

Goh-Schmutz Constant

Edgar Valdebenito

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ABSTRACT

In this note we give some formulas for the Goh-Schmutz constant

I. Introduction

The Goh-Schmutz constant is defined by

$$b = \int_0^{\infty} \frac{\ln(1+x)}{e^x - 1} dx \quad (1)$$

$$b = \int_0^{\infty} \ln(1 - \ln(1 - e^{-x})) dx \quad (2)$$

$$b = \int_0^{\infty} \frac{x e^{-x}}{(1 - e^{-x})(1 - \ln(1 - e^{-x}))} dx \quad (3)$$

$$b = - \sum_{n=1}^{\infty} \frac{e^n}{n} \text{Ei}(-n) \quad (4)$$

where $\text{Ei}(x)$ is the exponential integral.

$$b = 1.11786415118 \dots \quad (5)$$

In this note we give some formulas for b .

II. Formulas

Entry 1.

$$b = - \int_0^{\infty} \ln(1 - e^{1-e^x}) dx \quad (6)$$

$$b = \int_0^{\infty} \frac{\ln(1 + \ln(1+x))}{x(1+x)} dx \quad (7)$$

$$b = e \int_e^{\infty} \frac{\ln(\ln x)}{x(x-e)} dx \quad (8)$$

$$b = e \int_0^{\infty} \frac{\ln(\ln(x+e))}{x(x+e)} dx \quad (9)$$

$$b = e \int_0^{\infty} \frac{x e^x}{e^{e^x} - e} dx \quad (10)$$

$$b = \int_0^{\infty} \frac{1}{1+x} \ln\left(\frac{e^x}{e^x - 1}\right) dx = - \int_0^{\infty} \frac{\ln(1 - e^{-x})}{1+x} dx \quad (11)$$

$$b = - \int_0^1 \frac{\ln(1-x)}{(1-\ln(x))x} dx = - \int_0^1 \frac{\ln(x)}{(1-\ln(1-x))(1-x)} dx \quad (12)$$

$$b = \int_0^1 \frac{\ln(1 - \ln(x))}{1 - x} dx \quad (13)$$

$$b = \int_0^\infty \frac{\ln(1 + 2x)}{\sinh(x)} e^{-x} dx \quad (14)$$

$$b = \int_{-\infty}^\infty \frac{e^{-x} \ln(1 + e^{-x})}{e^{e^{-x}} - 1} dx \quad (15)$$

$$b = \int_1^\infty \frac{1}{e^{x-1} - 1} \cosh^{-1}\left(\frac{1+x^2}{2x}\right) dx \quad (16)$$

$$b = \int_0^\infty \frac{1}{e^x - 1} \sinh^{-1}\left(\frac{x(2+x)}{2(1+x)}\right) dx \quad (17)$$

$$b = \int_0^\infty \frac{\sinh^{-1}(x)}{e^{-1+x+\sqrt{1+x^2}} \left(1 + \frac{x}{\sqrt{1+x^2}}\right)} dx \quad (18)$$

$$b = 2 \int_0^\infty \frac{\sinh(2x) \ln(\cosh(x))}{e^{(\sinh(x))^2} - 1} dx \quad (19)$$

$$b = \int_0^\infty \frac{\ln(1 + 2x)}{e^x - 1} dx - \int_0^\infty \frac{\ln(1 + 2x)}{e^x + 1} dx \quad (20)$$

$$b = \int_{-\infty}^\infty \frac{\ln(1 + \ln(1 + e^x))}{1 + e^x} dx = \int_{-\infty}^\infty \frac{\ln(1 + \ln(1 + e^{-x}))}{1 + e^{-x}} dx \quad (21)$$

$$b = 4 \int_0^\infty \frac{\ln(1 + 2 \ln(\cosh(x)))}{\sinh(2x)} dx \quad (22)$$

$$b = 2 \int_0^\infty \tanh(x) (1 - 2 \ln(\tanh(x))) dx \quad (23)$$

$$b = 2 \int_0^\infty \frac{x}{(1+x^2)(1+\ln(1+x^2))} \ln\left(\frac{1+x^2}{x^2}\right) dx \quad (24)$$

$$b = -4 \int_0^\infty \frac{\tanh(x) \ln(\tanh(x))}{(1 + 2 \ln(\cosh(x)))} dx \quad (25)$$

$$b = \frac{1}{2} \int_0^\infty (\operatorname{csch}(x) + \operatorname{coth}(x) - 1) \ln(1+x) dx = \frac{1}{2} \int_0^\infty (\operatorname{csch}(e^x - 1) + \operatorname{coth}(e^x - 1) - 1) x e^x dx \quad (26)$$

$$b = \int_0^{\pi/2} (\sec(x) + \tan(x)) \ln(1 - \ln(\sin(x))) dx \quad (27)$$

$$b = \int_0^{\pi/2} (\csc(x) + \cot(x)) \ln(1 - \ln(\cos(x))) dx \quad (28)$$

$$b = \int_0^{\pi/2} \cot\left(\frac{x}{2}\right) \ln(1 - \ln(\cos(x))) dx \quad (29)$$

