dt} &= \varphi - \varphi^3/3 + q + I + d_1, \\ \frac{dq}{dt} &= 0.08(\varphi - 0.8q + d_2), \end{aligned}\right\} \tag{28}$$

where

$$\left.\begin{aligned} \varphi(t) &\triangleq \int_{-\infty}^{t} v(\tau)d\tau = \varphi(0) + \int_{0}^{t} v(\tau)d\tau, \\ q(t) &\triangleq \int_{-\infty}^{t} i(\tau)d\tau = q(0) + \int_{0}^{t} i(\tau)d\tau, \\ I(t) &\triangleq \int_{-\infty}^{t} J(\tau)d\tau = I(0) + \int_{0}^{t} J(\tau)d\tau, \end{aligned}\right\} \tag{29}$$

for $t \geq 0$, and d_1 and d_2 are the constants of integration. If we set $d_1 = 0$ and $d_2 = 0.7$, then we obtain the following equation:

$$\left.\begin{aligned} \frac{d\varphi}{dt} &= \varphi - \varphi^3/3 - q + I, \\ \frac{dq}{dt} &= 0.08(\varphi - 0.8q + 0.7), \end{aligned}\right\} \tag{30}$$

which is defined on the (q, φ)-plane. Note that from Eq. (29), we obtain

$$\varphi(0) \triangleq \int_{-\infty}^{0} v(\tau)d\tau = \int_{-\infty}^{0} \frac{d\varphi(\tau)}{d\tau} d\tau = \int_{-\infty}^{0} d\varphi(\tau), \tag{31}$$

which implies the total amount of past changes of φ. Similarly, $q(0)$ and $I(0)$ are the total amount of past changes of q and I.

Changing the variables φ and q with V and W respectively, we obtain the FitzHugh-Nagumo equation (13), that is,

$$\left.\begin{aligned} \frac{dV}{dt} &= V - \frac{V^3}{3} - W + I, \\ \frac{dW}{dt} &= 0.08(V + 0.7 - 0.8W), \end{aligned}\right\} \tag{32}$$

which is defined on the new coordinate (V, W)-plane. The relationship between Eqs. (16) and (13) is described as follows:

Memristor FitzHugh-Nagumo equation (16) $\xrightarrow{\text{integration}}$ FitzHugh-Nagumo equation (13)

Therefore, we conclude as follows:

Integrating with respect to time t transforms Eq. (16) into Eq. (13). In other words, this operation converts the memristor FitzHugh-Nagumo equation to the original FitzHugh-Nagumo equation.

Furthermore, the following point should be noted:

- The memristor FitzHugh-Nagumo equation (16) is defined by a set of three nonlinear differential equations in the three-dimensional (v, i, φ)-space.
- The FitzHugh-Nagumo equation (13) is defined by a set of two nonlinear differential equations on the two-dimensional (V, W)-plane.

  Therefore, they are not equivalent and can exhibit different behaviors.. We show its details in the next section.

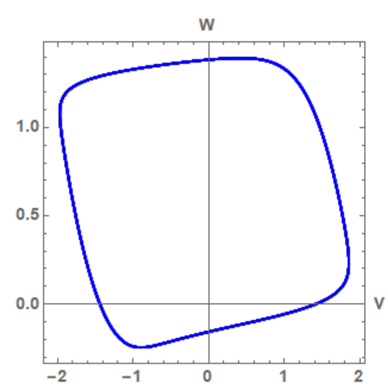
4.3 Dependence on initial flux conditions

This section discusses the dependence of the initial flux $\varphi(0)$ on the trajectories of the memristor FitzHugh-Nagumo equation. Let us first show the trajectory of the original FitzHugh-Nagumo equation, which is given by

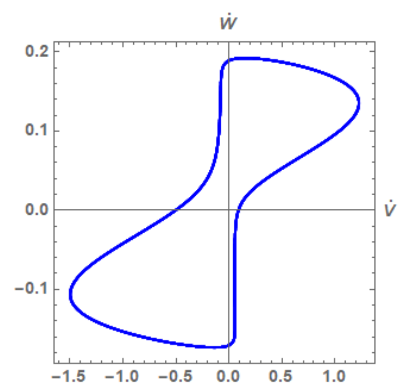
FitzHugh-Nagumo equation

$$\left.\begin{aligned} \frac{dV}{dt} &= V - \frac{V^3}{3} - W + I, \\ \frac{dW}{dt} &= 0.08(V + 0.7 - 0.8W), \end{aligned}\right\} \qquad (33)$$

where we set $I = 0.5$. As shown in Fig. 5, Eq. (33) exhibits a limit cycle when the initial conditions are $V(0) = 0.5$ and $W(0) = 0.5$. We show next the waveforms of $V(t)$ and $W(t)$ of the limit cycle in Fig. 6.

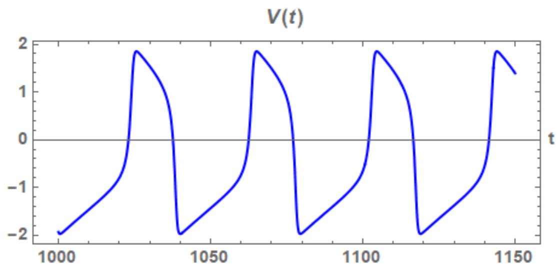


(a) Limit cycle on the (V, W)-plane.

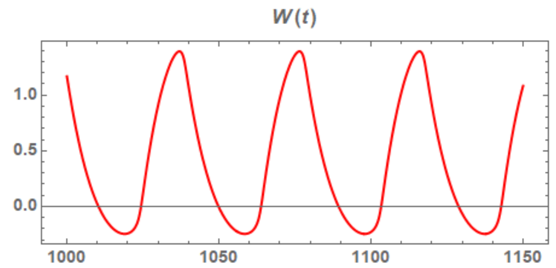


(b) Limit cycle on the $(\dot{V}, \dot{W})$-plane.

Figure 5: Limit cycle of the FitzHugh-Nagumo equation (33), where $(\dot{V}, \dot{W}) \triangleq \left(\frac{dV}{dt}, \frac{dW}{dt}\right)$. Initial conditions: $V(0) = 0.5$ and $W(0) = 0.5$. Parameter: $I = 0.5$.



(a) $V(t)$



(b) $W(t)$

Figure 6: Waveforms $V(t)$ and $W(t)$ of the limit cycle shown in Fig. 5. Initial conditions: $V(0) = 0.5$ and $W(0) = 0.5$. Parameter: $I = 0.5$.

Consider next the memristor FitzHugh-Nagumo equation

Memristor FitzHugh-Nagumo equation

$$\left.\begin{aligned} \frac{dv}{dt} &= -\left(\varphi^2 - 1\right)v - i + J, \\ \frac{di}{dt} &= 0.08\,(v - 0.8i)\,, \\ \frac{d\varphi}{dt} &= v, \\ \frac{dq}{dt} &= i, \end{aligned}\right\} \tag{34}$$

where we set $J = \frac{dI}{dt} = 0$ since $I = 0.5$ for $t \geq 0$ (see Eq. (15)). The last equation was added to plot the trajectory on the (φ, q)-plane. The equilibrium point that satisfies $\frac{dv}{dt} = \frac{di}{dt} = \frac{d\varphi}{dt} = \frac{dq}{dt} = 0$, is given by $(v, i, \varphi, q) = (0, 0, c_1, c_2)$, where c_1 and c_2 are some constants. The stability of the equilibrium point depends on φ. If $J \neq 0$, then Eq. (34) does not have the equilibrium point.

To compare the trajectories of the original FitzHugh-Nagumo equation (33) and the memristor FitzHugh-Nagumo equation (34), we calculate the initial conditions of Eq. (34) using the initial conditions of Eq. (33). From Eqs. (15) and (33), we can obtain the initial conditions for $v(t)$ and $i(t)$ as follows:

$$\left.\begin{aligned} v(0) &= \left.\frac{dV}{dt}\right|_{t=0} &= -W(0) + V(0) - V(0)^3/3.0 + 0.5, \\ i(0) &= \left.\frac{dW}{dt}\right|_{t=0} &= 0.08(V(0) + 0.7 - 0.8W(0)). \end{aligned}\right\} \tag{35}$$

By substituting $V(0) = 0.5$ and $W(0) = 0.5$, we obtain $v(0) = 0.375$ and $i(0) = 0.064$. The following results were obtained through computer simulations of Eq. (33):

- Figure 7 shows the trajectory of Eq. (34) for different initial flux conditions, i.e., for different value of $\varphi(0)$. If we compare the trajectories shown in Figs. 5 and 7, then we can find that Figs. 7 (j) and (k) closely resemble those in Fig. 5.
- Figure 8 shows the waveforms of $\varphi(t)$ and $q(t)$, which resemble to those in Fig. 6.

Note that changing the value of $\varphi(0)$ can cause the memristor FitzHugh-Nagumo equation (34) to behave significantly differently, as shown in Fig. 7. In other words, $\varphi(0)$ is considered the bifurcation parameter. Therefore, we conclude as follows:

> The behavior of the trajectory for the memristor FitzHugh-Nagumo equation (34) depends on the value of $\varphi(0)$, which is the total amount of past changes of φ.

In other words, it can be stated as follows:

> It is difficult to determine which trajectory will appear without knowing the initial condition $\varphi(0)$.

4.4 Trajectory change by a small pulse signal

In this section, we show that we can change the trajectory by applying a small pulse signal to the circuit through the current source $J(t)$. For example, assume that $J(t)$ in Eq. (34) is given by

$$J(t) = \begin{cases} 0, & \text{for } \ t < 5000, \\ 0.00002, & \text{for } \ 5000 \leq t \leq 5500, \\ 0, & \text{for } \ t > 5500. \end{cases} \tag{36}$$

The waveform of $J(t)$ is shown in Fig. 9.

As illustrated in Fig 10, applying a small pulse signal $J(t)$ changes the trajectory of Eq. (34) for $\varphi(0) = 0.27$. That is, the trajectory changes as follows:

> At first, the trajectory of Eq. (34) does not oscillate. After that, it oscillates chaotically, and then it finally forms a closed orbit on the (φ, q)-plane.

That is, we conclude as follows:

> A small single pulse can change the trajectory of the memristor FitzHugh-Nagumo equation (34).

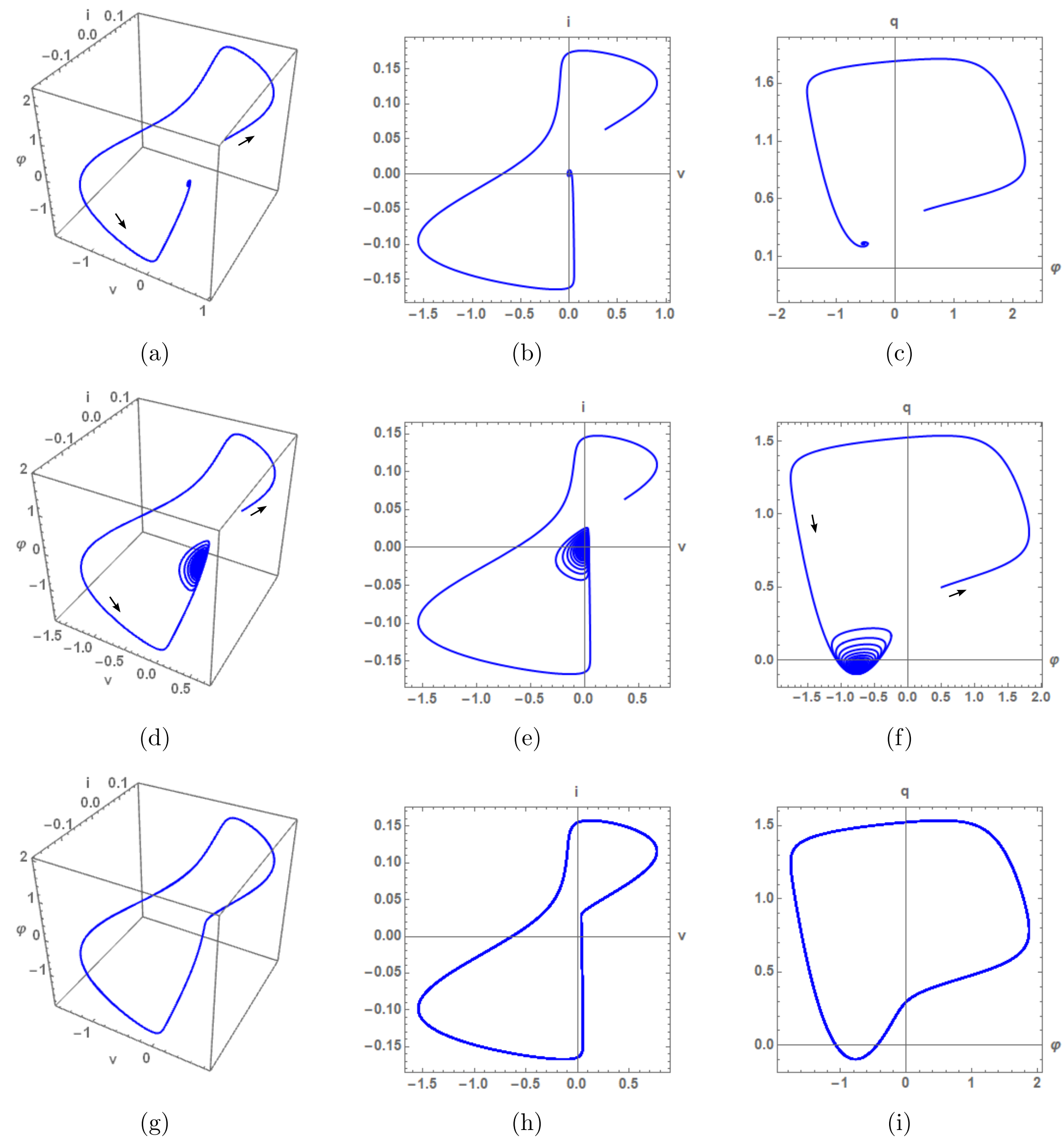


Figure 7: Trajectories in the (v, i, φ)-space (left), the (v, i)-plane (center), and the (φ, q)-plane (right) of the memristor FitzHugh-Nagumo equation (34), which depends on the initial condition $\varphi(0)$. Parameter: $J = 0$. Initial conditions: $v(0) = 0.375$, $i(0) = 0.064$, and $q(0) = 0.5$. The initial condition $\varphi(0)$ is given as follows: (a)-(c): $\varphi(0) = 0$. (d)-(f): $\varphi(0) = 0.27$. (g)-(h): $\varphi(0) = 0.27038$.

