A Structural Framework for Classical Dynamics I: Singular Limits and the Emergence of Discrete Motion

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Abstract

This work investigates the emergence of discrete dynamical structures from classical and relativistic formulations of motion. Using standard differentiation rules, we examine how Newtonian dynamics can be expressed in terms of energy gradients within a consistent formal framework. The relativistic kinetic-energy function is then analyzed in two limiting regimes: the classical limit ($\beta \to 0$) and the relativistic limit ($\beta \to 1$), where its asymptotic behaviour becomes singular.

In the singular regime, the dominant contributions of the kinetic-energy expansion exhibit a hierarchical structure that naturally leads to weighting factors of the form $(n+\frac{1}{2})$. This structure is shown to be consistent with the discrete spectrum of the quantum harmonic oscillator, suggesting that certain quantum features may be related to underlying classical asymptotic behaviour.

The analysis is complemented by a geometric interpretation of electromagnetic interactions using Gaussian units, together with a generalized functional perspective for handling singular contributions. Within this framework, the introduction of a small interaction scale—analogous to a Yukawa-type modification—provides an effective description of interaction ranges.

Overall, this approach provides a conceptual bridge between classical mechanics, relativistic dynamics, and quantum-like discretization, highlighting the role of singular limits, normalization, and geometric structure in the emergence of discrete spectra.

1 Introduction

The early 20th century marked a period of profound innovation in physics, from the discovery of the photoelectric effect [1] to the formulation of the electrodynamics of moving bodies [2], which established frameworks distinct from classical mechanics.

The simple harmonic oscillator, one of the most fundamental physical systems [3], provides a direct link between energy conservation and periodic motion. In quantum mechanics, however, energy is subject to statistical constraints, and this classical connection is modified, reflecting the time–energy uncertainty considerations discussed by Pauli [4]. Classical kinetic energy itself arises as the low-velocity limit of Einstein's relativistic expression [5].

While classical mechanics is typically formulated for massive particles, modern physics includes massless excitations and field-based descriptions, emphasizing the need for a consistent treatment of dynamical quantities across different frameworks [6, 7].

A useful perspective is therefore to examine how mechanical laws arise from more general structural relations between physical quantities. The interplay between relativistic and quantum descriptions has been extensively studied in phase space formulations [8, 6, 4], where the role of time and its definition becomes central [9].

The aim of this paper is to explore how classical dynamical relations can be recovered and extended from such structural considerations, with particular emphasis on the role of energy, differentiation, and interaction structure.

2 Low-Velocity Regime ($\beta \to 0$)

In Newtonian mechanics, the motion of a simple harmonic oscillator is characterized by its period $T = \frac{2\pi}{\omega}$ [3]. Using the energy conservation law for such a system,

$$E_{\text{tot}} = E_c + E_p = \frac{1}{2}m\|\vec{v}(t)\|^2 + \frac{1}{2}k\|\vec{x}(t)\|^2, \tag{1}$$

one can show that, since $x(t)$ depends smoothly on time t, variations of the kinetic energy can be related to the acceleration within the present framework (see Part II for detailed derivation). As long as E_{tot} is independent of position, Newton's second law emerges consistently:

$$m\vec{a} \sim \frac{\delta E_c}{\delta x} = -\sum F. \tag{2}$$

In special relativity, kinetic energy adopts a modified form. The gradient of this relativistic kinetic energy is generally nontrivial; however, a general Taylor expansion is possible:

$$E_c = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2}\right) \frac{{}^{\beta}\Xi_n}{2^{2n}} m v^2 \frac{v^{2n}}{c^{2n}}, \tag{3}$$

where ${}^{\beta}\Xi_n$ denotes the coefficients of the series expansion in $\beta = v^2/c^2$, with ${}^{0}\Xi_n = 1, 1, 2, 5, 14, \ldots$ when $\beta = 0$.

