

A Structural Framework for Classical Dynamics III: Effective Interaction Fields and the Three-Body Problem

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Abstract

This paper develops a geometric reconstruction of the three-body problem starting from a reformulation of the two-body interaction law. In the two-body case, the interaction is separated into two complementary contributions associated with the two masses, combined by an inverse-sum rule, and used to identify the origin of motion relative to the observed body. This reproduces the usual barycentric structure while making the geometric role of the interaction origin explicit.

The central question addressed here is whether this construction can be extended to the three-body problem. We show first that the naive extension fails for geometric reasons: the interaction of a given observed body is no longer supported on a single line of centres, but on two generically non-collinear channels. The difficulty is therefore not simply that more terms are present, but that the two-body interaction geometry is no longer globally available.

To overcome this obstruction, we construct a local three-body interaction law based on three ingredients: a local interaction scale, a local shape tensor encoding the two-channel geometry, and a frame/self structure that determines the local interaction length. The asymmetry between the two channels is incorporated through dimensionless correction factors associated with channel participation and channel sharing. The resulting local pull axis is then defined by a Newtonian-weighted combination of the two interaction directions, while the motion itself is governed by a scalar pull-only law.

The construction is tested on the principal benchmark solutions. It reduces correctly to the two-body problem when one channel disappears, reproduces the equal-mass equilateral Lagrange solution exactly, and the figure-eight choreography to numerical precision. These results indicate that the essential geometric problem of the three-body system lies in the reconstruction of the local pull axis rather than in the need for a fundamentally more complicated force law.

The main conclusion is that the two-body problem extends to three bodies not through a direct superposition of pairwise reductions, but through the reconstruction of a local interaction geometry in which axis selection and origin scaling must be separated. Within this framework, a scalar pull-only law remains sufficient once the correct local pull axis has been identified. The full geometric status of the corresponding origin distance in the generic case remains open and is discussed as a direction for future work.

1 Introduction

The two-body problem occupies a distinguished place in classical mechanics because its interaction geometry is exceptionally simple[1]. For two point masses interacting through a central force, the force on each body is directed along a single relative line. After separating the centre-of-mass motion, the dynamics may therefore be reduced to the motion of one effective coordinate together with its associated angular momentum. In this sense, the two-body problem admits a privileged interaction axis and a corresponding reduced description.

This simplicity does not extend directly to the three-body problem. For a given observed body, the total interaction is no longer determined by a single pairwise direction, but by the combination of two generically non-collinear interaction channels. The difficulty is therefore not merely quantitative — namely, that more terms must be added — but geometric: the interaction no longer selects a unique global axis. This is the basic obstruction that prevents the standard two-body reduction from carrying over unchanged to the three-body system.

This geometric obstruction is already implicit in classical mechanics. In the standard two-body treatment, the reduction to relative coordinates is possible because the full interaction is organized by one line of centres. In the three-body case[2], by contrast, each body is acted upon by several pairwise contributions whose directions do not, in general, coincide. The resulting motion cannot therefore be reduced globally to a single one-dimensional interaction coordinate. The problem is not the existence of force itself, but the absence of a unique geometric support for that force.

The present work starts from the observation that the two-body reduction can be reconstructed in a way that makes the role of the interaction origin explicit. Instead of treating the pair interaction only as one indivisible quantity, we separate it into two complementary contributions associated with the two masses involved in the interaction. These contributions combine by an inverse-sum rule, and the ratio between the total interaction and the contribution generated by the observed body identifies the origin of motion along the interaction axis. In the two-body case this reproduces the classical barycentric structure.

The main question of this paper is whether that logic can be extended to the three-body problem without abandoning the geometric meaning of the construction. Our answer is to retain the central idea of a locally defined interaction origin, while distinguishing three separate roles that become entangled in the many-body case: the geometric definition of an interaction axis, the construction of an interaction origin along that axis, and the scalar pull law governing the resulting motion. The purpose of the paper is not to modify the Newtonian pairwise force law, but to reorganize the local interaction structure in a way that remains compatible with it.

The resulting framework proceeds in three steps. First, the two-body problem is reconstructed in a form that makes the interaction origin explicit. Second, the obstruction to extending that construction directly to three bodies is identified. Third, a local three-body construction is introduced in which the pairwise interaction channels are combined into a frame/self structure, a distinguished local axis, and an effective origin of motion. This produces a pull-only description whose trajectory can be compared directly with the Newtonian one.

The point of view developed here is therefore geometric rather than purely algebraic. The two-body problem is solvable not only because there are fewer bodies, but because its interaction geometry is organized by a single axis. The three-body problem becomes difficult not only because there are more interactions, but because the interaction geometry is no longer globally reducible. The purpose of this paper is to show that, by rebuilding the interaction origin law locally, one can nevertheless recover a structured description of three-body motion that preserves the geometric meaning of the two-body case while extending it to a genuinely multi-channel setting.

This paper is organized as follows. Section 2 reconstructs the two-body problem from the split interaction contributions and derives the corresponding origin of motion. Section 4 introduces the three-body construction and develops the full local interaction framework. Section 5 discusses the geometric overlap structure and the role of the local axis. Finally, Section 8 summarizes the results and comments on the extent to which the construction should be regarded as a local origin law for many-body motion.

2 Reconstructing the Two-Body Problem

We begin with the simplest non-trivial case, namely a pair of point masses m_i and m_j separated by the distance[1]

$$X_{ij} := \|\mathbf{x}_j - \mathbf{x}_i\|, \quad \hat{\mathbf{r}}_{ij} := \frac{\mathbf{x}_j - \mathbf{x}_i}{X_{ij}}.$$

The purpose of this section is to reconstruct the usual two-body geometry from a split interaction picture in which the pair interaction is written as the combination of two complementary contributions.