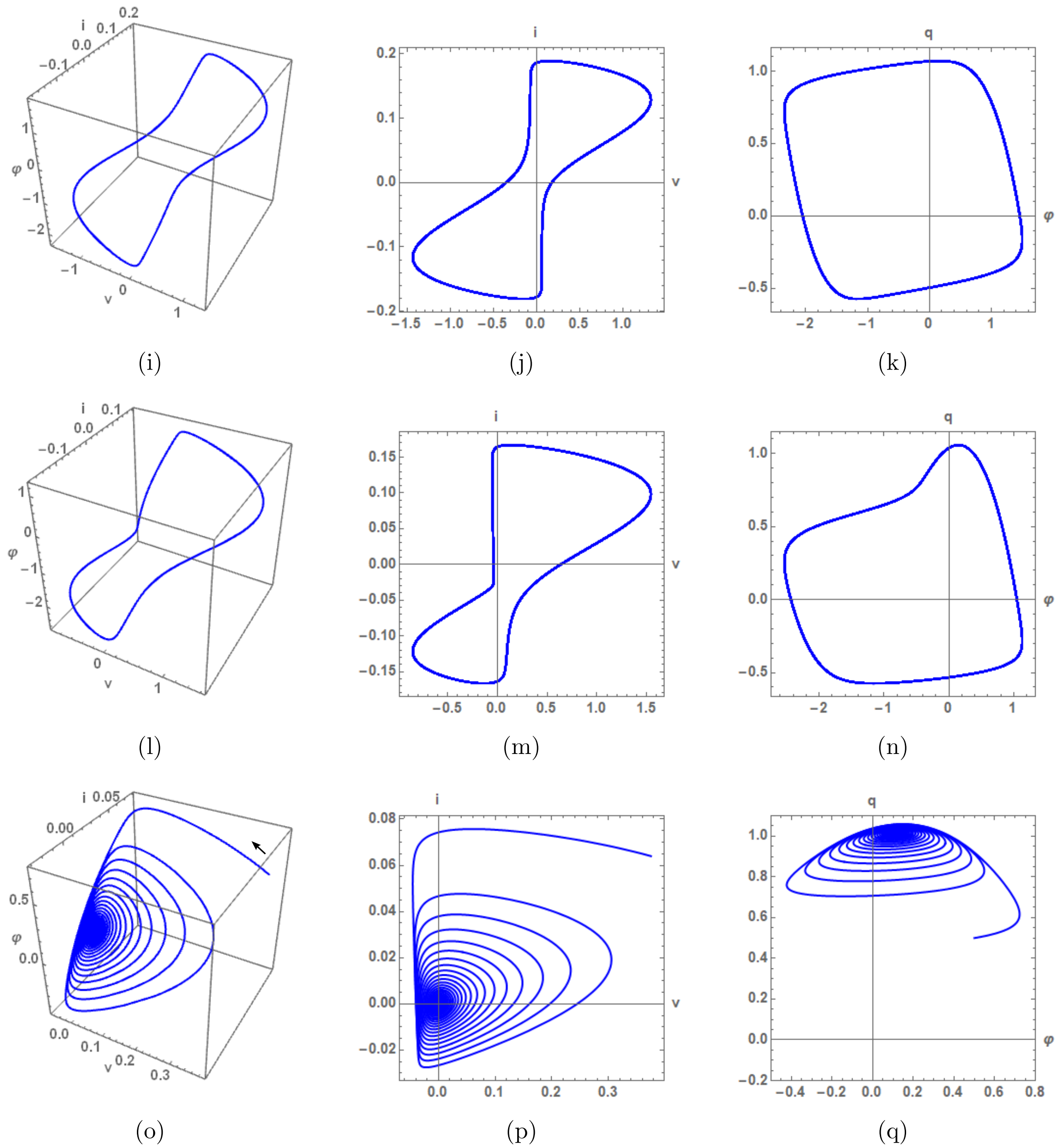
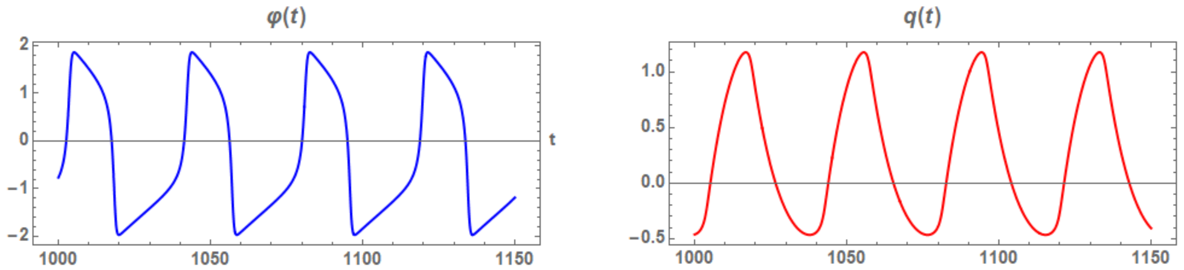


Figure 7: (Continued). Initial conditions: (i)-(k): $\varphi(0) = 0.7$. (l)-(n): $\varphi(0) = 1.3587$. (0)-(q): $\varphi(0) = 1.35875$.



(a) $\varphi(t)$ (b) $q(t)$

Figure 8: Waveforms $\varphi(t)$ and $q(t)$. Initial conditions: $v(0) = 0.375$, $i(0) = 0.064$, $q(0) = 0.5$, and $\varphi(0) = 0.7$.

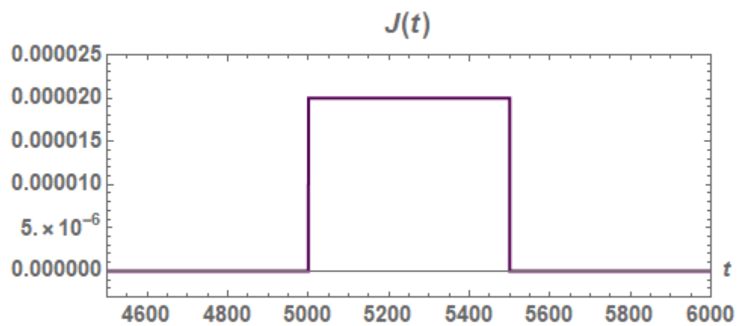
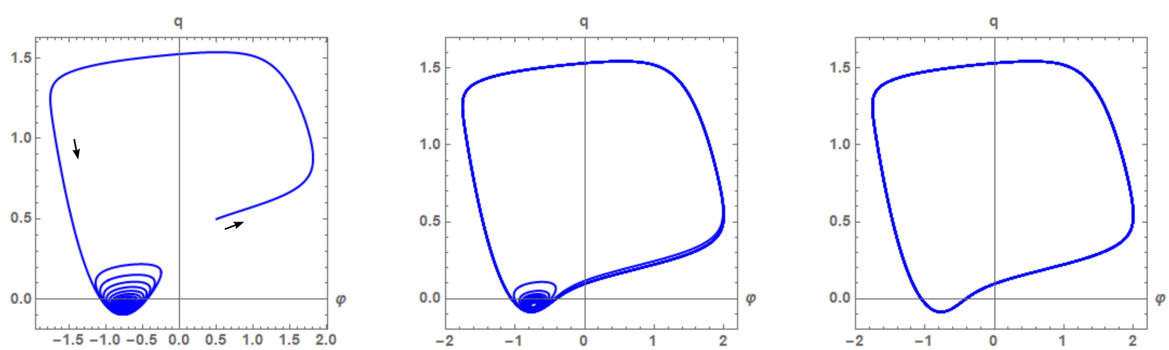


Figure 9: Wave form of the pulse signal $J(t)$, which is supplied the memristor FitzHugh-Nagumo circuit.



(a) $0 \leq t \leq 5000$ (b) $8000 \leq t \leq 9000$ (c) $9000 \leq t \leq 10000$

Figure 10: Trajectory change of Eq. (34) for the small pulse signal $J(t)$, which is defined by Eq.(36). At first, the trajectory of Eq. (34) does not oscillate (left). After that, it oscillates chaotically (center), and then it finally oscillates in the closed orbit on th (φ, q)-plane (right). Initial conditions: $v(0) = 0.375$, $i(0) = 0.064$, $q(0) = 0.5$, and $\varphi(0) = 0.27$.

4.5 Forced memristor FitzHugh-Nagumo equation

In this section, we apply the square wave $J(t)$ to the memristor FitzHugh-Nagumo circuit. The $J(t)$ shown in Fig. 11 is defined as follows:

$$J(t) = 0.005\,\mathrm{sign}\{\sin(0.22t)\} \tag{37}$$

where $\mathrm{sign}(x)$ gives -1, 0, or 1 depending on whether x is negative, zero, or positive.

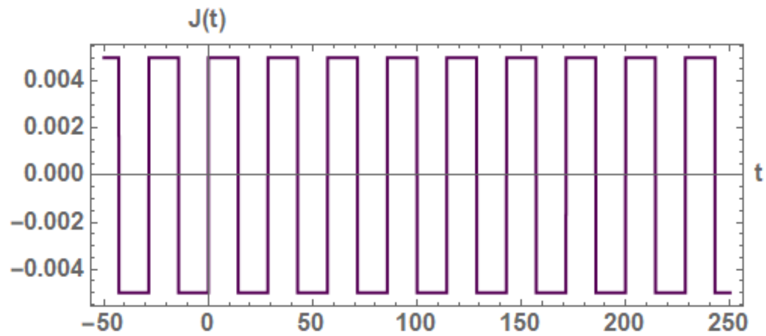


Figure 11: Wave form of the current source $J(t)$ which is supplied the memristor FitzHugh-Nagumo circuit.

In this case, the memristor FitzHugh-Nagumo equation (34) can be written as

Forced memristor FitzHugh-Nagumo equation

$$\left.\begin{aligned}\frac{dv}{dt} &= -\left(\varphi^2 - 1\right)v - i + 0.005\,\mathrm{sign}\{\sin(0.22t)\},\\ \frac{di}{dt} &= 0.08\,(v + 0.7 - 0.8i)\,,\\ \frac{d\varphi}{dt} &= v,\\ \frac{dq}{dt} &= i.\end{aligned}\right\} \tag{38}$$

We show the trajectories of Eq. (38) for the different initial conditions of $\varphi(0)$ in Fig. 12. Except for $\varphi(0)$, the initial conditions are given by $v(0) = 0.375$, $i(0) = 0.064$, and $q(0) = 0.5$. Observe that changing the value of $\varphi(0)$ can result in significantly different behaviors, as shown in Fig. 12. Therefore, we obtained the same results as in the previous section:

We cannot determine which trajectory will appear without knowing the initial condition $\varphi(0)$.

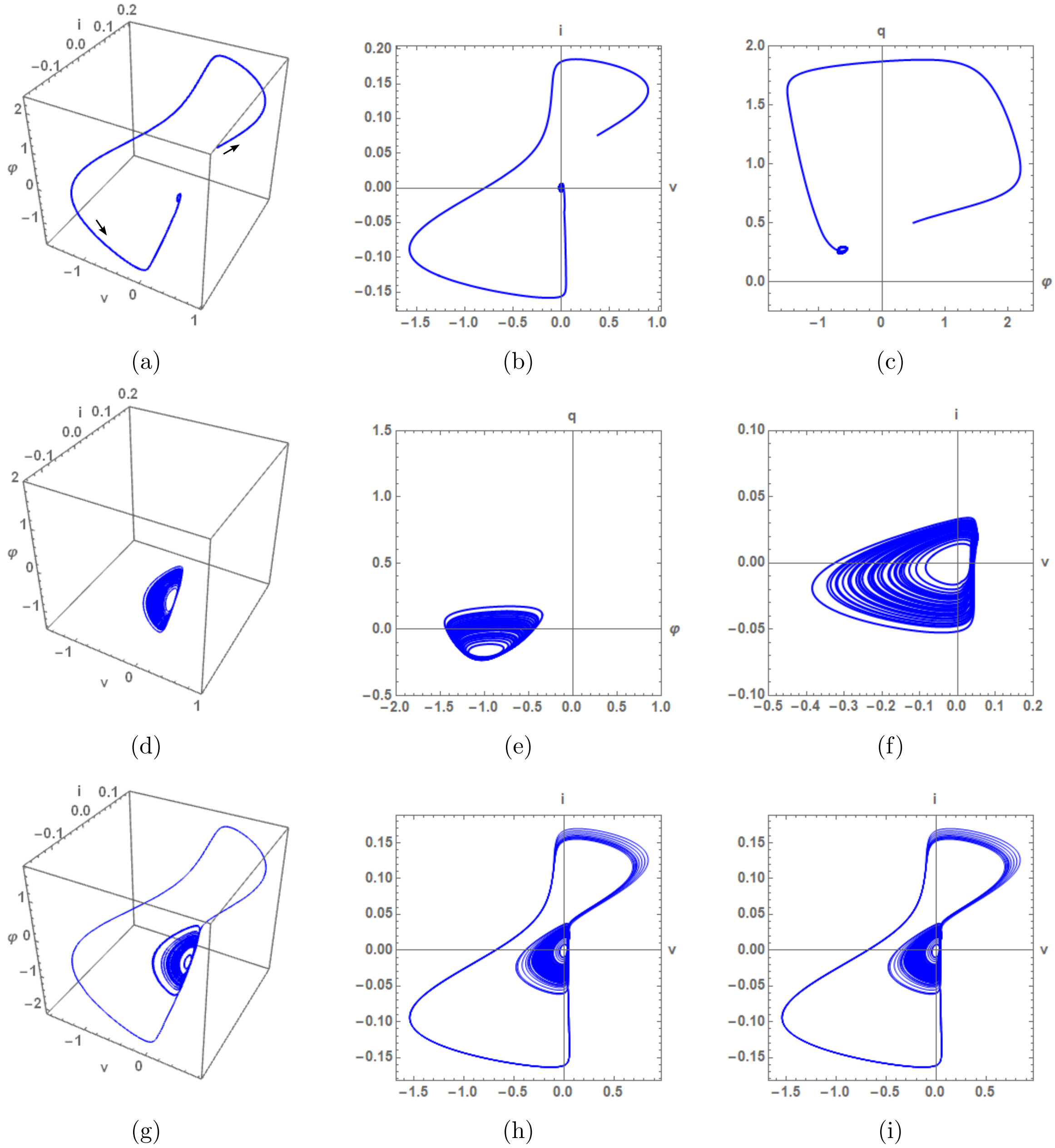


Figure 12: Trajectories on the (v, i, φ)-space (left), (v, i)-plane (center), and the (φ, q)-plane (right) of the forced memristor FitzHugh-Nagumo equation (38), which depends on the initial condition $\varphi(0)$. Parameter: $J = 0$. Initial conditions: $v(0) = 0.375$, $i(0) = 0.064$, and $q(0) = 0.5$. The initial condition $\varphi(0)$ is given as follows: (a)-(c): $\varphi(0) = 0$. (d)-(f): $\varphi(0) = 0.4692$. (g)-(h): $\varphi(0) = 0.4694$.