Several structural features arise from this expansion:

1. The series contains binomial coefficients $\binom{-1/2}{n}$, reflecting the structure of the relativistic correction.
2. When $v/c > 1$, the Taylor expansion formally produces complex branches, indicating the breakdown of the expansion in this regime.
3. Higher-order terms introduce nontrivial corrections to the relation between energy and motion.

From equation (1), Newton's second law may be viewed as a static relation. If promoted to a dynamical conservation condition (i.e., $\delta_\mu E_{\text{tot}} = 0$) within the context of Noether's theorem [10], one obtains conservation structures associated with the energy-momentum tensor $\delta_\mu T^{\nu\mu}$ (see Part II for details). In appropriate limits, these structures are related to known continuum descriptions.

Keeping E_{tot} constant, one can formally write:

$$\nabla E_c \sim m\vec{a} = \sum_{n=0}^{\infty} \frac{\left((n+\frac{1}{2})^2 + \frac{1}{2}n + \frac{1}{4}\right)}{2^{2n-1}} {}^{\beta}\Xi_n m \frac{\vec{a}x^{2n+1}}{L^{2n}}, \tag{4}$$

where $L = c/\omega$. Using the definition of angular velocity $\omega = d\theta/dt$ such that $\vec{v} = x\omega$, one finds:

$$\left(n + \sqrt{\frac{n}{2}} + 1\right)\omega_n \sim \sqrt{\frac{K}{m}\frac{2^{2n}}{{}^{\beta}\Xi_n}\beta^{-2n}}. \tag{5}$$

This expression suggests a discretized structure in the effective frequency when higher-order contributions are taken into account. Within the present framework, in the classical limit the higher-order terms scale as inverse powers of the expansion parameter and diverge for $n \geq 1$.

As a result, these contributions correspond to arbitrarily large frequency scales and do not participate in the effective dynamics. The system is therefore governed by the leading contribution $n = 0$, and no discrete structure is observed in the classical regime. with

$$\omega_0 = \sqrt{\frac{K}{m}}, \qquad \omega_n \to \infty \quad \text{for } n \geq 1. \tag{6}$$

This result indicates that, within the present framework, the oscillator exhibits a hierarchy of characteristic frequencies associated with the different contributions of the energy expansion, analogous to the discrete structure observed in quantum systems.

In the classical limit, these additional frequency scales are driven to arbitrarily large values and therefore decouple from the effective dynamics, leaving only the leading mode observable. This explains why classical motion appears continuous and suggests that an effective mass scale is required to describe particle-like behaviour, as in macroscopic systems and propagating excitations. It also suggests that extensions of this framework may involve additional a massive photon, which will be explored in the following sections.

3 Photon Mass from Yukawa-Type Interaction

A massif photon has never been observed but following Yukawa [11] theoretical work if the photon was to have a mass the Coulomb force would be rewritten:

$$F = -\frac{q_1 q_2}{r^2} e^{-\frac{c}{\hbar} mr} \tag{5}$$

One can see that if m (the mass of photons) is set at 0 then we get the coulomb force[1]. On the otherhand, if m is set to have any over value two problems occur, the first is that the exponential function is not a symmetric function $f(r) \neq f(-r)$. The second is that the limit towards ∞ of an exponential is itself ∞. This is clearly not possible, a charged particle in a distant galaxies would have a profound effect on earth. The first problem can easily be solved by the spherical symmetry found in electrodynamics. This is the effects occurring at r are those of r and those of $-r$. this means that Yukawa force should be rewritten:

$$F = -\frac{q_1 q_2}{2r^2} \left(e^{-\frac{c}{\hbar} mr} + e^{\frac{c}{\hbar} mr}\right) = -\frac{q_1 q_2}{r^2} \cosh\left(\frac{c}{\hbar} mr\right) \tag{6}$$

The second problem is slightly harder to solve as now all limits tends to infinity. This is where reasoning comes into play, as QM shows that there exists a cutoff point for which electrons can not collide with ions. Such a cross section can be imagined as the point which the rise in potential is larger than the step allowed from the space-time discontinuity. This puts a limit on the macroscopic force towards 0 and should also put a limit at a point r_c the critical length after which the force stops applying.