2.1 Split interaction contributions

For the pair (i, j) , we associate to each mass its own contribution to the interaction:

$$\Phi_{ij}^{(i)} := \frac{m_i}{X_{ij}^3}, \quad \Phi_{ij}^{(j)} := \frac{m_j}{X_{ij}^3}.$$

These two quantities are not yet the total interaction. Rather, they represent the two complementary parts out of which the pair interaction is reconstructed.

The central assumption is that the full two-body interaction (here called channels) is obtained by an inverse-sum rule [3]:

$$\Phi_{ij} := \left(\left(\Phi_{ij}^{(i)} \right)^{-1} + \left(\Phi_{ij}^{(j)} \right)^{-1} \right)^{-1}.$$

Substituting the definitions above gives

$$\Phi_{ij} = \left(\frac{X_{ij}^3}{m_i} + \frac{X_{ij}^3}{m_j} \right)^{-1} = \frac{m_i m_j}{m_i + m_j} \frac{1}{X_{ij}^3}.$$

Hence the total interaction takes the form

$$\Phi_{ij} = \frac{\mu_{ij}}{X_{ij}^3}, \quad \mu_{ij} := \frac{m_i m_j}{m_i + m_j},$$

where μ_{ij} is the usual reduced mass. In this way, the standard two-body quantity is recovered from the split interaction picture.

2.2 Observed-body ratio and origin of motion

We now show how the origin of motion appears directly from this construction. Let body i be the observed body. The relevant ratio is the total interaction divided by the contribution attached to the observed body:

$$\frac{\Phi_{ij}}{\Phi_{ij}^{(i)}} = \frac{\mu_{ij}/X_{ij}^3}{m_i/X_{ij}^3} = \frac{\mu_{ij}}{m_i} = \frac{m_j}{m_i + m_j}.$$

This ratio is dimensionless and gives the fraction of the relative separation that must be traversed from \mathbf{x}_i toward \mathbf{x}_j in order to reach the origin of motion.

We therefore define the origin associated with the observed body i by

$$\mathbf{o}_i^{(ij)} := \mathbf{x}_i + \frac{\Phi_{ij}}{\Phi_{ij}^{(i)}} X_{ij} \hat{\mathbf{r}}_{ij}.$$

Since $X_{ij} \hat{\mathbf{r}}_{ij} = \mathbf{x}_j - \mathbf{x}_i$, this becomes

$$\mathbf{o}_i^{(ij)} = \mathbf{x}_i + \frac{m_j}{m_i + m_j} (\mathbf{x}_j - \mathbf{x}_i) = \frac{m_i \mathbf{x}_i + m_j \mathbf{x}_j}{m_i + m_j}.$$

Thus the origin obtained from the split interaction construction is exactly the barycentric origin of the two-body system.

The same computation from the perspective of the other body gives

$$\frac{\Phi_{ij}}{\Phi_{ij}^{(j)}} = \frac{m_i}{m_i + m_j},$$

and therefore

$$\mathbf{o}_j^{(ij)} = \mathbf{x}_j + \frac{\Phi_{ij}}{\Phi_{ij}^{(j)}} X_{ji} \hat{\mathbf{r}}_{ji} = \frac{m_i \mathbf{x}_i + m_j \mathbf{x}_j}{m_i + m_j}.$$

Hence both observed-body constructions identify the same origin, as they must.

2.3 Geometric meaning

This derivation shows that the two-body problem admits a particularly simple interaction geometry. The pair interaction is supported on a single axis, namely the line joining the two masses, and the inverse-sum rule determines a unique distinguished point on that axis. The origin of motion is therefore not added by hand: it is selected directly by the ratio between the total interaction and the observed-body contribution.

Equivalently, the displacement from the observed body i to the origin may be written as

$$\mathbf{q}_i^{(ij)} := \mathbf{x}_i - \mathbf{o}_i^{(ij)} = -\frac{\Phi_{ij}}{\Phi_{ij}^{(i)}} X_{ij} \hat{\mathbf{r}}_{ij} = -\frac{m_j}{m_i + m_j} (\mathbf{x}_j - \mathbf{x}_i).$$

Its magnitude is

$$\|\mathbf{q}_i^{(ij)}\| = \frac{m_j}{m_i + m_j} X_{ij},$$

which is exactly the barycentric distance from body i to the centre of mass.

The importance of this reconstruction is that it isolates the mechanism by which the two-body origin is obtained:

1. split the pair interaction into two complementary contributions;
2. combine them by an inverse-sum rule;
3. divide the resulting interaction by the contribution of the observed body;
4. use the resulting dimensionless ratio to locate the origin along the interaction axis.

This four-step structure will serve as the template for the many-body generalization. The essential difficulty of the three-body problem will not be the inverse-sum principle itself, but the fact that the interaction is no longer supported on a single pairwise axis.

3 Why the Naive Extension Fails for Three Bodies

The two-body construction of the previous section succeeds because the interaction geometry is supported on a single axis. For a pair of bodies (i, j) , the relative displacement

$$\mathbf{r}_{ij} := \mathbf{x}_j - \mathbf{x}_i$$

selects a unique direction

$$\hat{\mathbf{r}}_{ij} := \frac{\mathbf{r}_{ij}}{\|\mathbf{r}_{ij}\|},$$

and the interaction origin lies somewhere on that line. The inverse-sum rule then determines how far one must move along this unique axis in order to reach the origin of motion.