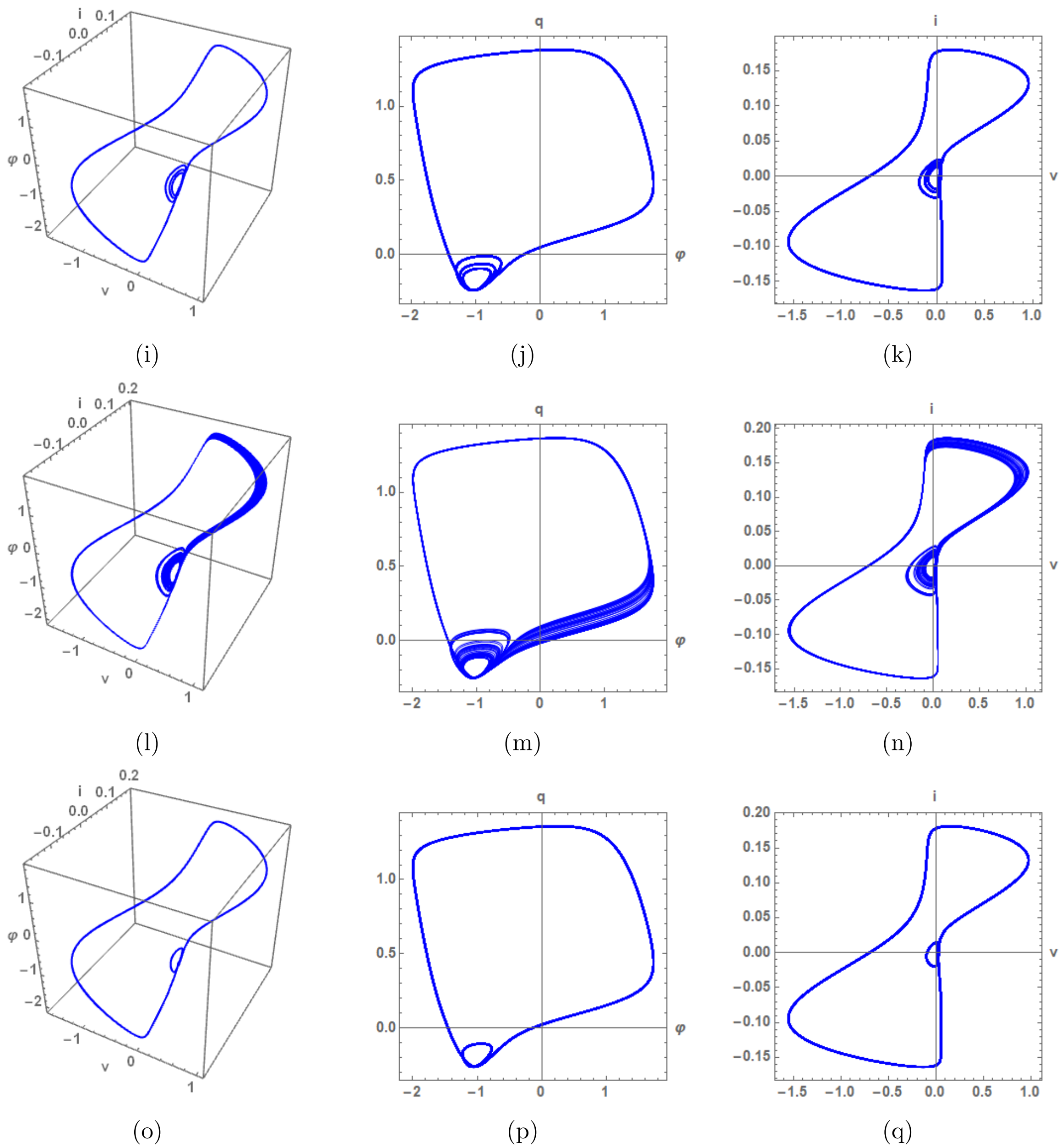


Figure 12: (Continued). The initial condition $\varphi(0)$ is given as follows: (i)-(k): $\varphi(0) = 0.48$. (l)-(n): $\varphi(0) = 0.498$. (0)-(q): $\varphi(0) = 10.505$.

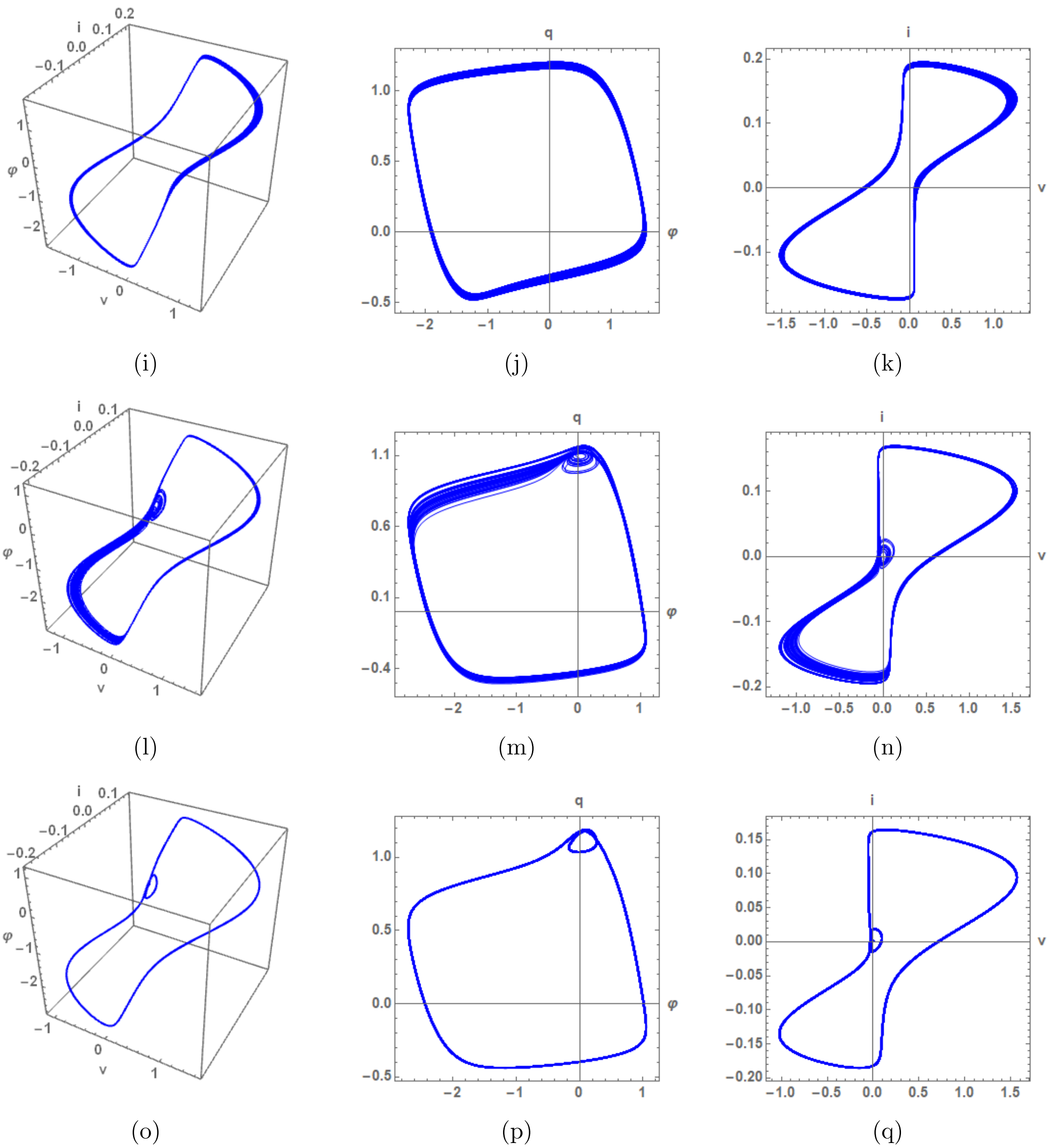
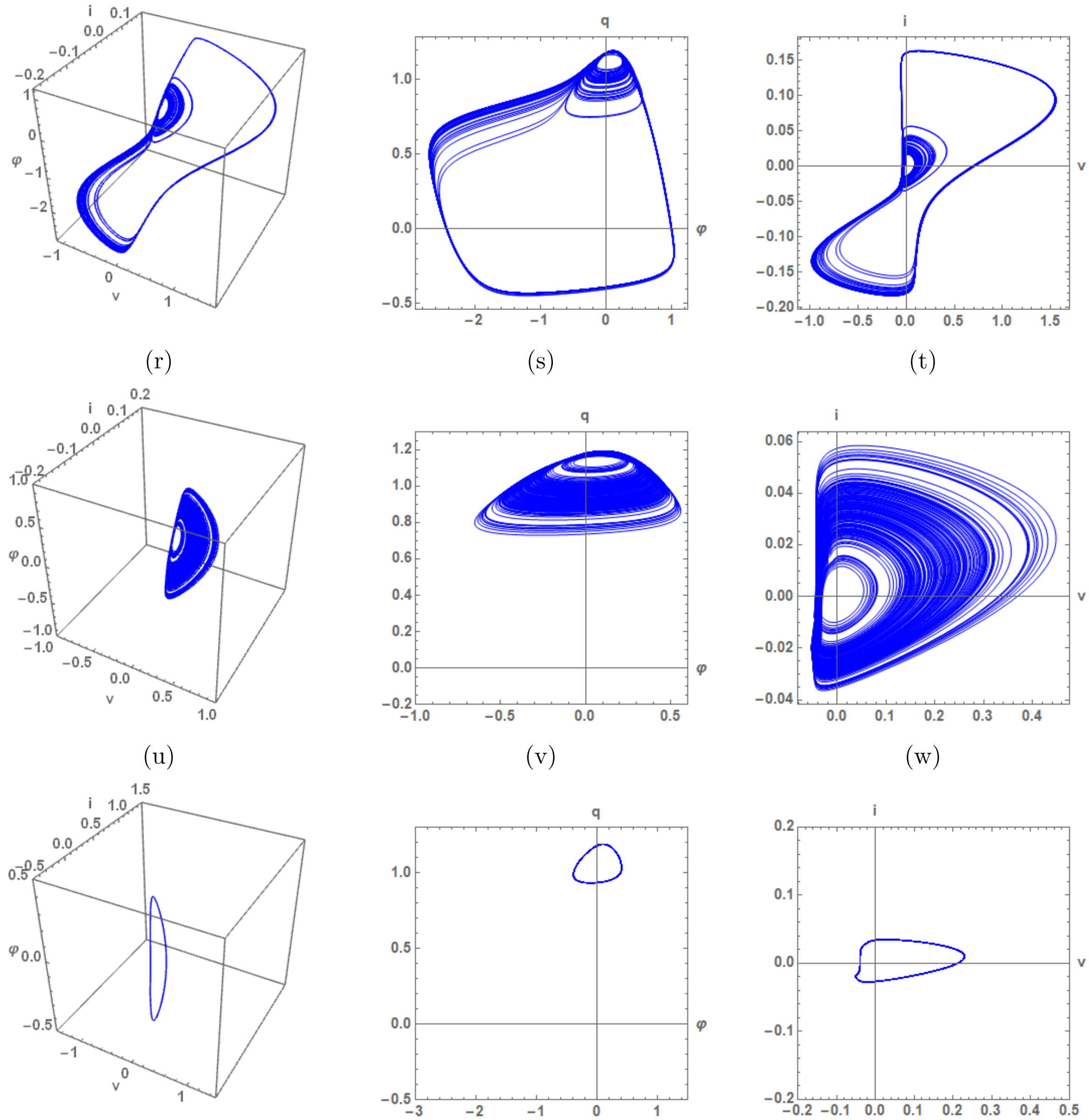


Figure 12: (Continued). The initial condition $\varphi(0)$ is given as follows: (i)-(k): $\varphi(0) = 0.8$. (l)-(n): $\varphi(0) = 1.405$. (0)-(q): $\varphi(0) = 1.45$.



(r) (s) (t)

(u) (v) (w)

(x) (y) (z)

Figure 12: (Continued). The initial condition $\varphi(0)$ is given as follows: (r)-(t): $\varphi(0) = 1.457$. (u)-(w): $\varphi(0) = 1.4597$. (x)-(z): $\varphi(0) = 1.46$.

4.6 Evolution of trajectory

In this section, we observe the evolution of trajectory by adding the small DC current (DC bias current) to the square wave.

Example 1.

Let us set $\varphi(0)$ to 0.5, and add a small negative DC bias current to the square wave when $t \geq 5000$. That is, $J(t)$ is given by

$$J(t) = \begin{cases} 0.005\,\mathrm{sign}\{\sin(0.22t)\}, & \text{for} \ \ t < 5000, \\ 0.005\,\mathrm{sign}\{\sin(0.22t)\} - 0.00001, & \text{for} \ \ t \geq 5000, \end{cases} \tag{39}$$

where the forced memristor FitzHugh-Nagumo equation s given by

$$\left.\begin{array}{rcl} \dfrac{dv}{dt} & = & -\Big(\varphi^2 - 1\Big)v - i + J(t), \\ \dfrac{di}{dt} & = & 0.08\,(v + 0.7 - 0.8i)\,, \\ \dfrac{d\varphi}{dt} & = & v, \\ \dfrac{dq}{dt} & = & i. \end{array}\right\} \tag{40}$$

Figure 13 shows the evolution of trajectory for Eq. (40) on the (φ, q)-plane. The initial conditions are $v(0) = 0.375$, $i(0) = 0.064$, $q(0) = 0.5$, and $\varphi(0) = 0.5$. Observe that the shape of the trajectory changes over time. Eventually, it begins to shrink. This resembles the refractory period of a neuron, during which the neuron returns to a resting state after a strong response.

In order for Eq. (40) to exhibit the large trajectory again, we have to modify $J(t)$ by adding a small positive DC bias current instead of a small negative DC bias current. That is, if $J(t)$ is given by

$$J(t) = \begin{cases} 0.005\,\mathrm{sign}\{\sin(0.22t)\}, & \text{for} \ \ t < 5000, \\ 0.005\,\mathrm{sign}\{\sin(0.22t)\} - 0.00001, & \text{for} \ \ t5000 \leq t \leq 10000, \\ 0.005\,\mathrm{sign}\{\sin(0.22t)\} + 0.00003, & \text{for} \ \ t > 100000. \end{cases} \tag{41}$$

Then we can observe the large trajectory shown in Fig. 14.

Example 2.