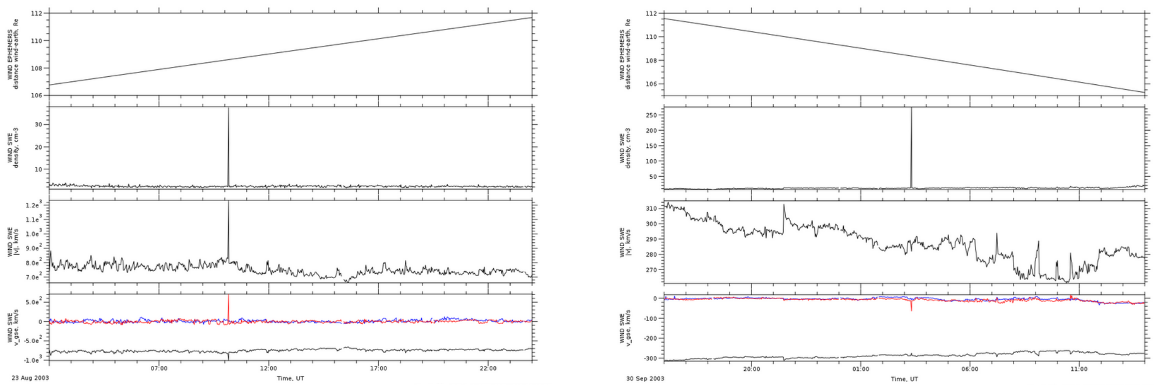
If Yukawa force was to be true one should look at the most charged object in the solar system and find a property that could be explained by massif photon. Thus looking at the sun, that is in its core, it would be positively charged, while the rest can be seen as quasi-neutral, one would expect a problem in our model to occur somewhere all around the sun between the iner-surface of the sun to the outer solar system. Such a phenomena is; the heating of the corona, setting r_c to be $7.10^8 m$.

Assuming a simple model in which the ambipolar diffusion allows the ions velocity to reach the velocity of the accelerated electrons. The force applied on the ions is then found to be $F_{1,2} = \frac{k_b}{4}\frac{dT_i}{dx}$. Through observation one can seen that the raise in the plasma's temperature at that particular point is about 1 degree per meter this means that $F_{1,2} = 3,45.10^{-24} N$ this force can be used in Yukawa force to get a wroth estimate of the photon's mass (this is done by assuming that the sun core is at $q = 77C$):

$$3,45.10^{-24} = \frac{1,38.10^{-6}}{4,9.10^{17}} \cosh(3.10^{42} m 7.10^8) \longrightarrow m \approx \frac{\text{acosh}(1,22)}{2,1.10^{51}} \approx 3,1.10^{-52} kg \tag{7}$$

If one was to choose an stronger force as the chosen force here is that of the ions acceleration and neglect all other forces such as ambipolar diffusion or the particles trapped in magnetic loops. The force that should be considered in (7) should be that needed to accelerate electrons (lets say 1N), one would find that the mass of a photon $m = 2,25.10^{-50} kg$ then one would need $F = 10^{68} N$ to reach a photon mass of $m = 10^{-49} kg$. It is also important to point out that acosh(1) = 0 this would mean that the sun would need to have a charge of $94C$ in this model for the mass of the photon to be nonexistent. This could be checked by looking at observation around the earth as its charge is $q = -5.10^5 C$, r_c is a constant so it does not change. But in the following graphs no noticeable decrease in the solar wind velocity has been observed. This may still be due to the instrument used. In figure (a) and (b) are 2 days where the satellite 'Wind' was at $r_c = 109,76 R_e$ collecting information on the solar wind. As the charge is distributed over the hole surface of the earth, this means that spikes of ions density and solar

[1] the minus term is taken away for convention in the coulomb force



(a) 23 Aug 2003 (b) 30 Sep 2003

Figure 1: The Satellite "Wind" existing (a) and interning (b) the Yukawa charge sphere generated by the earth

wind velocity should be found at $r_1 = 108.7R_e$ and $r_2 = 110.7R_e$. This is hardly observed and can be seen as being of the same order as the noise in most measurements although some spikes at r_1 can be observed and also an anomaly in v_z that could be explained as the satellite is not entering/exiting the sheath on the X axis. This outcome was expected as the earth surface charge is small.