A naive extension to the three-body problem would proceed as follows. For a given observed body i , one introduces two pairwise channels,

$$(i, j), \quad (i, k),$$

with the corresponding distances

$$X_{ij} := \|\mathbf{x}_j - \mathbf{x}_i\|, \quad X_{ik} := \|\mathbf{x}_k - \mathbf{x}_i\|,$$

and two unit directions

$$\hat{\mathbf{r}}_{ij} := \frac{\mathbf{x}_j - \mathbf{x}_i}{X_{ij}}, \quad \hat{\mathbf{r}}_{ik} := \frac{\mathbf{x}_k - \mathbf{x}_i}{X_{ik}}.$$

One may then define interaction contribution for each channel,

$$F_{ij} = \frac{m_i m_j}{X_{ij}^2}, \quad F_{ik} = \frac{m_i m_k}{X_{ik}^2},$$

attempt to determine an origin of motion for body i by analogy with the two-body case i.e. through determining CoM.

The difficulty is immediate: in the two-body problem, the interaction origin is determined along one line and thus the Forces, the origin of motion, and the CoM are constructed in superposition, whereas in the three-body problem there are generically two different interaction directions. Unless

$$\hat{\mathbf{r}}_{ij} \parallel \hat{\mathbf{r}}_{ik},$$

there is no single pairwise axis on which the origin can be placed. The obstruction is therefore geometric as they all map from 2 \rightarrow 3 bodies in different ways. Using this paper model, they can each be tracked separately.

3.1 What must be rebuilt

the three-body generalization cannot be obtained merely by repeating the two-body inverse-sum construction channel by channel. One must instead rebuild three objects simultaneously:

1. a *local axis* that organizes the two channels into one effective geometric direction;
2. a *local origin* defined relative to that axis;
3. a *reduced interaction law* acting with respect to that origin.

Only after such a local geometric structure has been defined does it make sense to ask for the analogue of the two-body origin law. The next section is devoted to precisely this reconstruction.

4 Three-Body Construction

We now present the full local three-body construction used in this model. The guiding principle is to preserve the frame/self logic of the two-body reconstruction while allowing the local interaction geometry to be determined by two non-collinear channels. The construction separates three roles:

1. the *geometric data* carried by the two pairwise directions,
2. the *frame/self scalar law* which determines the local interaction length,
3. the *pull axis and pull law* governing the resulting dynamics.

Throughout this section, i denotes the observed body, while j and k denote the other two bodies.

4.1 Geometric vectors and local interaction scale

We define the pairwise relative vectors

$$\mathbf{r}_{ij} := \mathbf{x}_j - \mathbf{x}_i, \quad \mathbf{r}_{ik} := \mathbf{x}_k - \mathbf{x}_i,$$

with corresponding distances

$$X_{ij} := \|\mathbf{r}_{ij}\|, \quad X_{ik} := \|\mathbf{r}_{ik}\|,$$

and unit directions

$$\hat{\mathbf{r}}_{ij} := \frac{\mathbf{r}_{ij}}{X_{ij}}, \quad \hat{\mathbf{r}}_{ik} := \frac{\mathbf{r}_{ik}}{X_{ik}}.$$

The first geometric object is the local two-channel interaction vector

$$\mathbf{d}_i := \hat{\mathbf{r}}_{ij} + \hat{\mathbf{r}}_{ik}.$$

Its norm

$$L_i := \|\mathbf{d}_i\|$$

defines the local interaction scale. Whenever $\mathbf{d}_i \neq \mathbf{0}$, we also define the associated geometric axis

$$\hat{\mathbf{e}}_i^{(d)} := \frac{\mathbf{d}_i}{L_i}.$$

The quantity L_i measures the coherence of the two channels. In the symmetric equilateral case one has $L_i = \sqrt{3}$, while L_i decreases as the two channels become more opposed.

4.2 Local shape tensor

To encode the two-channel geometry beyond the single vector \mathbf{d}_i , we define the local shape tensor

$$\mathbf{S}_i := \frac{1}{2} (\hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ij}^T + \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik}^T).$$

This tensor is purely geometric. It is symmetric, positive semidefinite, and its trace is equal to 1. It captures the anisotropy of the local two-channel configuration.

The scalar

$$\gamma_i := \left(\hat{\mathbf{e}}_i^{(d)} \right)^T \mathbf{S}_i \hat{\mathbf{e}}_i^{(d)}$$

measures the coherence of the shape tensor along the local bisector direction. In the present two-channel setting one has

$$\gamma_i = \frac{L_i^2}{4}.$$

4.3 Frame/self scalar construction

We now build the scalar frame/self quantities. The self contribution attached to the observed body is

$$\Phi_i^{\text{self}} := \frac{m_i}{L_i^3}.$$

The uncorrected channel contributions are

$$\Phi_{ij}^{(j)} := \frac{m_j}{L_i^3}, \quad \Phi_{ik}^{(k)} := \frac{m_k}{L_i^3}.$$

Due to their geometry, these are then modified by dimensionless geometric correction factors α_i and β_i , to be defined below, leading to

$$\Phi_{ij}^{(j),\text{eff}} := (1 + \alpha_i) \frac{m_j}{L_i^3}, \quad \Phi_{ik}^{(k),\text{eff}} := (1 + \beta_i) \frac{m_k}{L_i^3}.$$

The effective external interaction is defined by the inverse-sum rule

$$\Phi_i^{\text{ext}} := \left(\left(\Phi_{ij}^{(j),\text{eff}} \right)^{-1} + \left(\Phi_{ik}^{(k),\text{eff}} \right)^{-1} \right)^{-1},$$

and the frame interaction is then

$$\Phi_i^{\text{frame}} := \left(\left(\Phi_i^{\text{self}} \right)^{-1} + \left(\Phi_i^{\text{ext}} \right)^{-1} \right)^{-1}.$$

The scalar frame/self ratio therefore becomes

$$\frac{\Phi_i^{\text{frame}}}{\Phi_i^{\text{self}}} = \frac{\Phi_i^{\text{ext}}}{\Phi_i^{\text{self}} + \Phi_i^{\text{ext}}}.$$

Finally, the local frame/self length is defined by

$$\ell_i := \frac{\Phi_i^{\text{frame}}}{\Phi_i^{\text{self}}} L_i.$$

4.4 Geometric correction factors for interactions

The role of α_i and β_i is to modulate the two channel contributions according to the geometry of the local configuration, while preserving the two-body limit and the symmetric three-body limits.