Let us set $\varphi(0)$ to 0, and add a small positive DC bias current to the square wave when $t \geq 5000$. That is, $J(t)$ is given by

$$J(t) = \begin{cases} 0.005\,\mathrm{sign}\{\sin(0.22t)\}, & \text{for} \ \ t < 5000, \\ 0.005\,\mathrm{sign}\{\sin(0.22t)\} + 0.00001, & \text{for} \ \ t \geq 5000. \end{cases} \tag{42}$$

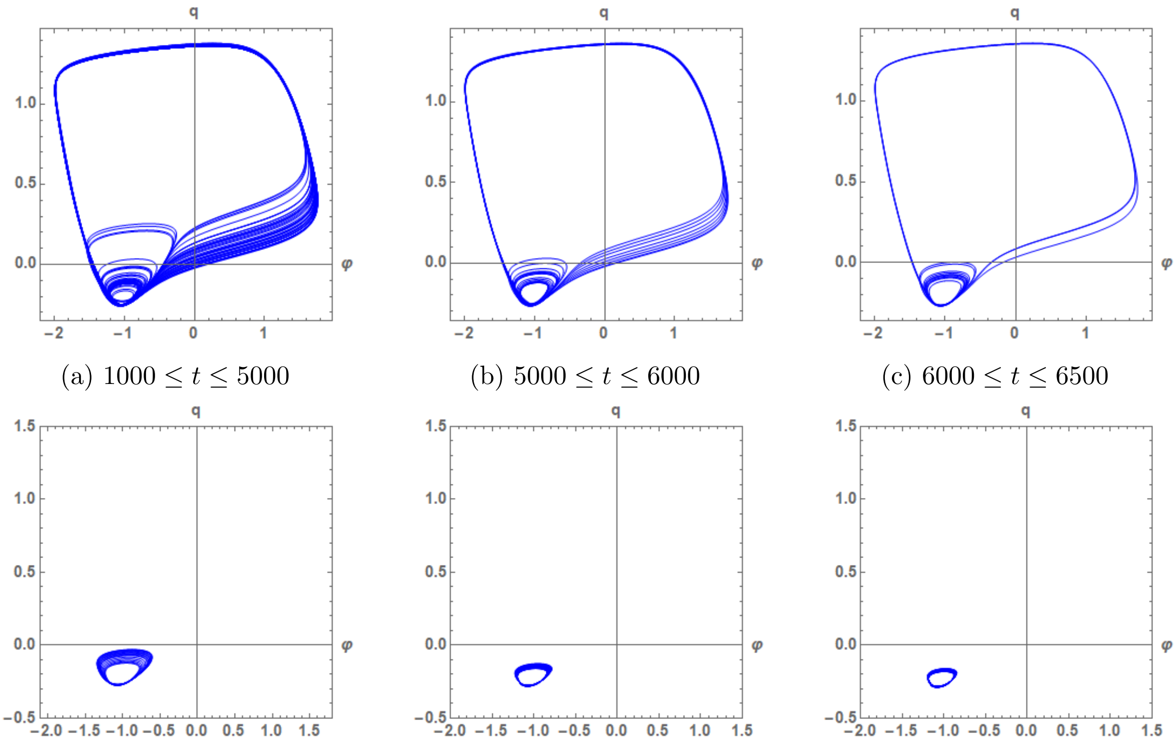
Figure 15 shows the evolution of trajectory for Eq. (40) on the (φ, q)-plane. The initial conditions are $v(0) = 0.375$, $i(0) = 0.064$, $q(0) = 0.5$, and $\varphi(0) = 0$. Not all of the trajectory evolution is shown since there are too many different trajectories. Similarly, observe that the shape of the trajectory changes over time. Eventually, it shrinks. This resembles the refractory period of neurons.

In order for Eq. (40) to exhibit the larger trajectory again, we add a small negative DC bias current instead of a small positive DC bias current. That is, if $J(t)$ is given by

$$J(t) = \begin{cases} 0.005\,\mathrm{sign}\{\sin(0.22t)\}, & \text{for} \ \ t < 5000, \\ 0.005\,\mathrm{sign}\{\sin(0.22t)\} + 0.00001, & \text{for} \ \ t5000 \leq t \leq 150000, \\ 0.005\,\mathrm{sign}\{\sin(0.22t)\} - 0.00001, & \text{for} \ \ t > 150000. \end{cases} \tag{43}$$

Then we can observe the large trajectory shown in Fig. 16.

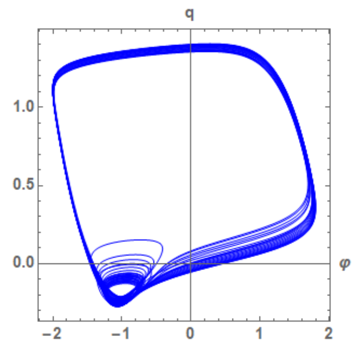
> To transition from a reduced trajectory to a larger one, a small DC bias current must be applied to the square wave. In other words, a small DC bias current is necessary to transition from refractory mode to excitation mode.



(a) $1000 \le t \le 5000$ (b) $5000 \le t \le 6000$ (c) $6000 \le t \le 6500$

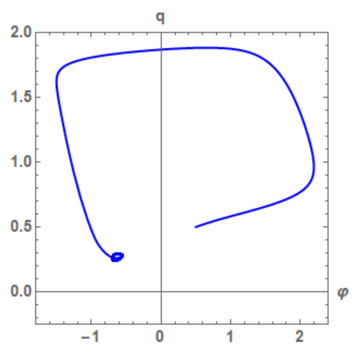
(d) $6500 \le t \le 7000$ (e) $7000 \le t \le 8000$ (f) $8000 \le t \le 9000$

Figure 13: Time evolution of trajectory for Eq. (40) on the (φ, q)-plane, where the value of $\varphi(0)$ is fixed to 0.5. Other initial conditions: $v(0) = 0.375$, $i(0) = 0.064$, and $q(0) = 0.5$.

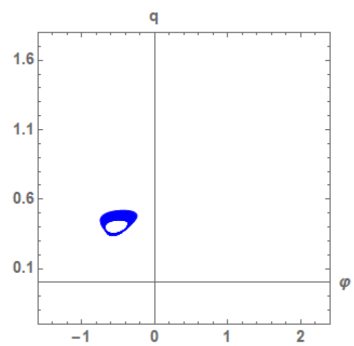


$11000 \leq t \leq 13000$

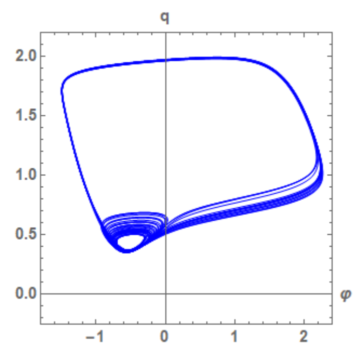
Figure 14: Trajectory of Eq. (40) where $J(t)$ is given by Eq. (41). We can observe the large trajectory again.



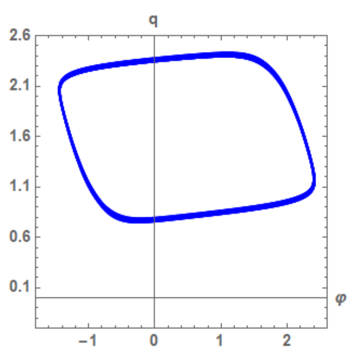
(a) $1000 \leq t \leq 5000$



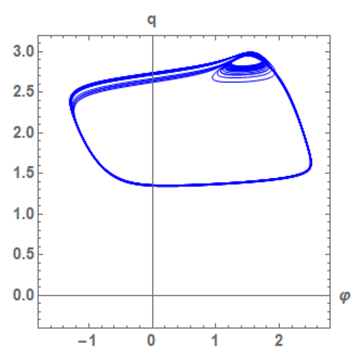
(b) $18000 \leq t \leq 20000$



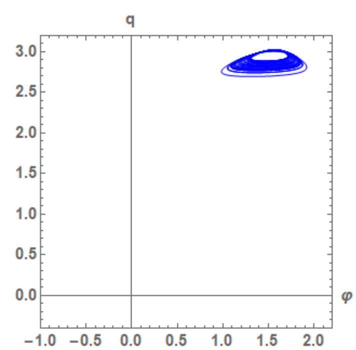
(c) $20000 \leq t \leq 22000$



(d) $70000 \leq t \leq 71000$

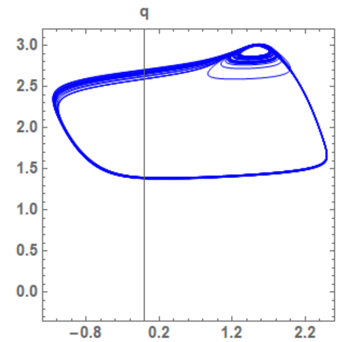


(e) $140000 \leq t \leq 142000$



(f) $145000 \leq t \leq 146000$

Figure 15: Time evolution of trajectory for Eq. (40) on the (φ, q)-plane, where the value of $\varphi(0)$ is fixed to 0. Other initial conditions: $v(0) = 0.375$, $i(0) = 0.064$, and $q(0) = 0.5$.



$155000 \le t \le 157000$

Figure 16: Trajectory of Eq. (40) for the current source $J(t)$ defined by Eq. (43). We can observe the large trajectory again.

4.7 Signal transmission between memristor FitzHugh-Nagumo circuits

In this section, we study the signal transmission between the two coupled memristor FitzHugh-Nagumo circuits. Consider the circuit shown in Fig. 17. Two identical memristor FitzHugh-Nagumo circuits are connected by a unity-gain buffer, which is an operational amplifier circuit with a voltage gain of 1. The current source J supplies only the left circuit. The dynamics of this circuit is given by

$$\left.\begin{aligned}
\frac{dv_1}{dt} &= -\left(\varphi_1{}^2 - 1\right)v_1 - i_1 + J, \\
\frac{di_1}{dt} &= 0.08\left(v_1 - 0.8 i_1\right), \\
\frac{d\varphi_1}{dt} &= v_1, \\
\frac{dq_1}{dt} &= i_1, \\
\frac{di_2}{dt} &= 0.08\left(v_1 - 0.8 i_2\right), \\
\frac{d\varphi_2}{dt} &= v_1, \\
\frac{dq_2}{dt} &= i_2,
\end{aligned}\right\} \tag{44}$$

where q_1 (q_2, respectively) is the charge on the left (right, respectively) circuit, φ_1 (φ_2, respectively) is the flux of the left (right, respectively) circuit, and K is the current flowing from the unity-gain buffer into the memristor FitzHugh-Nagumo circuit on the right. Note that $v_1 = v_2$ because the unity-gain buffer copies v_1 of the left circuit to v_2 of the right circuit. The initial conditions are given by

$$\left.\begin{aligned}
v_1(0) = 0.375, \quad i_1(0) = 0.064, \quad q_1(0) = 0.5, \quad \varphi_1(0) = 0.5, \\
i_2(0) = 0.1, \quad q_2(0) = 0.55, \quad \varphi_2(0) = 0.52.
\end{aligned}\right\} \tag{45}$$

Therefore, $i_1(0) \neq i_2(0)$, $q_1(0) \neq q_2(0)$, $\varphi_1(0) \neq \varphi_2(0)$. The current source J supplied to the left circuit is given by

$$J(t) = 0.005\, \mathrm{sign}\{\sin(0.22t)\}, \tag{46}$$

(see Fig. 18). The current K flowing from the unity-gain buffer into the memristor FitzHugh-Nagumo circuit on the right is given by

$$K = \frac{dv_1}{dt} + \left(\varphi_2{}^2 - 1\right)v_1 + i_2. \tag{47}$$

We show the trajectories of the coupled memristor FitzHugh-Nagumo circuits (44) in Fig. 19. Observe that the trajectory in Fig. 19(d) is shifted upward compared to the trajectory in Fig. 19(c). In this circuit, the currents i_1 and i_2 will synchronize asymptotically as shown in Fig. 20 [12]. However, if the initial conditions $\varphi_1(0)$ and $\varphi_2(0)$ are not identical, then the waveform of the current $K(t)$ will be distorted as shown in Fig. 21(a). If they are identical, then $K(t)$ will soon become almost equivalent to $J(t)$ as shown in Fig. 21(b) [13]. This can be explained as follows: From Eqs. (44) and (47), $J(t)$ and $K(t)$ can be written as

$$\left.\begin{array}{rcl} J(t) & = & \dfrac{dv_1(t)}{dt} + \left(\varphi_1(t)^2 - 1\right)v_1(t) + i_1(t), \\ K(t) & = & \dfrac{dv_1(t)}{dt} + \left(\varphi_2(t)^2 - 1\right)v_1(t) + i_2(t). \end{array}\right\} \tag{48}$$

Note that only the first term on the right-hand side of the two equations is the same. From Eq. (44), we obtain

$$\frac{d\varphi_1}{dt} - \frac{d\varphi_2}{dt} = 0. \tag{49}$$

Integrating the above equation from 0 to t, we obtain

$$\varphi_1(t) - \varphi_2(t) = \varphi_1(0) - \varphi_2(0). \tag{50}$$

If the initial conditions $\varphi_1(0)$ and $\varphi_2(0)$ are not identical, then we have

$$\varphi_1(t) \not\equiv \varphi_2(t). \tag{51}$$

for $t \geq 0$. Assume that the currents $i_1(t)$ and $i_2(t)$ will synchronize asymptotically, that is, $i_1(t) \approx i_2(t)$ for sufficiently large t, and $\varphi_1(0)$ and $\varphi_2(0)$ are not identical. In this case, the current $K(t)$ will not be nearly equal to $J(t)$. This is because the second term on the right side of the two equations in Eq. (48) will not become nearly equivalent. Next assume that $\varphi_1(0) = \varphi_2(0)$. Then, from Eq. (50), we obtain $\varphi_1(t) = \varphi_2(t)$. Therefore, when the currents $i_1(t)$ and $i_2(t)$ synchronize asymptotically, $K(t)$ becomes nearly equivalent to $J(t)$. Furthermore, the two FitzHugh-Nagumo circuits exhibit nearly identical behavior after they are synchronized. Thus, they are in a special relationship as a pair. We conclude as follows:

> If the initial conditions $\varphi_1(0)$ and $\varphi_2(0)$ are not identical, then the signal $J(t)$ will not be transmitted correctly to the memristor FitzHugh-Nagumo circuit on the right.

> We may be able to build a communication system as shown in [13, 14], if we can prepare two memristor FitzHugh-Nagumo circuits with the same initial flux condition, i.e., $\varphi_1(0) = \varphi_2(0)$.

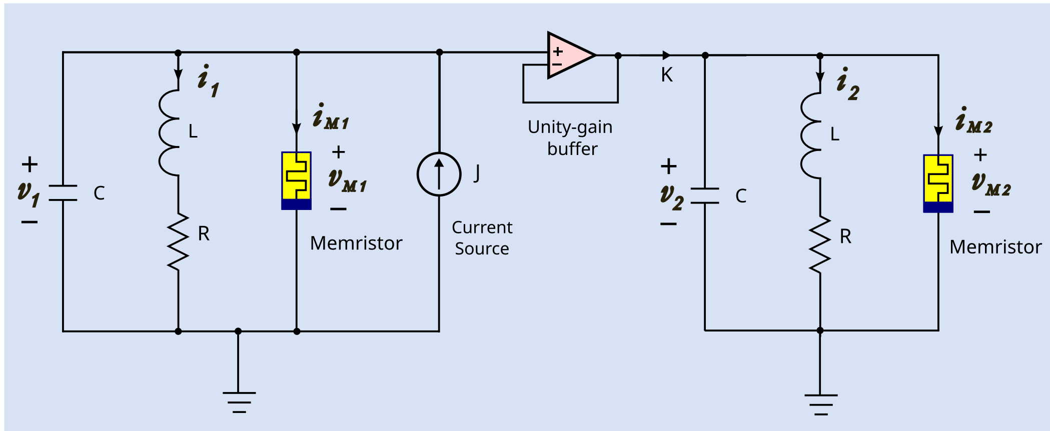


Figure 17: Two coupled memristor FitzHugh-Nagumm circuits via the unity-gain buffer. The current source J is supplied to the left circuit only. Parameters: $C = 1$, $L = \dfrac{1}{0.08}$, $R = 0.8$. The two voltage-controlled memristors are defined by Eq. (60).