This paper does not go in the detail of the plasma physics that would be needed to explain the heating of the sun's corona. This is to mean that assuming that accelerating electron will increase the temperature is a cruel approximation of meany important plasma phenomena. That is to say that this model can only say that photons should have a mass between $10^{-49} > m > 10^{-52}$. Right now one's best bet is to imagine that the Planck's constant $\hbar$ is the energy of a photon multiplied by the unitary unit of time such that normalizing it by the period gives back the wave's energy ($E = \hbar 2\pi\nu = mc^2 \frac{\vec{t}}{T}$). There is the period of oscillation and not the temperature. This would mean that $7,33.10^{-51}$ is the mass in kg of a single photon at any frequency.

4 Uni-Dimension Interaction

One may find that $\frac{1}{\sqrt{2}}$ makes a lot of appearance in quantum mechanics this is due to the fact that the probability amplitudes represent a stat, and the square of these probability amplitudes are the probability of this stat existing (and most exercises take the simple case of spin to explain superposition). In this paper there is also a recurrent $\frac{1}{\sqrt{2}}$ that has been hidden in (4). To solve this, one needs to understand how mechanics was constructed, this is to mean Newtonian gravitation force and then the similar electrodynamics.

$$F = \frac{m_1 m_2}{r^2} \qquad F = \frac{q_1 q_2}{r^2} \tag{8}$$

The observation of these force were done from a distance of the source and then an mathematical function was made to fit these observation. This means that the physical effect of the spherical symmetry, used to obtain (6), of this force has not been taken into account. This is to say that a more correct writing of these forces should be:

$$F = \frac{2m_1 m_2}{2r^2} \qquad F = \frac{2q_1 q_2}{2r^2} \tag{9}$$

This does not look impotent at first hand but one may be more acquainted to the less elegant form of the coulomb force:

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \tag{10}$$

Here ϵ_0 is a non-constant that can be set to 1, it comes from pre-Einsteinian physics where the need for ether in Maxwellian physics arose. The 4π on the other hand may have a more profound mathematical mining, it often arises when integrating over a sphere but this paper will argue for an other possible cause. Using the simple fact that $4\pi = 2.2\pi$ one can argue that the first '2' comes from the rewritten form of the Coulomb force and that 2π comes from the need to normalized over all forces on the plan source-observer that can have a 2π rotation. This would mean that a factor '2' on top is missing, one would have to write $q = \sqrt{2}q$ to solve this problem.

The explanation for the 2π may need more detail than at first look. One needs to understand that the electric potential for is a particle in empty space is; the charge of this particle normalized on the perimeter of the interaction at a point r. As the electric field fully diverge from this particle, the electric interaction is of uni-dimension. In simpler words, the perimeter that is needed is one of a line. This can be found with the elliptic perimeter:

$$P_{ellipse} = 2\pi\sqrt{\frac{R_1^2 + R_2^2}{2}} \longrightarrow P_{line} = 2\pi\frac{R_1}{\sqrt{2}} \tag{11}$$

Where $R_1 = r$ is the major axis and $R_2 = 0$ is the minor axes. The electric potential $V(r)$ can then be written as:

$$V(r) = \frac{q\sqrt{2}}{2\pi r} \longrightarrow F = \frac{q_1 q_2\sqrt{2}}{2\pi r^2} = \frac{q_1 q_2 2\sqrt{2}}{4\pi r^2} \tag{12}$$

A $\sqrt{2}$ rises here to. it is important that $1N = 1N$ still so the charge in this unite is not equal to the charge in the International System units.