We first define the symmetric participation factor

$$\nu_i := \frac{4m_j m_k}{(m_j + m_k)^2}.$$

This satisfies

$$\nu_i = 1 \quad \text{if } m_j = m_k, \quad \nu_i \rightarrow 0 \quad \text{if } m_j \rightarrow 0 \text{ or } m_k \rightarrow 0.$$

Next, we define the local overlap amplitude

$$h_i := 1 + \nu_i \left(\frac{2}{4\gamma_i - 1} - 1 \right).$$

Since $\gamma_i = L_i^2/4$, this is equivalent to

$$h_i = 1 + \nu_i \left(\frac{2}{L_i^2 - 1} - 1 \right).$$

We then define the channel-sharing scalar

$$q_i^{\text{share}} := \exp\left(\nu_i(c_{ij} - c_{ik})\right),$$

where

$$c_{ij} := \hat{\mathbf{r}}_{ij} \cdot \hat{\mathbf{e}}_i^{(d)}, \quad c_{ik} := \hat{\mathbf{r}}_{ik} \cdot \hat{\mathbf{e}}_i^{(d)}.$$

The effective correction factors are then

$$1 + \alpha_i := h_i \frac{q_i^{\text{share}} m_j + m_k}{m_j + m_k},$$

$$1 + \beta_i := h_i \frac{q_i^{\text{share}} m_j + m_k}{q_i^{\text{share}} (m_j + m_k)}.$$

By construction:

- in the two-body limit, $\nu_i \rightarrow 0$, so $\alpha_i \rightarrow 0$ and $\beta_i \rightarrow 0$,
- in the symmetric equilateral case, $h_i = 1$ and $q_i^{\text{share}} = 1$, so $\alpha_i = \beta_i = 0$.

4.5 Newtonian-weighted local axis

The decisive step is to separate the *length* of the local pull vector from its *axis*. The frame/self construction determines the length ℓ_i , but the local pull axis is determined by the Newtonian channel weights.

We therefore define the Newtonian-weighted axis generator

$$\mathbf{g}_i^{\text{axis}} := (1 + \alpha_i) \frac{m_j}{X_{ij}^2} \hat{\mathbf{r}}_{ij} + (1 + \beta_i) \frac{m_k}{X_{ik}^2} \hat{\mathbf{r}}_{ik}.$$

Its associated unit direction is

$$\hat{\mathbf{e}}_i^{(N)} := \frac{\mathbf{g}_i^{\text{axis}}}{\|\mathbf{g}_i^{\text{axis}}\|}.$$

This is the local axis used. It is no longer the simple geometric bisector $\hat{\mathbf{e}}_i^{(d)}$, but rather the Newtonian-weighted pull direction corrected by the geometric factors α_i, β_i .

4.6 The exact origin

The local pull vector is finally defined by combining the frame/self length and the Newtonian-weighted axis:

$$\mathbf{q}_i := -\ell_i \hat{\mathbf{e}}_i^{(N)}.$$

The local origin is then

$$\mathbf{o}_i := \mathbf{x}_i - \mathbf{q}_i.$$

This is the exact origin map used in the numerical implementation.

It is useful to note explicitly that the frame/self construction determines only the scalar length ℓ_i , while the Newtonian-weighted combination determines the axis. Thus the local origin law takes the form

$$\mathbf{o}_i = \mathbf{x}_i + \left(\frac{\Phi_i^{\text{frame}}}{\Phi_i^{\text{self}}} L_i \right) \frac{\mathbf{g}_i^{\text{axis}}}{\|\mathbf{g}_i^{\text{axis}}\|}.$$

4.7 Scalar pull law

The dynamics is then governed by a pull-only scalar law. First define the effective pull vector

$$\mathbf{F}_i := Gm_i \left((1 + \alpha_i) \frac{m_j}{X_{ij}^2} \hat{\mathbf{r}}_{ij} + (1 + \beta_i) \frac{m_k}{X_{ik}^2} \hat{\mathbf{r}}_{ik} \right).$$

The scalar pull coefficient is chosen by projection onto \mathbf{q}_i :

$$K_i := -\frac{\mathbf{F}_i \cdot \mathbf{q}_i}{\|\mathbf{q}_i\|^2}.$$

The acceleration is then

$$\mathbf{a}_i = -\frac{K_i}{m_i} \mathbf{q}_i.$$

Since \mathbf{q}_i is collinear with $\mathbf{g}_i^{\text{axis}}$, the acceleration is a pure pull along the local origin axis. In the symmetric equilateral case this reproduces the exact Newtonian motion. In the figure-eight case it was found numerically that the same construction also reproduces the Newtonian trajectory to numerical precision.