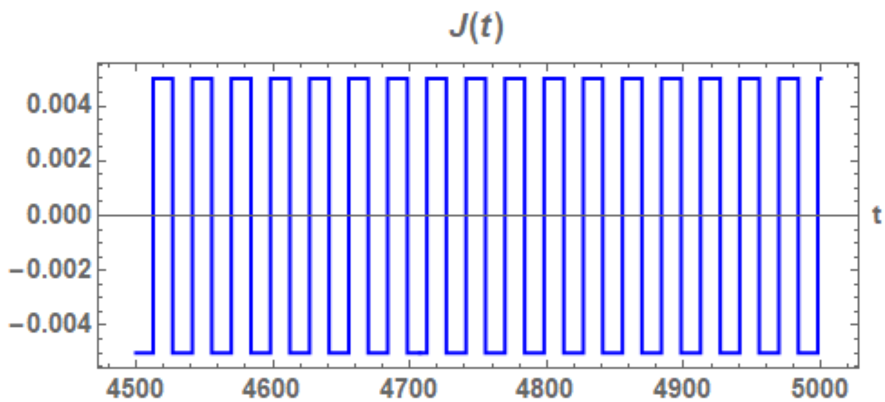
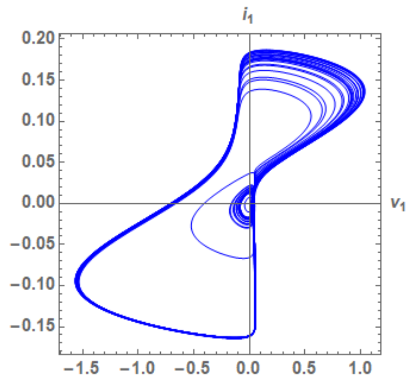
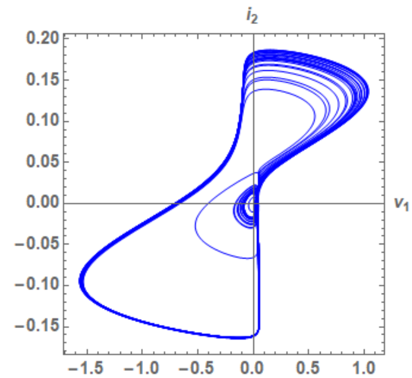


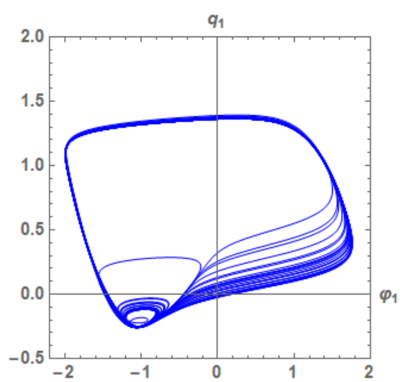
Figure 18: Waveform of $J(t)$ supplied to the left memristor FitzHugh-Nagumm circuit.



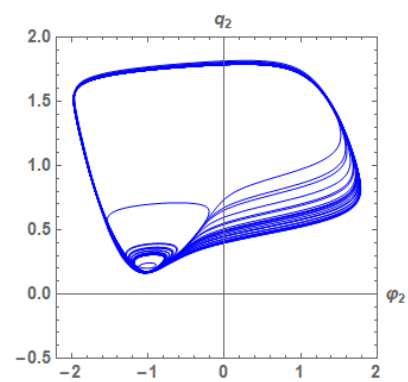
(a) (v_1, i_1)-plane



(b) (v_2, i_2)-plane



(c) (φ_1, q_1)-plane



(d) (φ_2, q_2)-plane

Figure 19: Trajectories of the coupled memristor FitzHugh-Nagumo circuits (44). (a) and (c) Trajectories of the memristor FitzHugh-Nagumo circuit on the left. (b) and (d) Trajectories of the memristor FitzHugh-Nagumo circuit on the right. Note that the trajectory in Fig. 19(d) is shifted upward compared to the trajectory in Fig. 19(c).

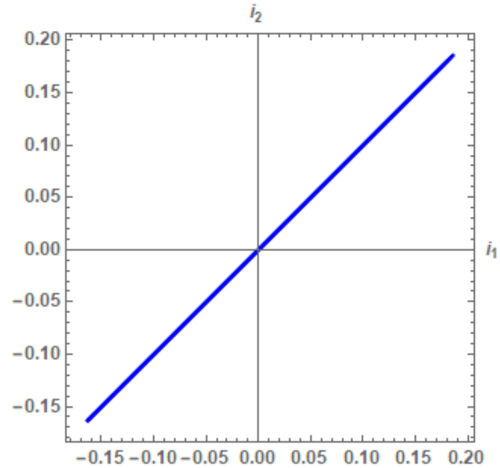
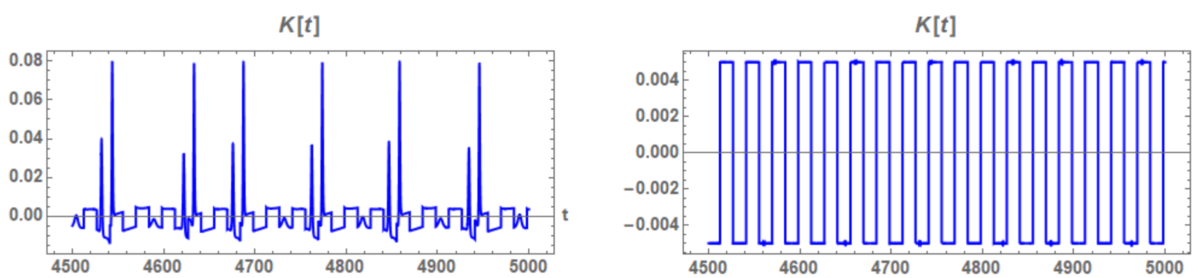


Figure 20: Synchronization of $i_1(t)$ and $i_2(t)$, which exhibits 45-degree line.



(a) $\varphi_1(0) = 0.5$, $\varphi_2(0) = 0.52$.

(b) $\varphi_1(0) = \varphi_2(0) = 0.5$.

Figure 21: Waveform of the current $K(t)$ flowing from the unity-gain buffer into the memristor FitzHugh-Nagumo circuit on the right. (a) If the initial conditions $\varphi_1(0)$ and $\varphi_2(0)$ are not identical, then the waveform of the current $K(t)$ will be distorted. (b) If they are identical, then $K(t)$ will soon become almost equivalent to $J(t)$.

5 Memristor Chua circuit

In this section, we derive the memristor Chua circuit equation by taking the time derivative of the original Chua circuit equation. Next, we show that the memristor Chua circuit equation yields nearly identical results to those of the memristor FitzHugh-Nagumo equation.

5.1 Chua Circuit

Consider the Chua circuit shown in Fig. 22 which exhibits chaotic behavior (see [15] for more details). The circuit contains five elements: two linear capacitors (C_1 and C_2), one linear inductor (L), one linear resistor (R), and one nonlinear resistor.

$$\left.\begin{array}{rcl} C_1 \dfrac{dV_1}{dt} & = & \dfrac{V_2 - V_1}{R} - f(V_1), \\ C_2 \dfrac{dV_2}{dt} & = & y - \dfrac{V_2 - V_1}{R}, \\ L \dfrac{dI}{dt} & = & -V_2, \end{array}\right\} \tag{52}$$

where the symbols V_1, V_2, and I denote the voltage across the capacitor C_1, the voltage across the capacitor C_2, and the current through the inductor L, respectively. The $v - i$ characteristic of the nonlinear resistor is given by

$$I_R = f(V_R) = \frac{1}{16} {V_R}^3 - \frac{7}{6} V_R, \tag{53}$$

where i_R and v_R are the current through and voltage across the nonlinear resistor.

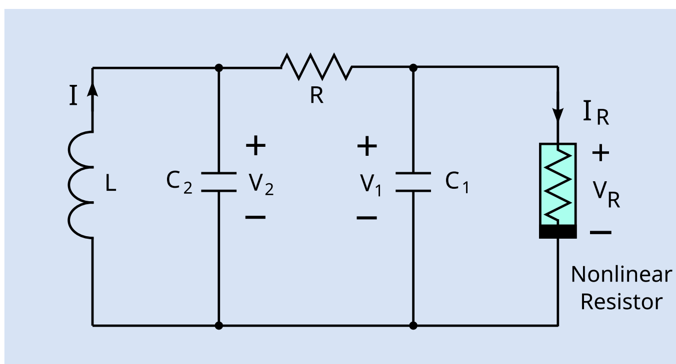


Figure 22: Chua circuit contains five elements: two linear capacitors (C_1 and C_2), one linear inductor (L), one linear resistor (R), and one nonlinear resistor. V_1 and V_2, denote the voltages across C_1 and C_2, respectively, and I denotes the current through L. The $v - i$ characteristic of the nonlinear resistor is given by $I_R = f(V_R) = \dfrac{1}{16} {V_R}^3 - \dfrac{7}{6} V_R$, where I_R and V_R are the current through and voltage across the nonlinear resistor, respectively.

If we set

$$\left.\begin{array}{llll} C_1 = \dfrac{1}{\alpha}, & C_2 = 1, & L = \dfrac{1}{\beta}, & R = 1, \\ x = V_1, & y = V_2, & I = z, & \end{array}\right\} \tag{54}$$

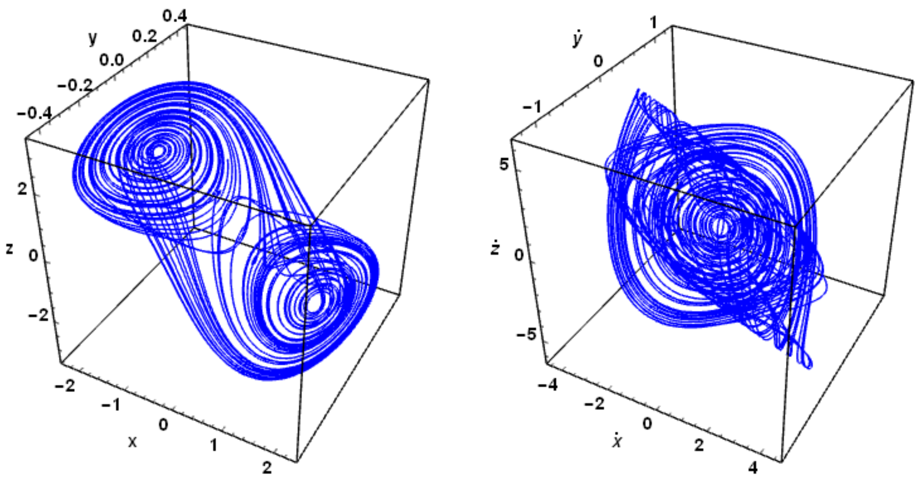
then the dynamics of the Chua circuit is defined by [15, 16]

$$\left.\begin{aligned}\frac{dx}{dt} &= \alpha\big(y - x - f(x)\big),\\ \frac{dy}{dt} &= x - y + z,\\ \frac{dz}{dt} &= -\beta y,\end{aligned}\right\} \tag{55}$$

where $f(x)$ is given by

$$f(x) = \frac{1}{16}x^3 - \frac{7}{6}x. \tag{56}$$

Equation (55) exhibits chaotic behavior and many well-known bifurcation phenomena (see [15, 16] for more details). We plot the trajectory of Eq. (55) in Fig. 23, where the initial conditions are given by $x(0) = y(0) = z(0) = 0.1$, and the parameters are given by $\alpha = 10$ and $\beta = 14$.



(a) (x, y, z)-space. (b) $(\dot{x}, \dot{y}, \dot{z})$-space.

Figure 23: Chaotic trajectory of Eq. (55), where $(\dot{x}, \dot{y}, \dot{z}) \triangleq \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$. Initial conditions: $x(0) = 0.1$, $y(0) = 0.1$, $z(0) = 0.1$. Parameters: $\alpha = 10$, $\beta = 14$.

5.2 Memristor Chua circuit

Differentiating Eq. (55) with respect to time t, we obtain the memristor Chua circuit equation

$$\left.\begin{aligned}
\frac{dv_1}{dt} &= \alpha\left\{i - v_1 - \left(\frac{3}{16}{\varphi_1}^2 - \frac{7}{6}\right) v_1\right\},\\
\frac{dv_2}{dt} &= v_1 - v_2 + i,\\
\frac{di}{dt} &= -\beta v_2,\\
\frac{d\varphi_1}{dt} &= v_1,\\
\frac{d\varphi_2}{dt} &= v_2,\\
\frac{dq}{dt} &= i,
\end{aligned}\right\} \tag{57}$$

where

$$v_1 \triangleq \frac{dx}{dt}, \quad v_2 \triangleq \frac{dy}{dt}, \quad i \triangleq \frac{dz}{dt}, \tag{58}$$

and

$$\varphi_1(t) \triangleq \int_{-\infty}^{t} v_1(\tau)d\tau, \quad \varphi_2(t) \triangleq \int_{-\infty}^{t} v_2(\tau)d\tau, \quad q(t) \triangleq \int_{-\infty}^{t} i(\tau)d\tau. \tag{59}$$

The last two equations of Eq. (57) are added in order to plot the trajectory in the $(\varphi_1, \varphi_2, q)$-space. The state-dependent Ohm's law and its associated state equation for the the voltage-controlled memristor is given by

$$\left.\begin{aligned}
i_M &= G(\varphi_M)\, v_M,\\
\frac{d\varphi_M}{dt} &= v_M,
\end{aligned}\right\} \tag{60}$$

where v_M is the voltage across the memristor, i_M is the current through the memristor, φ_M is the flux of the memristor, and $G(\varphi_M)$ is given by

$$G(\varphi_M) = \frac{3}{16}{\varphi_M}^2 - \frac{7}{6}. \tag{61}$$

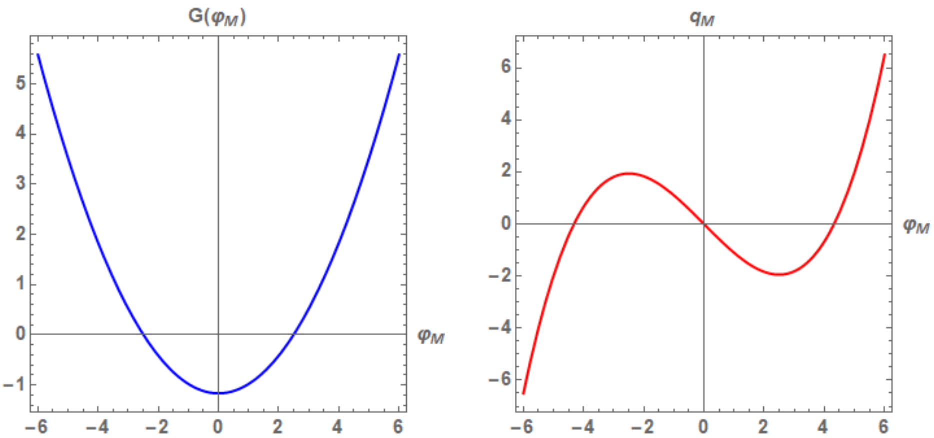
Note that $v_M = v_1$ and $\varphi_M = \varphi_1$. The constitutive relation of the memristor is obtained by integrating Eq. (61) with respect to time t. This integration yields:

$$q_M = \frac{1}{16}{\varphi_M}^3 - \frac{7}{6}\varphi_M + d_M, \tag{62}$$

where d_M is a constant of integration and φ_M and q_M are the flux and charge of the memristor, and they are defined by

$$\varphi_M(t) = \int_{-\infty}^{t} v_M(\tau)d\tau \quad \text{and} \quad q_M(t) = \int_{-\infty}^{t} i_M(\tau)d\tau. \tag{63}$$

The memductance (61) and the constitutive relation (62) are plotted in Fig. 24, where we set $d_M = 0$ for the sake of simplicity.