5 Unit Normalisation

One can ask why it is more elegant to use the Gauss units in electrodynamics. First let it be said that the Heaviside–Lorentz units used will trying to explain the $\sqrt{2}$ has its charm as one does not need to keep track of the 2π but at the end of this section the importance of this term will be visible.

Although Coulomb, Ampere and Volts are in the SI units, the short answer for why one should use Gauss units in electrodynamics is to stay in "true" SI units and help one easily understand the meaning of their findings. for example in electrostatic one may be familiar with Ohm's law:

$$R = \frac{V}{I} \tag{13}$$

Where R is in Ohm, V in Volts and I in Ampere. These units can be decomposed in the following way Volts are Joules per Coulomb so $N.m/C$ and Ampere are simply Coulomb per seconds. This brings one to say that the resistance has the units of $N.m.s/C^2$. One may also know Ohm's law in conductive fluids under the form of $J = E\sigma$, where using the charge conservation law with Heaviside-Lorentz units:

$$\frac{\delta\rho}{\delta t} + \nabla.J = 0 \longrightarrow \frac{\delta\rho}{\delta t} + \sigma\rho = 0 \tag{14}$$

One can see that σ has units of s^{-1} this is already meaningful as it shows that the electrical conductivity of a plasma $\sigma = \frac{ne^2}{\nu m}$ is the electric frequency of this plasma. Coming back to SI units this means that the resistivity in ohm meter are equivalent to a time (an other way to look at it is as a open circuit takes an infinite amount off time for electrons to travel). One can then find the unit of Coulomb.

$$s = \frac{kg.m^3}{C^2.s} \longrightarrow C = \sqrt{\frac{kg.m^3}{s^2}} \tag{15}$$

This may seem like nonsense since this much could have been found through Coulomb force. But it is important as, now one would expect to find the photon's base energy as this is the elementary particle being exchanged in such interactions thus $C = \sqrt{m^3.s^{-2}.c^{-2}.kg.c^2} = \sqrt{m}.\sqrt{kg.c^2}$. To put it in to words, a charge caries half the energy of a base photon and half the strength of the photon in meter. Putting this in the coulomb force gives the following:

$$N = \frac{kg.c^2.m}{m^2} \longrightarrow E = \frac{kg.c^3.m.s}{m^2} \approx h\omega \tag{16}$$

To go from a force to an energy one needs to integrate on the distance traveled, as a photon goes at the speed of light this is equivalent to multiplying by the speed of light and the time to travel the distance. Transforming this equation to isolate Planck's constant with a unitary time leaves the strength of the photon times its speed divided by the distance between charge squared, this is the frequency of thus photon. One needs to remember that in Gauss unite there is 2π hanging around that allow one to find $E = h\nu$.

This may not be simpler but Gauss units shine in Maxwell equations as it is there as:

$$\nabla . E = 2\pi\rho \qquad \nabla . B = 0 \qquad c(\nabla \times E) = -\frac{\delta B}{\delta t} \qquad c(\nabla \times B) = 2\pi J + \frac{\delta E}{\delta t} \tag{17}$$

One may find that the divergence of the electric field is the full charge density, the divergence of the magnetic field is null as there is no magnetic charge all magnetic lines need to be close. The speed of light times the rotation of the electric field yields the magnetic field time variation, and the speed of light times the rotation of the magnetic field is the full currant to which is added the electric field time variation.

With the $\sqrt{2}$ found previously this becomes:

$$\nabla . E = \frac{2\pi}{\sqrt{2}}\rho \qquad \nabla . B = 0 \qquad c(\nabla \times E) = -\frac{\delta B}{\delta t} \qquad c(\nabla \times B) = \frac{2\pi}{\sqrt{2}} J + \frac{\delta E}{\delta t} \tag{18}$$

6 Singular Limits and Asymptotic Mode Structure

To analyze the limiting behaviour of functions with singularities, it is convenient to work within a distributional framework and focus on their asymptotic structure near points of divergence.