4.8 Summary of the construction

The complete local three-body law for the observed body i may therefore be summarized as follows:

1. Build the geometric vectors

$$\mathbf{r}_{ij}, \mathbf{r}_{ik}, \hat{\mathbf{r}}_{ij}, \hat{\mathbf{r}}_{ik}, \mathbf{d}_i, L_i.$$

2. Build the shape tensor

$$\mathbf{S}_i = \frac{1}{2} (\hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ij}^T + \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik}^T)$$

and the coherence scalar

$$\gamma_i = \left(\hat{\mathbf{e}}_i^{(d)} \right)^T \mathbf{S}_i \hat{\mathbf{e}}_i^{(d)}.$$

3. Define $\nu_i, h_i, q_i^{\text{share}}$, and from them α_i, β_i .

4. Construct the corrected channel fields

$$\Phi_{ij}^{(j),\text{eff}}, \quad \Phi_{ik}^{(k),\text{eff}},$$

together with $\Phi_i^{\text{ext}}, \Phi_i^{\text{frame}}$, and the length

$$\ell_i = \frac{\Phi_i^{\text{frame}}}{\Phi_i^{\text{self}}} L_i.$$

5. Define the Newtonian-weighted axis

$$\hat{\mathbf{e}}_i^{(N)} = \frac{(1 + \alpha_i) \frac{m_j}{X_{ij}^2} \hat{\mathbf{r}}_{ij} + (1 + \beta_i) \frac{m_k}{X_{ik}^2} \hat{\mathbf{r}}_{ik}}{\left\| (1 + \alpha_i) \frac{m_j}{X_{ij}^2} \hat{\mathbf{r}}_{ij} + (1 + \beta_i) \frac{m_k}{X_{ik}^2} \hat{\mathbf{r}}_{ik} \right\|}.$$

6. Define the local origin vector

$$\mathbf{q}_i = -\ell_i \hat{\mathbf{e}}_i^{(N)}, \quad \mathbf{o}_i = \mathbf{x}_i - \mathbf{q}_i.$$

7. Evolve the system under the scalar pull law

$$\mathbf{a}_i = -\frac{K_i}{m_i} \mathbf{q}_i, \quad K_i = -\frac{\mathbf{F}_i \cdot \mathbf{q}_i}{\|\mathbf{q}_i\|^2}.$$

This is the three-body construction in its totality.

5 Consistency Checks and Benchmark Solutions

In this section we summarize the principal consistency checks satisfied by the construction introduced above. The purpose of these checks is twofold. First, they test whether the local origin law reduces correctly to previously understood situations. Second, they clarify which parts of the construction should be regarded as exact and which parts remain interpretive or diagnostic.

The three basic tests considered here are:

1. the two-body reduction,
2. the equilateral Lagrange solution,
3. the figure-eight choreography.

5.1 Two-body reduction

A minimal consistency requirement is that the three-body construction reduce to the two-body construction whenever one interaction channel disappears. In the present framework, this means that if one of the source bodies becomes ineffective as an interaction partner, then the local correction factors should switch off and the interaction geometry should collapse to a single channel.

Concretely, for the observed body i , suppose that one channel becomes negligible, for example

$$m_k \rightarrow 0.$$

Then the participation factor

$$\nu_i = \frac{4m_j m_k}{(m_j + m_k)^2}$$

satisfies

$$\nu_i \rightarrow 0.$$

As a consequence,

$$h_i \rightarrow 1, \quad q_i^{\text{share}} \rightarrow 1, \quad \alpha_i \rightarrow 0, \quad \beta_i \rightarrow 0.$$

Thus the corrected channel contributions reduce to the uncorrected ones, and the effective external interaction collapses to the surviving interaction channel. In this sense, the many-channel correction mechanism is turned off in the two-body limit.

The geometric content also collapses appropriately. With only one active source channel, the local pull axis is just the single pairwise direction, and the pull-only law reduces to the standard two-body structure. This check is therefore satisfied exactly at the level of construction.

It is important to note, however, that this is an *internal consistency check* on the reduction of the local law. It does not by itself prove that the full many-body origin law is uniquely fixed by the two-body case. Rather, it shows that the present construction is compatible with the two-body reduction in the appropriate limit.

5.2 Equilateral Lagrange solution

The second benchmark is the equal-mass equilateral Lagrange solution. This case is crucial because it provides a nontrivial three-body motion with maximal symmetry. In this regime the construction simplifies dramatically and serves as a stringent test of whether the local axis, the origin law, and the pull coefficient reduce correctly.

For the equilateral configuration, the two channel directions relative to the observed body are symmetric, so the geometric data satisfy

$$X_{ij} = X_{ik}, \quad m_j = m_k, \quad \hat{\mathbf{r}}_{ij} \text{ and } \hat{\mathbf{r}}_{ik} \text{ are symmetric about the local bisector.}$$

As a result,

$$q_i^{\text{share}} = 1, \quad \alpha_i = \beta_i = 0,$$

and the local shape tensor reduces to its symmetric form. In particular, one finds

$$L_i = \sqrt{3}, \quad \gamma_i = \frac{L_i^2}{4} = \frac{3}{4}, \quad h_i = 1.$$

Thus the correction structure switches off exactly in the symmetric Lagrange case, and the frame/self ratio reduces to its uncorrected form.

At the same time, the Newtonian-weighted axis coincides with the geometric bisector axis, since both channels contribute equally. Hence the local pull vector is aligned with the symmetry axis and the scalar pull law reproduces the exact Newtonian acceleration. In particular, the numerical implementation yields machine-precision agreement with the Newtonian Lagrange orbit.

Therefore, in the equilateral Lagrange benchmark, the following are exact:

- the pull axis,
- the scalar pull coefficient,
- the resulting trajectory.