(a) Memductance $G(\varphi_M)$ (b) Constitutive relation

Figure 24: Memductance: $G(\varphi_M) = \frac{3}{16}{\varphi_M}^2 - \frac{7}{6}$ and constitutive relation: $q_M = \frac{1}{16}{\varphi_M}^3 - \frac{7}{6}\varphi_M$ of the voltage-controlled memristor.

The memristor Chua circuit equation (57) can be realized by using the circuit in Fig. 25. Replacing the voltage-controlled memristor with the nonlinear resistor yields the original Chua circuit shown in Fig. 22 [17]. The following point should be noted:

- The Chua circuit equation (52) is defined by a set of three nonlinear differential equations in the three-dimensional (V_1, V_1, I)-plane.
- The memristor Chua circuit equation (57) is defined by a set of six nonlinear differential equations in the six-dimensional $(v_1, v_2, i, \varphi_1, \varphi_2, q)$-space.

  Therefore, they are not equivalent and can exhibit different behaviors..

5.3 Dependence on initial flux conditions

In this section, we show the trajectories of Eq. (57) for various values of $\varphi_1(0)$ in Fig. 26. Let us first calculate the initial conditions for Eq. (57). Using Eqs. (55) and (58), we can get the initial conditions for $v_1(t)$, $v_2(t)$,and $i(t)$ as follows:

$$\left.\begin{array}{rcccl} v_1(0) &=& \left.\dfrac{dx}{dt}\right|_{t=0} &=& \alpha\big(y(0) - x(0) - f(x(0))\big), \\ v_2(0) &=& \left.\dfrac{dy}{dt}\right|_{t=0} &=& x(0) - y(0) + z(0), \\ i(0) &=& \left.\dfrac{dz}{dt}\right|_{t=0} &=& -\beta y(0). \end{array}\right\} \qquad (64)$$

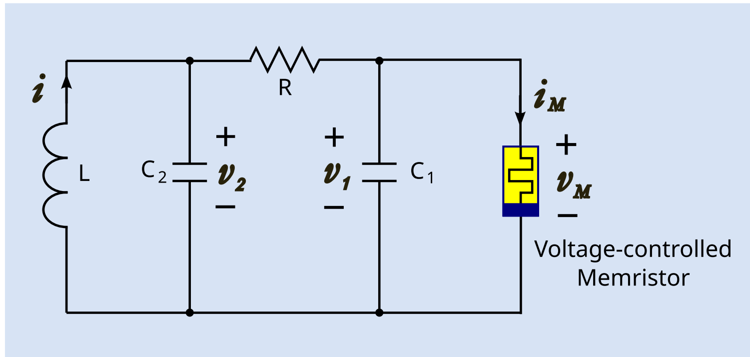


Figure 25: Memristor Chua circuit. Parameters: $C_1 = \dfrac{1}{\alpha}$, $C_2 = 1$, $L = \dfrac{1}{\beta}$, $R = 1$. The voltage-controlled memristor is defined by Eqs. (60) and (61).

where the parameters are given by

$$\alpha = 10, \quad \beta = 14. \tag{65}$$

If we set $x(0) = y(0) = z(0) = 0.1$, then we obtain

$$v_1(0) = \frac{11194}{9600} \approx 1.16604, \;\; v_2(0) = 0.1, \;\; i(0) = -1.4. \tag{66}$$

The other initial conditions except for $\varphi_1(0)$ are fixed as follows:

$$\varphi_2(0) = q(0) = 0.1, \tag{67}$$

The results of the computer simulations are shown in in 26. We plotted the trajectory in the $(\varphi_1, \varphi_2, q)$-space, since it is similar to that of the original Chua circuit. Observe that altering the value of $\varphi_1(0)$ can cause the memristor Chua circuit equation (57) to exhibit significantly different behaviors, as illustrated in 26. In other words, $\varphi_1(0)$ is considered to be the bifurcation parameter.

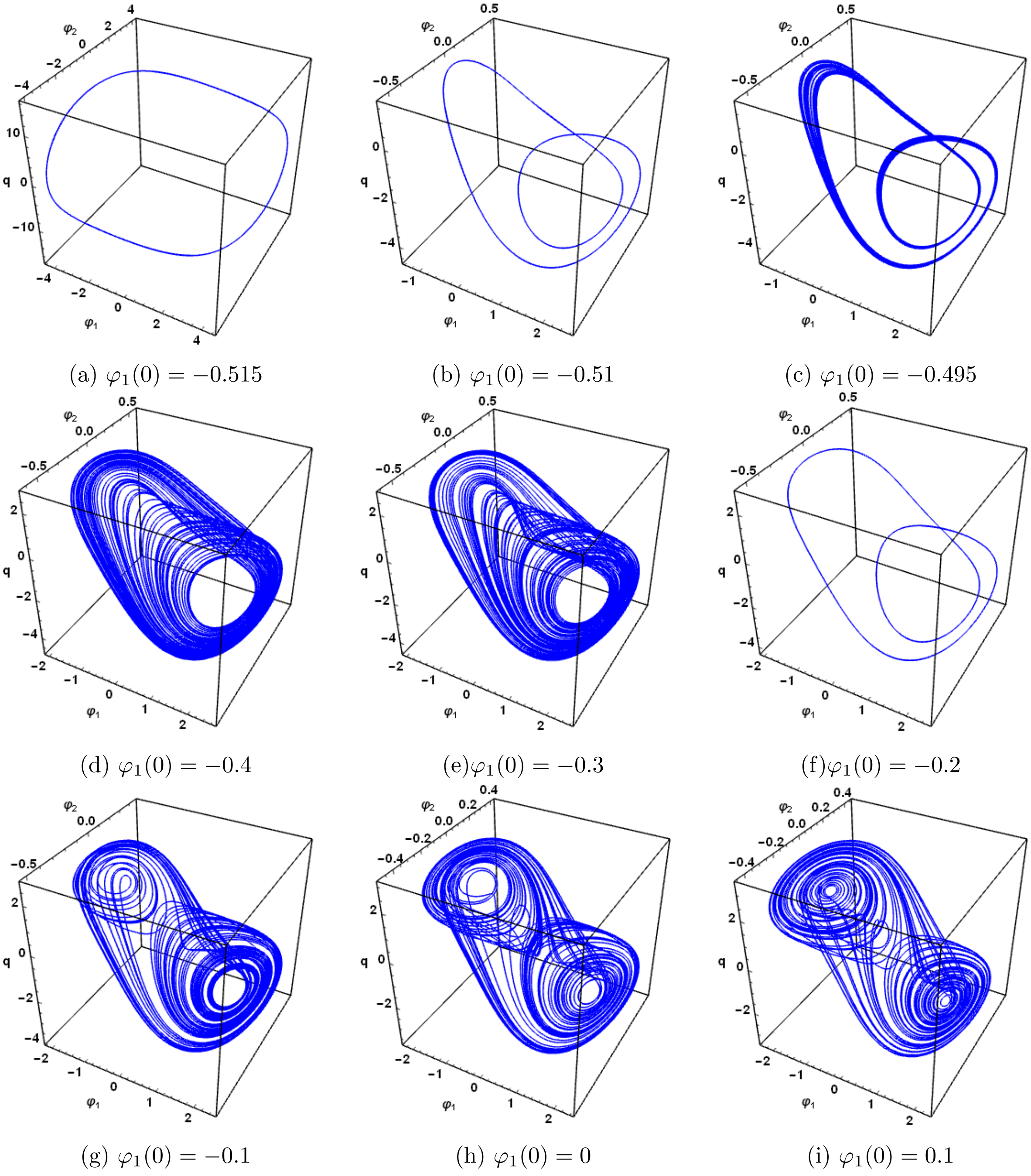
That is, we conclude as follows:

> The behavior of the trajectory for the memristor Chua circuit equation (57) depends on the value of $\varphi_1(0)$ which is the total amount of past changes of φ_1.

In other words, it can be stated as follows:

> We cannot determine which trajectory will appear without knowing the initial condition $\varphi_1(0)$.

Finally, we plot the trajectory of Eq. (57) for $\varphi_1(0) = 0.1$ in Fig. 27. Note that the chaotic trajectories in Figs. 23 and 27 are similar.

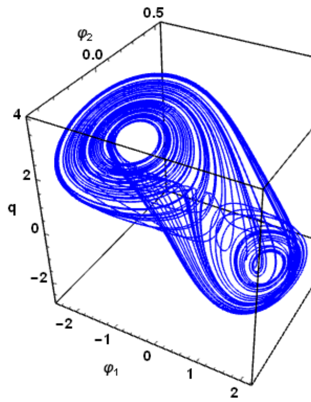


(a) $\varphi_1(0) = -0.515$ (b) $\varphi_1(0) = -0.51$ (c) $\varphi_1(0) = -0.495$

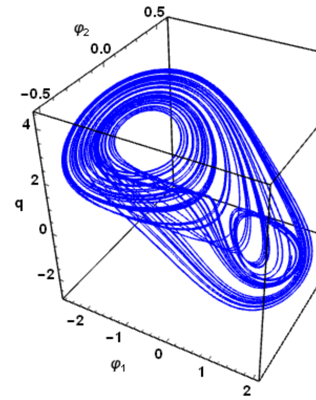
(d) $\varphi_1(0) = -0.4$ (e)$\varphi_1(0) = -0.3$ (f)$\varphi_1(0) = -0.2$

(g) $\varphi_1(0) = -0.1$ (h) $\varphi_1(0) = 0$ (i) $\varphi_1(0) = 0.1$

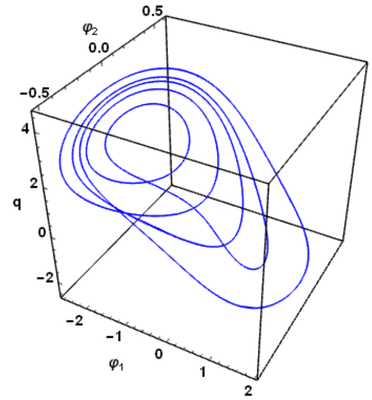
Figure 26: Trajectories of Eq. (57) in the $(\varphi_1, \varphi_2, q)$-space.



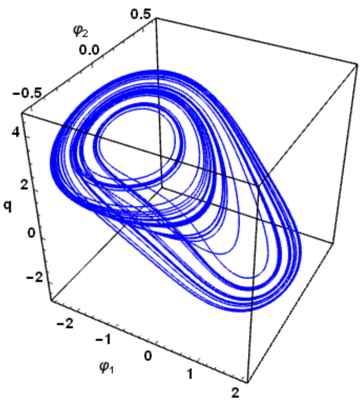
(j) $\varphi_1(0) = 0.3$



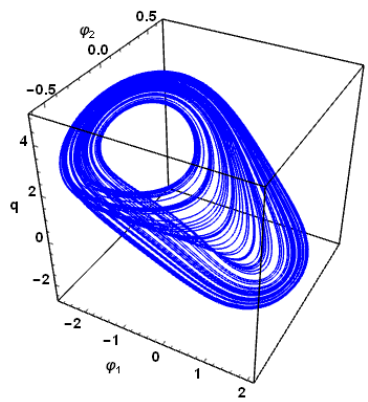
(k) $\varphi_1(0) = 0.5$



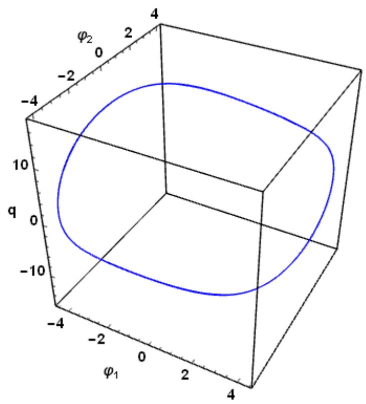
(l) $\varphi_1(0) = 0.6$



(m) $\varphi_1(0) = 0.60535$

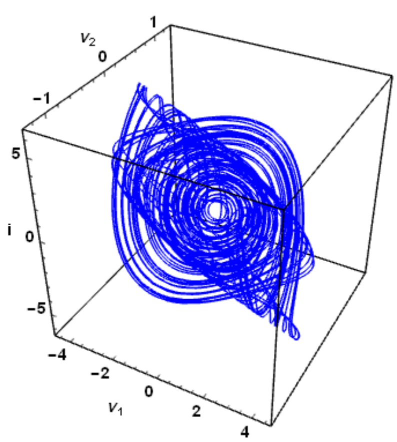


(n) $\varphi_1(0) = 0.8$

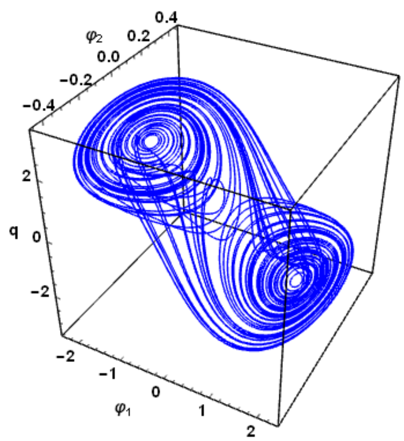


(o) $\varphi_1(0) = 0.9$

Figure 26: (Continued). Trajectories of Eq. (57) in the $(\varphi_1, \varphi_2, q)$-space.



(a) (v_1, v_2, i)-space.



(b) $(\varphi_1, \varphi_2, q)$-space.

Figure 27: Chaotic trajectory of Eq. (57) for $\varphi_1(0) = 0.1$. Th other initial conditions are given by Eqs. (66) and (67). The parameters are given by Eq. (65)

5.4 Trajectory change in response to a single pulse

This section shows that the trajectory can be altered by supplying a small-amplitude single pulse to the circuit via the current source $J(t)$ as shown in Fig. 28. The function $J(t)$ is defined as follows.

$$J(t) = \begin{cases} 0 & \text{for} \quad t < a, \\ -0.0008 & \text{for} \quad a \le t \le b, \\ 0 & \text{for} \quad t > b, \end{cases} \tag{68}$$

where we set $a = 3500$ and adjust the pulse length by changing the parameter b. In this case, the memristor Chua circuit equation is given by

$$\left.\begin{aligned} \frac{dv_1}{dt} &= \alpha \left\{ i - v_1 - \left(\frac{3}{16} {\varphi_1}^2 - \frac{7}{6} \right) v_1 + J(t) \right\}, \\ \frac{dv_2}{dt} &= v_1 - v_2 + i, \\ \frac{di}{dt} &= -\beta v_2, \\ \frac{d\varphi_1}{dt} &= v_1, \\ \frac{d\varphi_2}{dt} &= v_2, \\ \frac{dq}{dt} &= i, \end{aligned}\right\} \tag{69}$$

where $J(t)$ is the current source defined by Eq. (68).

Figure 29 shows the trajectory change in response to the pulse signal $J(t)$. We set $\varphi_1(0) = -0.4$ and the other initial conditions are given by Eqs. (66) and (67), and the parameters are given by Eq. (65). Note that we can modify the trajectory by adjusting the parameter b. We conclude as follows:

> We can alter the trajectory of the memristor Chua circuit equation (57) by changing the width $\Delta t = b - a$ of the small-amplitude single pulse.