Functions such as $1/x$, $1/x^2$, or more generally functions with poles are not square-integrable and therefore do not belong to standard Hilbert spaces. However, their behaviour near singular points can be characterized in terms of distributions. In particular, families of translated singular functions,

$$\frac{1}{x}, \quad \frac{1}{x-n}, \qquad n \in Z, \tag{7}$$

can be understood through their action on test functions. Each term contributes a localized singular structure at its pole.

In this sense, one may associate to such families a distributional representation capturing their dominant singular behaviour. For example, the family $\{1/(x-n)\}$ induces a sequence of localized contributions at $x = n$, which can be represented schematically as

$$\sum_n \frac{1}{x-n} \sim \sum_n c_n \, \delta(x-n), \tag{8}$$

where c_n are weights determined by the local behaviour near each singularity, and "$\sim$" denotes equivalence at the level of leading-order singular contributions.

Higher-order poles exhibit stronger singular behaviour. For instance,

$$\frac{1}{(x-n)^2} \tag{9}$$

corresponds, in distributional terms, to derivatives of delta functions or to higher-weight localized contributions. More generally, the order of the pole determines the strength of localization and the associated weight in the distributional representation.

More generally, consider the function obtained from the relativistic kinetic energy through differentiation with respect to its argument. This leads to expressions exhibiting non-integer singular behaviour, i.e.

$$f(x) = \frac{1}{(1-x^2)\left(1 + \frac{1}{(1-x^2)^{1/2}}\right)}. \tag{10}$$

The point $x = 1$ is a singular point where the function develops a branch-type divergence.

Proposition (Asymptotic Mode Hierarchy). *Let $f(x)$ be a function with a branch-point singularity at $x = 1$, admitting a Puiseux expansion of the form*

$$f(x) \sim \sum_{n=0}^{\infty} a_n (1-x)^{-\alpha_n}, \tag{11}$$

with strictly increasing exponents α_n. Then the leading asymptotic behaviour near $x \to 1$ can be organized into a discrete hierarchy of modes indexed by the exponents α_n.

Proof. By standard results on Puiseux expansions, a function with a branch-point singularity admits a local representation in fractional powers of $(1 - x)$. The dominant behaviour near the singular point is determined by the smallest exponent α_0, while subleading terms are ordered by increasing α_n.

Since the exponents $\{\alpha_n\}$ form a discrete sequence, the asymptotic contributions can be indexed accordingly. Each term in the expansion defines a distinct level in the hierarchy, characterized by its divergence rate $(1 - x)^{-\alpha_n}$.

Therefore, the asymptotic structure of $f(x)$ near $x = 1$ is naturally decomposed into a discrete set of modes labeled by n, completing the proof.

In the specific case of the function $f(x)$ above, the Puiseux expansion generates exponents of the form

$$\alpha_n = n + \frac{1}{2}, \tag{12}$$

so that the asymptotic hierarchy is indexed by

$$\left(n + \frac{1}{2}\right). \tag{13}$$

This establishes that singular structures of this type give rise to a discrete hierarchy of asymptotic modes. Each mode corresponds to a distinct level of divergence and can be interpreted as a weighted localized contribution in a generalized (distributional) sense.

As a consequence, singular behaviour near branch points provides a natural mechanism for generating discrete structures. In particular, the appearance of $(n + \frac{1}{2})$-type indices connects directly to spectral structures encountered in quantum systems, such as the harmonic oscillator.

7 Relativistic Limit and Singular Structure ($\beta \to 1$)

Using the result obtained in Section 2, we now analyze the singular limit $\beta \to 1$ and its consequences for the effective frequency.