In this case the local origin also coincides with the expected symmetry centre, so the origin law is exact in both geometric and dynamical senses.

5.3 Figure-eight choreography

The third benchmark is the equal-mass figure-eight solution. This case is qualitatively different from the Lagrange solution because the local interaction geometry is no longer static or symmetric. The two source channels vary in both length and direction, the local overlap changes in time, and the observed body passes through configurations in which the two channels become highly asymmetric.

This benchmark is particularly important because it reveals which parts of the present construction are essential and which are merely auxiliary. Numerical tests show that once the local axis is built from the Newtonian-weighted channel combination,

$$\hat{\mathbf{e}}_i^{(N)} = \frac{(1 + \alpha_i) \frac{m_j}{X_{ij}^2} \hat{\mathbf{r}}_{ij} + (1 + \beta_i) \frac{m_k}{X_{ik}^2} \hat{\mathbf{r}}_{ik}}{\left\| (1 + \alpha_i) \frac{m_j}{X_{ij}^2} \hat{\mathbf{r}}_{ij} + (1 + \beta_i) \frac{m_k}{X_{ik}^2} \hat{\mathbf{r}}_{ik} \right\|},$$

the scalar pull law reproduces the figure-eight trajectory to numerical precision.

This result has an immediate interpretation. Under the scalar pull law

$$\mathbf{a}_i = -\frac{K_i}{m_i} \mathbf{q}_i, \quad K_i = -\frac{\mathbf{F}_i \cdot \mathbf{q}_i}{\|\mathbf{q}_i\|^2},$$

the trajectory depends only on the *axis* of \mathbf{q}_i , not on its length. Thus, once the axis is aligned with the true Newtonian pull direction, the scalar pull law recovers the exact trajectory even if the norm of \mathbf{q}_i varies nontrivially. This is precisely what is observed in the figure-eight benchmark.

The trajectory is exact to numerical precision, but the norm of the local origin displacement can become large or drift substantially. In other words, the pull axis is correctly reconstructed, while the absolute location of the origin is separated from the origin of motion expected in Newtonian dynamics.

Accordingly, for the figure-eight solution, the following are exact:

- the trajectory,
- the pull axis,
- the scalar pull dynamics.
- the absolute magnitude of the local origin displacement $\|\mathbf{q}_i\|$,
- the absolute location of the local origin \mathbf{o}_i ,
- any interpretation of this origin as a unique physical centre of motion in the generic three-body case.

6 Discussion

The results above suggest a clear physical interpretation of the present construction. The essential observation is that the difficulty of the three-body problem is not merely the presence of more interaction terms, but the absence of a single pairwise interaction axis. The two-body problem is reducible because the full interaction is geometrically rank-one: there is one line of centres, and both the interaction magnitude and the interaction origin live on that line. In the three-body problem, the local interaction geometry is generically supported by two non-collinear channels. Any viable reduction must therefore specify both a local axis and a local origin law.

The present work shows that these two roles are not determined in the same way. First, the *pull axis* is fixed very rigidly. Once the Newtonian-weighted channel direction is used, the scalar pull law reproduces the benchmark trajectories exactly. This indicates that the decisive geometric quantity is the local Newtonian pull axis, not merely the simple bisector of the two source channels. In this sense, the axis law is much more tightly constrained than the origin length law.

Second, the new *origin length* is shown to reproduce the trajectory. Under the scalar pull law, the trajectory depends only on the axis of the local pull vector, not on its magnitude. This means that different choices of $\|\mathbf{q}_i\|$ can reproduce the same trajectory provided they share the same axis.

This distinction is, in our view, the central conceptual outcome of the present construction. The model identifies a distinguished local pull geometry that is dynamically exact, with a new unique physical origin law in the generic three-body case.

A second important point concerns the role of the correction factors α_i and β_i . These factors are not arbitrary fitting parameters. They encode two distinct pieces of information:

- the degree to which the local interaction is genuinely multi-channel, through the participation factor ν_i ,
- the asymmetry between the two channels, through the sharing factor q_i^{share} .

They therefore belong to the geometric part of the construction, not to the force law itself. Their role is to modulate the local channel contributions in a way that preserves the two-body limit, the symmetric three-body limit, and the generic asymmetric regime.

In summary, the present construction establishes an exact local pull-axis law and origin of motion for the three-body problem, and shows that these are sufficient to recover the principal benchmark solutions within a scalar pull-only dynamics. The accompanying Python code has further been extended to track four-body and five-body systems based on known stable configurations. This extension was obtained by generalizing the construction in a natural way to its higher-body form and by implementing the dynamics along all three spatial axes. In these tests, the reconstructed trajectories agree with the reference solutions up to errors of numerical precision. What remains open is to determine whether the full class of classical configurations is contained within the present framework, and to clarify how relativistic effects may be incorporated and interact with the model.

7 Passage to the Many-Body Case

The three-body construction described above suggests a natural extension to the general N -body problem. The guiding principle is to preserve the same decomposition of roles:

1. a local geometric description of the interaction channels,
2. a frame/self construction determining the local interaction scale,
3. a local pull axis,
4. a scalar pull-only law for the resulting motion.

What changes in the many-body case is that the observed body i is no longer coupled to two source channels, but to $M = N - 1$ channels.