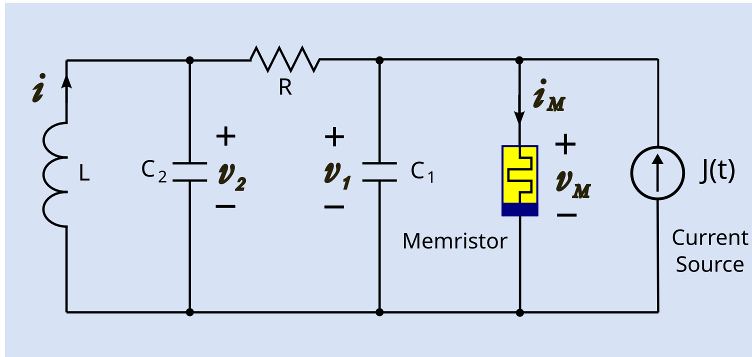
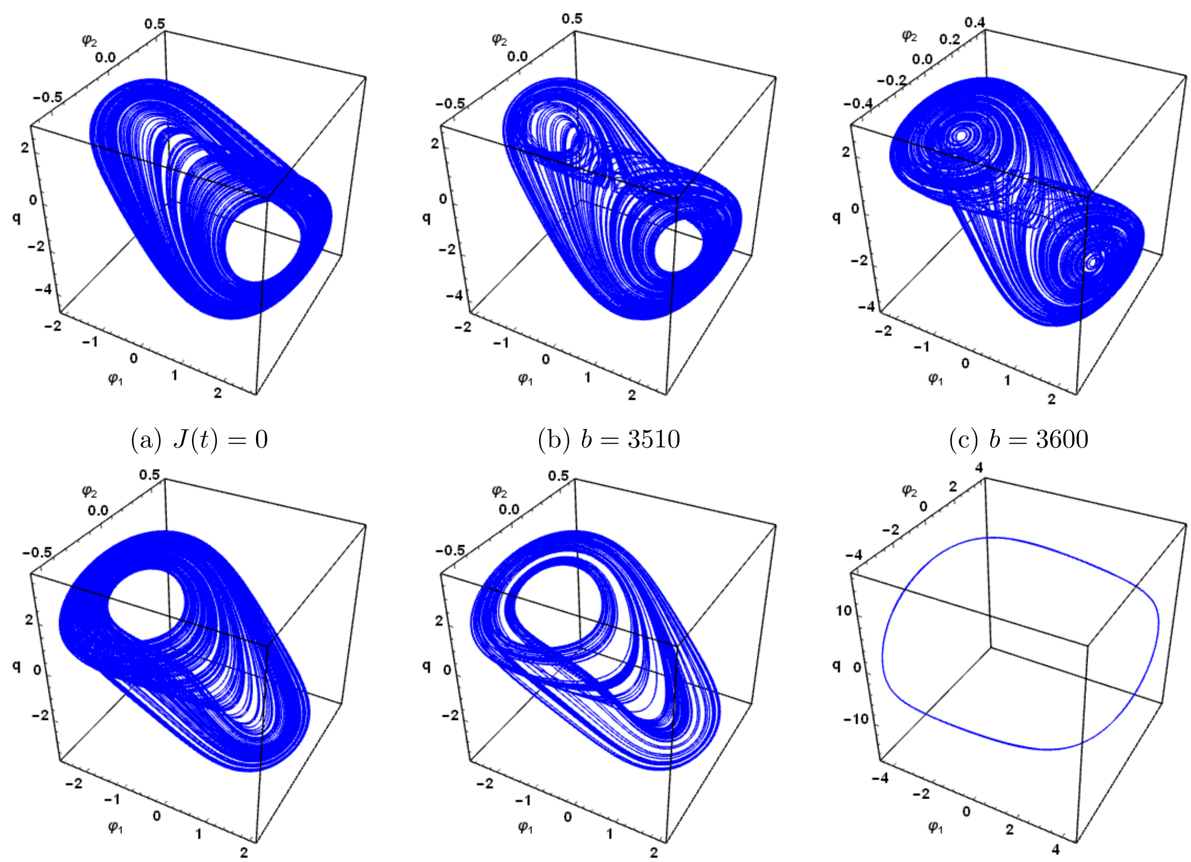


Figure 28: Memristor Chua circuit with the current source $J(t)$. Parameters: $C_1 = \dfrac{1}{\alpha}$, $C_2 = 1$, $L = \dfrac{1}{\beta}$, and $R = 1$, where $\alpha = 10,\ \beta = 14$. The current source $J(t)$ is given by Eq. (68).



(a) $J(t) = 0$ (b) $b = 3510$ (c) $b = 3600$

(c) $b = 3690$ (d) $b = 3700$ (e) $b = 3706$

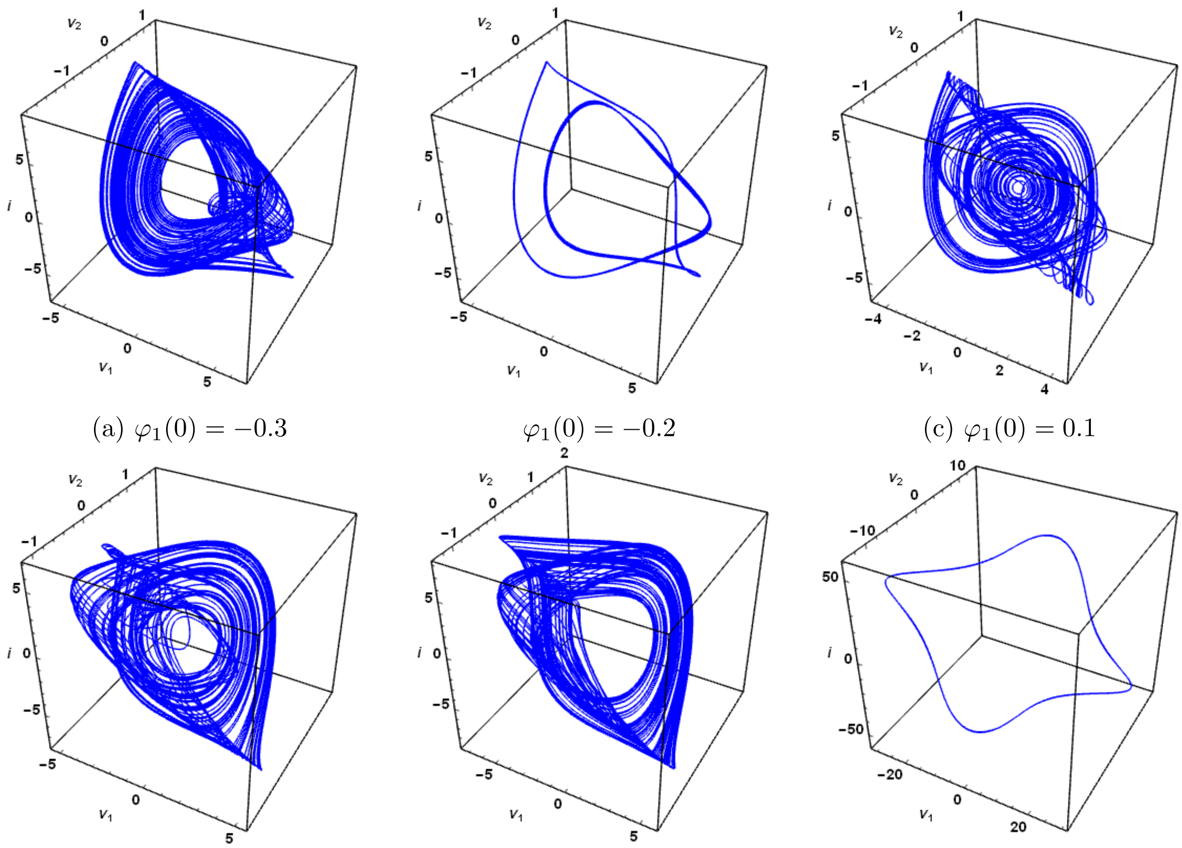
Figure 29: Trajectory change of Eq. (57) in response to the single pulse $J(t)$, which is defined by Eq. (68). The trajectories are plotted in the $(\varphi_1, \varphi_2, q)$-space for the time period $5500 \le t \le 6000$.

5.5 Forced Chua circuit equation

In this section, we apply the square wave $J(t)$ to the memristor Chua circuit. The $J(t)$ for Eq. (69) is defined as follows:

$$J(t) = 0.0001\,\mathrm{sign}\{\sin(0.22t)\}, \tag{70}$$

Similarly, changing the value of $\varphi_1(0)$ can result in significantly different behaviors. However, we could not find the interesting attractors in the $(\varphi_1, \varphi_2, q)$-space, since they resemble to those shown in Figs. 26 and 29. Instead, we show the attractor in the (v_1, v_2, i)-space.



(a) $\varphi_1(0) = -0.3$ $\varphi_1(0) = -0.2$ (c) $\varphi_1(0) = 0.1$

(c) $\varphi_1(0) = 0.4$ (d) $\varphi_1(0) = 0.7$ (e) $\varphi_1(0) = 0.85$

Figure 30: Trajectories of Eq. (57) for different values of $\varphi_1(0)$ in the (v_1, v_2, i)-space. They are plotted for the time period $2800 \leq t \leq 3000$.

5.6 Signal transmission between memristor Chua circuits

In this section, we study the signal transmission between the two coupled memristor Chua circuits. Consider the circuit shown in Fig. 31. The two memristor Chua circuits are connected by a unity-gain buffer, which is an operational amplifier circuit with a voltage gain of 1. The current source $J(t)$ is supplied to the left circuit only, and $K(t)$ is the current flowing from the unity-gain buffer into the memristor Chua circuit on the right. Note that $v_{1a} = v_{1b} = v_{Ma} = v_{Mb}$ because the unity-gain buffer copies v_{1a} of the left circuit to v_{1b} of the right circuit.

The dynamics of this circuit is given by

$$\left.\begin{aligned}
\frac{dv_{1a}}{dt} &= \alpha \left\{ i_a - v_{1a} - \left(\frac{3}{16} {\varphi_{1a}}^2 - \frac{7}{6} \right) v_{1a} + J(t) \right\}, \\
\frac{dv_{2a}}{dt} &= v_{1a} - v_{2a} + i_a, \\
\frac{di_a}{dt} &= -\beta v_{2a}, \\
\frac{d\varphi_{1a}}{dt} &= v_{1a}, \\
\frac{d\varphi_{2a}}{dt} &= v_{2a}, \\
\frac{dq_a}{dt} &= i_a, \\
\frac{dv_{2b}}{dt} &= v_{1b} - v_{2b} + i_b, \\
\frac{di_b}{dt} &= -\beta v_{2a}, \\
\frac{d\varphi_{1b}}{dt} &= v_{1b}, \\
\frac{d\varphi_{2b}}{dt} &= v_{2b}, \\
\frac{dq_b}{dt} &= i_b,
\end{aligned}\right\} \tag{71}$$

where $v_{1a}(t) = v_{1b}(t)$ for $t \geq 0$. The current source $J(t)$ supplied to the left circuit is given by

$$J(t) = 0.0001 \, \mathrm{sign}\{\sin(0.22t)\}, \tag{72}$$

where $\mathrm{sign}(x)$ gives -1, 0, or 1 depending on whether x is negative, zero, or positive (see Fig. 32). The parameters are given by

$$\alpha = 10, \beta = 14, \tag{73}$$

and the initial conditions are given by

$$\left.\begin{aligned}
& v_{1a}(0) = \frac{11194}{9600}, && v_{2a}(0) = 0.1, && i_a(0) = -1.4, \\
& && v_{2b}(0) = 0.11, && i_b(0) = 0.14, \\
& \varphi_{1a}(0) = 0.1, && \varphi_{2a} = 0.1, && q_a = 0.1, \\
& \varphi_{1b}(0) = 0.1, && \varphi_{2a} = 0.12, && q_a = 0.13.
\end{aligned}\right\} \tag{74}$$

The current $K(t)$ is given by

$$K(t) = \frac{1}{\alpha}\left(\frac{dv_{1a}(t)}{dt}\right) - \left\{ i_b(t) - v_{1a}(t) - \left(\frac{3}{16}\varphi_{1a}(t)^2 - \frac{7}{6}\right) v_{1a}(t) \right\}. \tag{75}$$

As shown in Fig. 33, the voltages v_{2a} and v_{2b} and the currents i_a and i_b will synchronize asymptotically in this circuit [12].

Next, we observe whether $J(t)$ is transmitted to the right memristor Chua circuit via $K(t)$. If the initial conditions $\varphi_{1a}(0)$ and $\varphi_{1b}(0)$ are not identical, then the waveform of the current $K(t)$ will be distorted as shown in Fig. 34(a). However, if the initial conditions are identical, then $K(t)$ will quickly become nearly equivalent to $J(t)$, as illustrated in Fig. 34(b).

This can be explained as follows: The currents $J(t)$ and $K(t)$ can be written as

$$\left.\begin{aligned} J(t) &= \frac{1}{\alpha}\left(\frac{dv_{1a}(t)}{dt}\right) - i_a(t) + v_{1a}(t) + \left(\frac{3}{16}\varphi_{1a}(t)^2 - \frac{7}{6}\right) v_{1a}(t), \\ K(t) &= \frac{1}{\alpha}\left(\frac{dv_{1a}(t)}{dt}\right) - i_b(t) + v_{1a}(t) + \left(\frac{3}{16}\varphi_{1b}(t)^2 - \frac{7}{6}\right) v_{1a}(t). \end{aligned}\right\} \tag{76}$$

From Eq. (71), we obtain

$$\frac{d\varphi_{1a}}{dt} - \frac{d\varphi_{1b}}{dt} = 0, \tag{77}$$

since $v_{1a} = v_{1b}$. Integrating the above equation from 0 to t, we obtain

$$\varphi_{1a}(t) - \varphi_{1b}(t) = \varphi_{1a}(0) - \varphi_{1b}(0). \tag{78}$$

If the initial conditions $\varphi_{1a}(0)$ and $\varphi_{1b}(0)$ are not identical, then we have

$$\varphi_{1a}(t) \not\equiv \varphi_{1b}(t). \tag{79}$$

for $t \geq 0$. Assume that the currents $i_a(t)$ and $i_b(t)$ will synchronize asymptotically, and the voltages $v_{2a}(t)$ and $v_{2b}(t)$ will also synchronize asymptotically. If the initial conditions $\varphi_{1a}(0)$ and $\varphi_{1b}(0)$ are not identical, then the current $K(t)$ will not nearly equal to $J(t)$. This is because the second and the fourth terms on the right side of the two equations in Eq. (76) will not become nearly equivalent. Next assume that $\varphi_{1a}(0) = \varphi_{1b}(0)$. Then, from Eq. (78), we obtain $\varphi_{1a}(t) = \varphi_{1b}(t)$. When the currents $i_a(t)$ and $i_b(t)$ synchronize asymptotically, $K(t)$ becomes nearly equivalent to $J(t)$. Furthermore, the two memristor Chua circuits exhibit nearly identical behavior after they are synchronized. Thus, they are in a special relationship as a pair.

We conclude that if we use the two coupled memristor Chua circuits for secure communication, then the requirement of identical initial flux conditions enhances secure communication.[5] This is because the same circuit parameters are sufficient for the two coupled original Chua circuits [13, 14].