Recall that, under the assumption of constant total energy, the gradient of the kinetic energy can be written in the form

$$\nabla E_c \sim m\vec{a} = \sum_{n=0}^{\infty} C_n(\beta)\, m\, \frac{\vec{a}\, x^{2n+1}}{L^{2n}}, \tag{14}$$

where $L = c/\omega$ and the coefficients $C_n(\beta)$ are determined by the relativistic expansion.

In the regular regime $0 \leq \beta < 1$, these coefficients are obtained from the Taylor expansion of the relativistic kinetic-energy function about the chosen value of β. This yields a well-defined hierarchy of contributions, from which the corresponding effective frequencies may be recovered.

The situation changes qualitatively in the limit $\beta \to 1$. At this point the relativistic kinetic-energy function becomes singular, so the Taylor expansion is no longer valid. The appropriate description is instead given by a Puiseux-type expansion of the singular part of ∇E_c near the asymptotic point.

For the singular structure considered in Section 6, this expansion organizes the dominant terms into a hierarchy indexed by

$$\left(n + \frac{1}{2}\right). \tag{15}$$

These factors do not arise by postulate, but from the order of the leading divergent contributions in the Puiseux expansion.

Substituting this asymptotic hierarchy back into the energy-gradient equation, and using the relation

$$\vec{v} = x\omega, \tag{16}$$

one obtains a corresponding hierarchy of effective frequencies. In this regime, the oscillator is therefore no longer characterized by a single frequency, but by a family of characteristic frequencies ω_n determined by the singular structure of the expansion.

- In the classical limit, the higher-order terms scale as inverse powers of the expansion parameter and diverge for $n \geq 1$. As a result, these contributions correspond to arbitrarily large frequency scales and do not participate in the effective dynamics. The system is therefore governed by the leading contribution $n = 0$, and no discrete structure is observed in the classical regime.
- For intermediate relativistic regimes $0 < \beta < 1$, the frequencies $\{\omega_n\}$ remain retrievable from the Taylor expansion at the corresponding value of β, so the hierarchy is present but varies smoothly with β.
- In the singular limit $\beta \to 1$, however, the Puiseux expansion replaces the Taylor expansion, and the hierarchy becomes asymptotically discrete. The resulting structure may be written in the form

  $$\left(n + \frac{1}{2}\right)\omega = \omega_0,$$

  where ω_0 is fixed by the interaction scale and the normalization of the leading term.

Thus, within the present framework, the discrete frequency structure is obtained by solving the oscillator through the chain

$$\nabla E_c \;\longrightarrow\; \text{expansion at } \beta \;\longrightarrow\; \omega_n, \tag{17}$$

so that the quantum-like hierarchy appears as a consequence of the structure of the relativistic kinetic-energy function.

8 Quantized Angular Frequency

With the understating given by Gauss unites one can now try and solve the problem first found in quantum mechanics that of discreet electrodynamics. For this one needs to solve (1) with the coulomb force as ∇E_p:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\beta)(x-\beta)^n}{n!} 2m\vec{a}(L) + \frac{q_1 q_2 \sqrt{2}}{2\pi r^2} = 0 \quad \text{where} \quad f(x) = \frac{1}{(1-x^2)\left(1 + \frac{1}{(1-x^2)^{\frac{1}{2}}}\right)} \tag{23}$$

Here L is the distance traveled by light for unitary time. This is assuming two things; first, the total energy stays constant which may not be the case if they are exchanging photons and if this is the case the following calculation should be made for this change in total energy and not the potential energy. Second, is that the assumption is that the interaction is happening at the speed of light and thus the β is chosen to be equal to 1 and one needs to use the inner product space contained in vertical asymptote of the kinetic energy function. So using (22) to find the null force of a photon from which the energy can be integrate:

$$m\vec{a} = -\sum_{n=0}^{\infty} \frac{q_1 q_2 \sqrt{2}}{2\pi r^2} \frac{1}{2\left(n + \frac{1}{2}\right)\delta(1)} \longrightarrow m_\gamma c^2 = \sum_{n=0}^{\infty} \frac{q_1 q_2 \sqrt{2}}{2\pi r} \frac{1}{\left(n + \frac{1}{2}\right)} \tag{24}$$