7.1 Pairwise geometric data

For an observed body i , and for every source body $a \neq i$, define

$$\mathbf{r}_{ia} := \mathbf{x}_a - \mathbf{x}_i, \quad X_{ia} := \|\mathbf{r}_{ia}\|, \quad \hat{\mathbf{r}}_{ia} := \frac{\mathbf{r}_{ia}}{X_{ia}}.$$

The Newtonian channel weights are

$$W_{ia} := \frac{m_a}{X_{ia}^2}.$$

The associated normalized channel weights are

$$\omega_{ia} := \frac{W_{ia}}{\sum_{b \neq i} W_{ib}}, \quad \sum_{a \neq i} \omega_{ia} = 1.$$

7.2 Local shape tensor

The local geometric interaction structure is encoded in the shape tensor

$$\mathbf{S}_i := \sum_{a \neq i} \omega_{ia} \hat{\mathbf{r}}_{ia} \hat{\mathbf{r}}_{ia}^T.$$

By construction,

$$\text{tr}(\mathbf{S}_i) = 1.$$

This tensor is the natural many-body generalization of the two-channel shape tensor introduced in the three-body case.

7.3 Newtonian-weighted local axis

The local pull axis is determined by the Newtonian-weighted sum of the channel directions:

$$\mathbf{g}_i^{\text{axis}} := \sum_{a \neq i} W_{ia} \hat{\mathbf{r}}_{ia},$$

and the corresponding unit axis is

$$\hat{\mathbf{e}}_i := \frac{\mathbf{g}_i^{\text{axis}}}{\|\mathbf{g}_i^{\text{axis}}\|}.$$

This is the direct many-body extension of the local pull axis used in the final three-body code.

7.4 Geometric invariants

The first invariant is the coherence of the shape tensor along the local pull axis:

$$\gamma_i := \hat{\mathbf{e}}_i^T \mathbf{S}_i \hat{\mathbf{e}}_i.$$

The remaining geometric information is contained in the intrinsic invariants of the shape tensor, for example

$$J_{2,i} := \text{tr}(\mathbf{S}_i^2), \quad J_{3,i} := \det(\mathbf{S}_i).$$

In the three-body case, the overlap law could be written in terms of γ_i alone. In the general many-body case, the full overlap geometry should be allowed to depend on the full local shape tensor, or equivalently on a complete set of invariants such as $(\gamma_i, J_{2,i}, J_{3,i})$.

7.5 Participation factor

The many-body generalization of the three-body participation factor is

$$\nu_i^{(N)} := \frac{M}{M-1} \left(1 - \sum_{a \neq i} p_{ia}^2 \right), \quad p_{ia} := \frac{m_a}{\sum_{b \neq i} m_b},$$

where $M = N - 1$ is the number of source channels seen by the observed body i .

This quantity satisfies:

- $\nu_i^{(N)} \rightarrow 0$ when only one source channel dominates,
- $\nu_i^{(N)} = 1$ for equal source masses.

Thus $\nu_i^{(N)}$ measures how genuinely many-channel the local interaction is.

7.6 Many-body overlap amplitude

The many-body overlap amplitude is written abstractly as

$$h_i = 1 + \nu_i^{(N)} (H(\mathbf{S}_i) - 1),$$

where $H(\mathbf{S}_i)$ is a geometric overlap law determined by the local shape tensor.

In the three-body case, this reduces to the previously obtained formula

$$H(\mathbf{S}_i) = \frac{2}{4\gamma_i - 1}.$$

In the generic many-body case, one should allow

$$H(\mathbf{S}_i) = H(\gamma_i, J_{2,i}, J_{3,i}) \approx H(\mathbf{S}_i) = \frac{2}{4\gamma_i - 1} \frac{1 + a_1 \Delta_{2,i} + a_2 \Delta_{3,i} + a_3 \Delta_{2,i}^2 + a_4 \Delta_{2,i} \Delta_{3,i}}{1 + b_1 \Delta_{2,i} + b_2 \Delta_{3,i}},$$

with

$$\gamma_i = \hat{\mathbf{e}}_i^T \mathbf{S}_i \hat{\mathbf{e}}_i, \quad J_{2,i} = \text{tr}(\mathbf{S}_i^2), \quad J_{3,i} = \det(\mathbf{S}_i),$$

and

$$\Delta_{2,i} := J_{2,i} - (2\gamma_i^2 - 2\gamma_i + 1), \quad \Delta_{3,i} := J_{3,i}.$$

or equivalently a function of the eigenvalues of \mathbf{S}_i . Thus the passage from three bodies to many bodies is encoded entirely in the replacement of the three-body overlap law by a universal many-body overlap functional H .

7.7 Frame/self construction

The self contribution remains

$$\Phi_i^{\text{self}} := \frac{m_i}{L_i^3},$$

where L_i is the chosen local interaction scale.

In the present implementation, the scale is still obtained from the three-body-type geometric sum

$$\mathbf{d}_i = \sum_{a \neq i} \hat{\mathbf{r}}_{ia}, \quad L_i = \|\mathbf{d}_i\|,$$

although in the generic many-body case a refined definition of L_i may eventually be required.

The effective channel fields are then taken in the generic form

$$\Phi_{ia}^{(a),\text{eff}} = (1 + \alpha_{ia}) \frac{m_a}{L_i^3},$$

where the coefficients α_{ia} encode the channel-by-channel redistribution implied by the overlap law. The effective external interaction is then constructed by the inverse-sum rule

$$\Phi_i^{\text{ext}} = \left(\sum_{a \neq i} \left(\Phi_{ia}^{(a), \text{eff}} \right)^{-1} \right)^{-1}.$$

The frame interaction is

$$\Phi_i^{\text{frame}} = \left(\left(\Phi_i^{\text{self}} \right)^{-1} + \left(\Phi_i^{\text{ext}} \right)^{-1} \right)^{-1}.$$

The corresponding frame/self length is

$$\ell_i := \frac{\Phi_i^{\text{frame}}}{\Phi_i^{\text{self}}} L_i.$$

7.8 Local origin law

The many-body local origin vector is then constructed from the frame/self length and the Newtonian-weighted local pull axis:

$$\mathbf{q}_i := -\ell_i \hat{\mathbf{e}}_i, \quad \mathbf{o}_i := \mathbf{x}_i - \mathbf{q}_i.$$

This is the direct many-body extension of the final three-body origin law used in the code.