> If the initial conditions $\varphi_{1a}(0)$ and $\varphi_{1b}(0)$ are not identical, then the signal $J(t)$ will not be transmitted correctly to the memristor Chua circuit on the right.

[5] We will explain secure communication by introducing the fictional characters Alice and Bob. Alice transmits the signal $J(t)$ using the memristor Chua circuit on the left. Bob receives the signal $K(t)$ from the memristor Chua circuit on the right. If Bob sets $\varphi_{1b}(0) = \varphi_{1a}(0)$, then he will receive a signal nearly equivalent to $J(t)$. Otherwise, he receives a distorted signal. The most critical factor for ensuring secure communication is having the identical initial flux condition, except for the requirement of two identical memristor Chua circuits.

As shown in [13, 14], we may be able to build a secure communication system by preparing two identical memristor Chua circuits with the same initial flux condition (i.e., $\varphi_{1a}(0) = \varphi_{1b}(0)$). This condition enhances the security of the communication system.

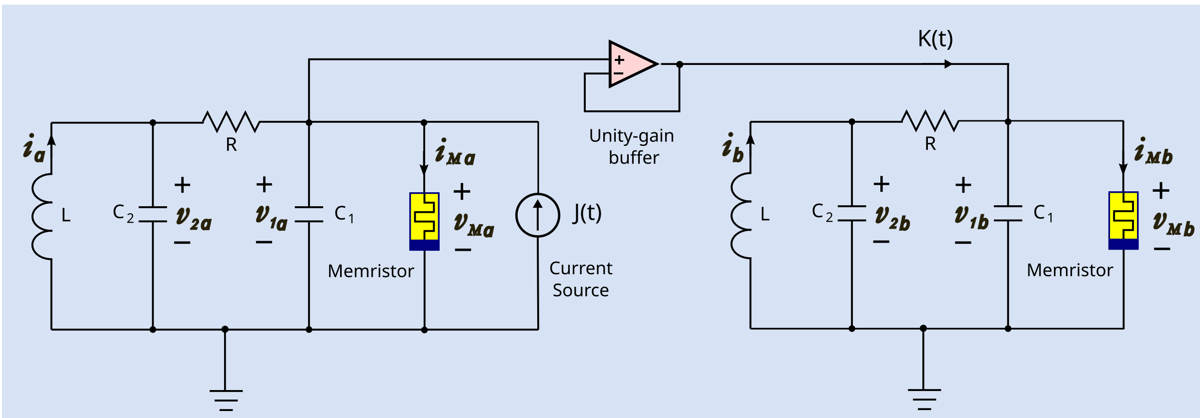


Figure 31: Two coupled memristor Chua circuits via the unity-gain buffer. The current source $J(t)$ is supplied to the left circuit only.

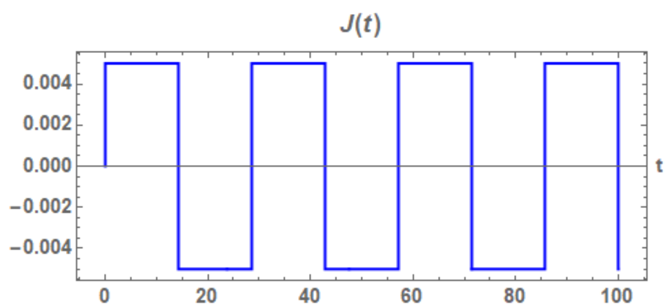
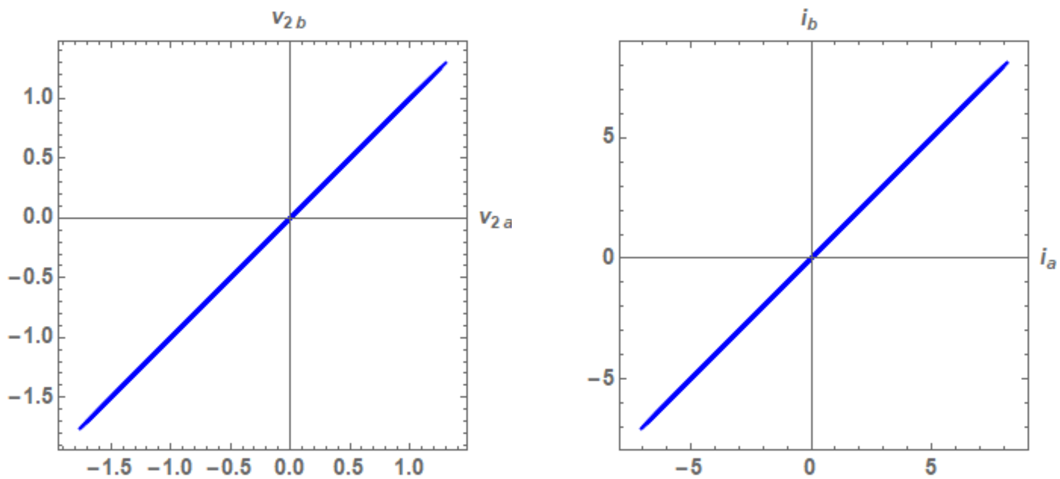


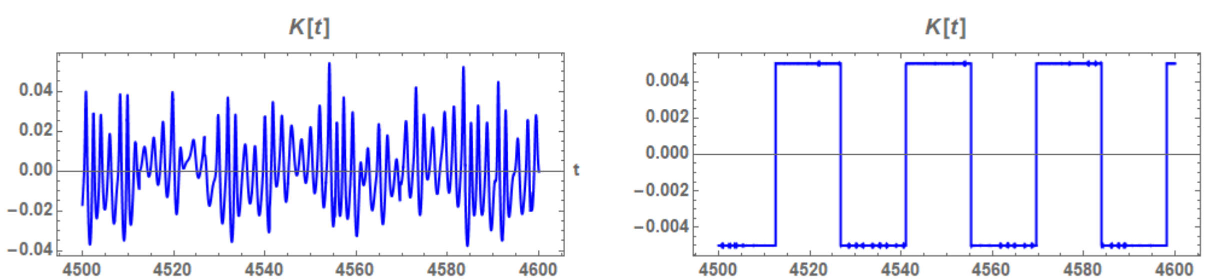
Figure 32: Waveform of $J(t)$ supplied to the left memristor Chua circuit.



(a) (v_{2a}, i_{2b})-plane

(b) (i_a, i_b)-plane

Figure 33: Synchronization of the voltages v_{2a} and v_{2b} and the currents i_a and i_b, which exhibits 45-degree line.



(a) $\varphi_{1a}(0) = 0.1$, $\varphi_{1b}(0) = 0.12$.

(b) $\varphi_1(0) = \varphi_2(0) = 0.1$.

Figure 34: Waveform of the current $K(t)$ flowing from the unity-gain buffer into the memristor Chua circuit on the right. It is plotted for the time period $4500 \leq t \leq 4600$. (a) If the initial conditions $\varphi_{1a}(0)$ and $\varphi_{1b}(0)$ are not identical, then the waveform of the current $K(t)$ will be distorted. (b) If they are identical, then $K(t)$ will soon become almost equivalent to $J(t)$.

6 Conclusion

We derived the memristor FitzHugh-Nagumo and the memristor Chua circuit equations by taking the time derivative of the original equations. We showed that the behavior of these equations is heavily dependent on the initial flux condition. Additionally, we showed that they exhibit various interesting behaviors that can be modified by applying a small DC bias current. These equations are expected to exhibit even more interesting behavior if the memductance possesses more complex nonlinear characteristics. Furthermore, differentiating the equations of various nonlinear systems may enable us to derive interesting memristor systems.

Acknowledgment

I would like to thank Professor Leon Chua of the University of California, Berkeley, for sending me many important papers on memristors over 10 years ago.

References

[1] Izhikevich, E.M. & FitzHugh, R. [2006] FitzHugh-Nagumo model, Scholarpedia, **1**(9):1349.

[2] Nagumo, J., Arimoto, S., & Yoshizawa, S. [1962] "An Active Pulse Transmission Line Simulating Nerve Axon," *Proceedings of the IRE.* **50** (10), 2061–2070.

[3] Chua, L.O. [1971] "Memristor–The missing circuit element," *IEEE Trans. Circuit Th.* **CT-18** (5), 507-519.

[4] Chua, L.O. & Kang, S.M. [1976] "Memristive devices and systems," *Proc. IEEE* **64**(2), 209-223.

[5] Chua, L.O. [2012] "The fourth element," *Proc. IEEE* **100**(6), 1920-1927.

[6] Strukov, D., Snider, G., Stewart, D., & Williams, R.S. [2008] "The missing memristor found," *Nature* **453**, 80-83.

[7] Chua, L. [2015] "Everything you want to know about memristors, but are afraid to ask," *Radioengineering* **24**(2), 319-368.

[8] Itoh, M. [2023] "Non-Volatility Property and Pinched Hysteresis Loops of 2-terminal Devices and Memristors," *viXra*: 2302.0055 [Category: Mathematical Physics].

[9] Itoh, M. [2025] "Chaotic Maps of Discretized Memristor Circuit Equations," *viXra*: 2410.0079 [Category: Mathematical Physics].

[10] Zhang, J. & Liao, X. [2017] "Synchronization and chaos in coupled memristor-based FitzHugh-Nagumo circuits with memristor synapse," *AEU - International Journal of Electronics and Communications*, **75**, 82-90.

[11] Zhang, J. & Liao, X. [2019] "Effects of initial conditions on the synchronization of the coupled memristor neural circuits," *Nonlinear Dyn*, **95**, 1269–1282.

[12] Pecora, L. & Carroll, T. [1990] "Synchronization in chaotic systems," *Phys. Rev. Lett.*, **64**(8), 821–824.

[13] Itoh, M., Murakami, H., & Chua, L.O. [1994] "Communication Systems via Chaotic Modulations," IEICE Trans. Fundamentals, **E77-A**(6), 1000-1006.

[14] Itoh, M. & Murakami, H. [1995] "New Communication Systems via Chaotic Synchronization and Modulations," IEICE Trans. Fundamentals, **E78-A**(3), 285-290.

[15] Madan, R.N. [1993] *Chua's Circuit: A Paradigm for Chaos* (World Scientific, Singapore).

[16] Hirsch, M.W., Smale, S., and Devaney, R.L. [2003] *Differential Equations, Dynamical Systems, and an Introduction to Chaos, Second Edition* (Elsevier Academic Press, Amsterdam).

[17] Itoh, M. & Chua, L.O. [2013] "Duality of Memristor Circuits," *International Journal of Bifurcation and Chaos*, **23**(1), (1330001-1)-(1330001-50).

Appendix

(1) Nonlinear capacitor

Consider the voltage-controlled nonlinear capacitor, which is defined by

$$Q_C = f(V_C), \tag{80}$$

where V_C and Q_C are the voltage and charge of the nonlinear capacitor. Differentiating Eq. (80) with respect to time t, we obtain

$$\frac{dQ_C}{dt} = \frac{\partial f(V_C)}{\partial V_C}\frac{dV_C}{dt}. \tag{81}$$

Define the new state variable:

$$\left.\begin{array}{lcl} i_C & \triangleq & \dfrac{dQ_C}{dt}, \\ v_C & \triangleq & \dfrac{dV_C}{dt}, \\ \varphi_C & \triangleq & V_C. \end{array}\right\} \tag{82}$$

Then, from Eq. (81), we obtain the equations which define the memristor

$$\left.\begin{array}{lcl} i_C & = & F(\varphi_C)v_C \\ \dfrac{d\varphi_C}{dt} & = & v_C \end{array}\right\} \tag{83}$$

where

$$F(\varphi_C) \triangleq \frac{\partial f(V_C)}{\partial V_C} = \frac{\partial f(\varphi_C)}{\partial \varphi_C} \tag{84}$$

Note that $\varphi_C \triangleq V_C$. Thus, we conclude the following:

> The voltage-controlled nonlinear capacitor is considered to be the voltage-controlled memristor in the new coordinate (v_C, i_C, φ_C)-space.

(2) Nonlinear inductor

Consider the current controlled nonlinear inductor, which is defined by

$$\Phi_L = g(I_L), \tag{85}$$

where I_L and Φ_L are the current and flux of the nonlinear inductor. Differentiating Eq. (80) with respect to time t, we obtain

$$\frac{d\Phi_L}{dt} = \frac{\partial g(I_L)}{\partial I_L}\frac{dI_L}{dt}. \tag{86}$$

Define the new state variable:

$$\left.\begin{array}{rcl} v_L & \triangleq & \dfrac{d\Phi_L}{dt}, \\ i_L & \triangleq & \dfrac{dI_L}{dt}, \\ q_L & \triangleq & I_L. \end{array}\right\} \tag{87}$$

Then, from Eq. (86), we obtain the equations which define the memristor

$$\left.\begin{array}{rcl} v_L & = & G(q_L)i_L \\ \dfrac{dq_L}{dt} & = & i_L \end{array}\right\} \tag{88}$$

where

$$G(q_L) \triangleq \frac{\partial g(I_L)}{\partial I_L} = \frac{\partial g(q_L)}{\partial q_L} \tag{89}$$

Note that $q_L \triangleq I_L$. Thus, we conclude the following:

The current-controlled nonlinear inductor is considered to be the current-controlled memristor in the new coordinate (i_L, v_L, q_L)-space.

(3) Linear circuit elements

Consider the dynamics of a linear capacitor, inductor, resistor, and DC voltage source (battery). These components are defined as follows:

$$C\frac{dV}{dt} = I, \quad L\frac{dI}{dt} = V, \quad V = RI, \quad V = E, \tag{90}$$

where V and I are the voltage and current, respectively, and C, L, R, and E are some constants. They are defined on the (V, I)-plane.

Define the new state variables:

$$v \triangleq \frac{dV}{dt}, \quad i \triangleq \frac{dI}{dt}. \tag{91}$$

Differentiating Eq. (90), we obtain

$$C\frac{dv}{dt} = i, \quad L\frac{di}{dt} = v, \quad v = Ri, \quad v = 0. \tag{92}$$

Therefore, the same relationships are obtained on the (v, i)-plane, except for the DC voltage source (battery). We can consider v and i to be the new voltage and current, respectively. Note that in the case of the DC voltage source (battery), we obtain $v = 0$. This implies that the battery acts as an ideal short circuit, allowing current to flow while creating no potential difference on the (v, i)-plane.

The same linear relationship holds for a linear capacitor, inductor, and resistor on the differential coordinate system, that is, the (v, i)-plane.

Furthermore, the linear resistor is written as follows from Eq. (90):

$$V = RI \quad \Longrightarrow \quad \frac{1}{R}\frac{d\varphi}{dt} = I \tag{93}$$

where $V = \dfrac{d\varphi}{dt}$ and φ is a flux. If we set $A \triangleq \dfrac{1}{R}$ and define the new state variables:

$$U \triangleq \varphi \ \text{ and } \ W \triangleq I, \tag{94}$$

then, from Eq. (93), we obtain the following relationship

$$A\frac{dU}{dt} = W, \tag{95}$$

which is analogous to the current-voltage relationship observed in capacitors and inductors. In other words, depending on the coordinate system used for observation, a resistor can be considered either a capacitor or an inductor. We can therefore conclude as follows:

The identification of circuit elements depends on the coordinate system used for observation.