where $m_\gamma c^2$ is the mass of the photon wave. One can recall from the previous section that interaction between charges caries the base mass of a photon such that Planck's constant can be found meaning that the strength length x_s divided by the distance r of both charge is the angular frequency:

$$m_\gamma c^2 = \sum_{n=0}^{\infty} \frac{x_s.\sqrt{2}.m.c^2}{2\pi r} \frac{1}{\left(n+\frac{1}{2}\right)} = h\nu \tag{25}$$

thus ν is:

$$\nu = \frac{m_\gamma c^2}{h} = \sum_{n=0}^{\infty} \frac{x_s\sqrt{2}}{2\pi r} \frac{1}{\left(n+\frac{1}{2}\right)} \tag{26}$$

The first term has been discussed and the second comes from Einstein's kinetic equation this is the term that generate discreet energy. For $\beta = 1$ this is $\frac{1}{\left(n+\frac{1}{2}\right)}$ and thus the only physical answer is:

$$\left(n+\frac{1}{2}\right)\nu = \frac{x_s\sqrt{2}}{2\pi r} \longrightarrow \left(n+\frac{1}{2}\right)\omega = \frac{x_s\sqrt{2}}{r} \tag{27}$$

9 Conclusion

This work develops a structural framework for relating classical, relativistic, and quantum-like descriptions of motion through the behaviour of the relativistic kinetic-energy function in different regimes.

The central observation is that the same underlying energy function exhibits qualitatively different structures depending on the value of $\beta = v^2/c^2$. For $0 \leq \beta < 1$, the function is regular and admits a Taylor expansion, from which a hierarchy of effective frequency contributions can be obtained. In this regime, the higher-order terms remain smoothly parametrized by β, and the resulting dynamics is effectively continuous.

The situation changes in the singular limit $\beta \to 1$. At this point the Taylor expansion is no longer valid, and the appropriate local description is given instead by a Puiseux-type expansion of the singular part of ∇E_c. The dominant terms of this expansion organize into a hierarchy indexed by factors of the form

$$\left(n+\frac{1}{2}\right), \tag{18}$$

which then induce a corresponding hierarchy of characteristic frequencies. In this sense, the discrete frequency structure is obtained by solving the oscillator through the chain

$$\nabla E_c \longrightarrow \text{singular expansion at } \beta \to 1 \longrightarrow \omega_n. \tag{19}$$

Within the present framework, the classical regime corresponds to the case in which the higher-order contributions are driven to arbitrarily large frequency scales and therefore do not participate in the effective dynamics. The motion is then governed by the leading contribution alone, so no discrete structure is resolved in standard classical treatments. By contrast, in the singular relativistic regime, the asymptotic hierarchy becomes dynamically relevant and produces a discrete frequency structure of the form

$$\left(n+\frac{1}{2}\right)\omega = \omega_0. \tag{20}$$

A second theme of the paper is that interaction laws carry an intrinsic geometric structure. The introduction of an effective interaction scale, together with the interpretation of inverse-square laws in reduced-dimensional settings, shows how normalization factors and characteristic lengths enter naturally into the description. In this context, Yukawa-type parameters should be understood as effective interaction scales rather than as evidence for a fundamental photon mass.

Taken together, these results suggest that certain quantum-like discrete features may be understood, within this framework, as arising from the asymptotic structure of relativistic energy rather than being imposed independently. The present approach does not replace standard quantum mechanics, but proposes a structural route by which discrete spectra can emerge from classical and relativistic ingredients.

Several aspects remain to be developed further. In particular, a more rigorous treatment of the generalized functional setting, a sharper derivation of the frequency normalization constant ω_0, and a more explicit comparison with standard quantization procedures would strengthen the framework. These questions are natural directions for future work.

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