7.9 Scalar pull-only law

Finally, define the effective pull vector

$$\mathbf{F}_i := G m_i \sum_{a \neq i} (1 + \alpha_{ia}) \frac{m_a}{X_{ia}^2} \hat{\mathbf{r}}_{ia}.$$

The scalar pull coefficient is then

$$K_i := -\frac{\mathbf{F}_i \cdot \mathbf{q}_i}{\|\mathbf{q}_i\|^2}.$$

The local dynamics is governed by the scalar pull-only law

$$\mathbf{a}_i = -\frac{K_i}{m_i} \mathbf{q}_i.$$

7.10 Summary

The passage from three bodies to N bodies is therefore obtained by extending the following meaningful equations:

$$\begin{aligned}
\mathbf{S}_i &= \sum_{a \neq i} \omega_{ia} \hat{\mathbf{r}}_{ia} \hat{\mathbf{r}}_{ia}^T, \\
\hat{\mathbf{e}}_i &= \frac{\sum_{a \neq i} W_{ia} \hat{\mathbf{r}}_{ia}}{\left\| \sum_{a \neq i} W_{ia} \hat{\mathbf{r}}_{ia} \right\|}, \\
\nu_i^{(N)} &= \frac{M}{M-1} \left(1 - \sum_{a \neq i} p_{ia}^2 \right), \\
h_i &= 1 + \nu_i^{(N)} (H(\mathbf{S}_i) - 1), \\
\Phi_i^{\text{ext}} &= \left(\sum_{a \neq i} \left(\Phi_{ia}^{(a), \text{eff}} \right)^{-1} \right)^{-1}, \\
\Phi_i^{\text{frame}} &= \left(\left(\Phi_i^{\text{self}} \right)^{-1} + \left(\Phi_i^{\text{ext}} \right)^{-1} \right)^{-1}, \\
\ell_i &= \frac{\Phi_i^{\text{frame}}}{\Phi_i^{\text{self}}} L_i, \\
\mathbf{q}_i &= -\ell_i \hat{\mathbf{e}}_i, \\
\mathbf{o}_i &= \mathbf{x}_i - \mathbf{q}_i, \\
\mathbf{a}_i &= -\frac{K_i}{m_i} \mathbf{q}_i.
\end{aligned}$$

The only genuinely new object required in the many-body passage is the universal overlap functional $H(\mathbf{S}_i)$. All the remaining ingredients already extend naturally from the three-body case.

8 Conclusion

The starting point of this work was the observation that the essential simplicity of the two-body problem is geometric. In the two-body case, the interaction is supported on a single line of centres, and the corresponding origin of motion can be reconstructed directly from a split interaction law. By separating the interaction into two complementary contributions associated with the two masses, combining them by an inverse-sum rule, and dividing by the contribution of the observed body, one recovers the barycentric origin of motion. This makes explicit a geometric structure that is usually implicit in the standard reduced-mass treatment.

The main problem addressed in this paper was whether that logic can be extended to the three-body problem. The answer seems to be yes.

It is *not* possible to extend the two-body construction naively. The obstruction is geometric: for a given observed body, the interaction is no longer organized by a single pairwise axis, but by two generically non-collinear channels. As a result, the direct two-body logic fails. Separate pairwise origin constructions do not in general agree, and a scalar reduction by itself cannot determine a unique direction.

However, it *is* possible to rebuild a local geometric interaction law for the three-body problem. The construction developed in this paper does so by distinguishing three roles that become entangled in the many-body setting:

1. the local geometry of the interaction channels,
2. the frame/self structure determining the local interaction scale,
3. the pull law governing the resulting motion.

The resulting model proceeds by building a local interaction scale L_i , a local shape tensor \mathbf{S}_i , correction factors α_i, β_i encoding channel participation and asymmetry, a frame/self scalar length, and finally a Newtonian-weighted local pull axis. The decisive result is that, once the correct local pull axis is used, a scalar pull-only law is sufficient to reproduce the principal benchmark trajectories.

The consistency checks confirm this structure. In the two-body limit, the correction mechanism switches off and the usual pairwise geometry is recovered. In the equal-mass equilateral Lagrange solution, the construction is exact in both geometry and dynamics. In the figure-eight choreography, the final construction reproduces the Newtonian trajectory to numerical precision. These results show that the essential three-body difficulty lies in the local interaction geometry, not in the need for a fundamentally more complicated force law.

The central conclusion is therefore the following: the geometrically meaningful extension of the two-body problem to three bodies is achieved not by directly summing or displacing pairwise origin laws, but by rebuilding a local interaction geometry that separates axis selection from origin scaling. Within this framework, the local pull axis can be identified exactly, and once that axis is known, a scalar pull-only law is sufficient to recover the benchmark three-body motions.

In this sense, the present work establishes a local geometric reconstruction of the three-body problem that preserves the conceptual spirit of the two-body origin law while extending it to a genuinely multi-channel setting. What remains open is whether the inclusion of relativistic effects can be carried out in a straightforward manner, and whether the same decomposition into participation, sharing, overlap, and local pull axis that extends naturally in the three-body case also applies systematically to higher-body systems, excluding the stable four-body and five-body configurations that have shown to work in previous tests. These questions are left for future work.

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