$$b = 2 \int_0^{\pi/2} \cot(x) \ln(1 - 2 \ln(\cos(x))) dx \quad (30)$$

$$b = 2 \int_0^{\pi/2} \tan(x) \ln(1 - 2 \ln(\sin(x))) dx \quad (31)$$

$$b = 2 \int_0^{\pi/4} \tan(2x) \ln(1 - \ln(2) - 2 \ln(\sin(x))) dx \quad (32)$$

$$b = \int_0^{\pi/2} \tan(x) \ln(1 - \ln(1 - \cos(x))) dx \quad (33)$$

$$b = \int_0^{\pi/2} \cot(x) \ln(1 - \ln(1 - \sin(x))) dx \quad (34)$$

Entry 2.

$$b = \sum_{n=1}^{\infty} \ln(n+1) \ln\left(\frac{1 - e^{-n-1}}{1 - e^{-n}}\right) + \sum_{n=0}^{\infty} \int_0^1 \frac{1}{e^{x+n} - 1} \ln\left(1 + \frac{x}{1+n}\right) dx \quad (35)$$

Entry 3.

$$b = \frac{\pi^2}{12} + \frac{1}{2} - \ln(2) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{(2n)!} \sum_{k=0}^{2n-1} \binom{2n-1}{k} (-1)^k \left(\frac{2^{k+1} \ln(2)}{k+1} - \frac{2^{k+1} - 1}{(k+1)^2} \right) + \int_1^{\infty} \frac{\ln(1+x)}{e^x - 1} dx \quad (36)$$

Remark: B_n are the Bernoulli numbers: $B_n = \left\{ \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \dots \right\}$.

Entry 4.

$$b = - \int_0^u \frac{\ln(1 - e^{-x})}{1+x} dx + \sum_{n=1}^{\infty} \frac{e^n}{n} \Gamma(0, (1+u)n), \quad u \geq 0 \quad (37)$$

Remark: $\Gamma(x, y)$ is the incomplete Gamma function.

Entry 5.

$$b = e \int_0^{\infty} \frac{W(x)}{(e^{e^{W(x)}} - e)(1+W(x))} dx = e \int_0^{\infty} \frac{x e^{-W(x)}}{(e^{e^{W(x)}} - e)(1+W(x))} dx \quad (38)$$

where $W(x)$ is the Lambert W-function.

Entry 6.

$$b = \int_0^1 \frac{\ln(1+x)}{e^x - 1} dx + 2 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{2k+1} \left(\frac{e^{-(n-k+1)}}{n-k+1} + \sum_{m=0}^{2k} \binom{2k+1}{m} (n-k+1)^{2k-m} (-2)^{2k+1-m} e^{2(n-k+1)} \Gamma(m-2k, 3(n-k+1)) \right) \quad (39)$$

Remark: $\Gamma(x, y)$ is the incomplete Gamma function.

Entry 7.

$$b = \sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1} c(n) + \int_u^{\infty} \frac{\ln(1+x)}{e^x - 1} dx, \quad 0 \leq u < 1 \quad (40)$$

where

$$c(n) = \frac{(-1)^n}{n+1} - \sum_{k=1}^n \frac{c(n-k)}{(k+1)!}, \quad c(0) = 1 \quad (41)$$

$$c(n) = \left\{ 1, -1, \frac{2}{3}, -\frac{11}{24}, \frac{253}{720}, -\frac{413}{1440}, \frac{7331}{30240}, -\frac{635}{3024}, \frac{672229}{3628800}, \dots \right\} \quad (42)$$

Entry 8.

$$b = 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^{\infty} \frac{x^{2n+1}}{(2+x)^{2n+1} (e^x - 1)} dx \quad (43)$$

Entry 9.

$$b = \int_0^u \frac{\ln(1+x)}{e^x - 1} dx + \sum_{n=1}^{\infty} 2^{n-1} \int_u^{\infty} \frac{\ln(1+x)}{(e^x + 1)^n} dx, \quad u > 0 \quad (44)$$

Entry 10. for $u > 1$ we have

$$b = \int_0^u \frac{\ln(1+x)}{e^x - 1} dx + \sum_{n=1}^{\infty} \frac{g(un)}{n} - \sum_{n=2}^{\infty} \frac{e^{-un}}{n} \ln(n) + \sum_{k=0}^{\infty} \sum_{n=0}^k \frac{(-1)^n (k-n+1)^n}{n+1} \Gamma(-n, u(k-n+1)) \quad (45)$$

where

$$g(y) = \int_y^{\infty} e^{-x} \ln(x) dx, \quad y \geq 0 \quad (46)$$

Remark: $\Gamma(x, y)$ is the incomplete Gamma function.

Entry 11. for $u > 0$ we have

$$b = \int_0^u \frac{\ln(1+x)}{e^x - 1} dx + \sum_{n=0}^{\infty} 2^n \int_u^{\infty} \frac{\ln(1+x)}{e^{2^n x} + 1} dx \quad (47)$$

$$b = \int_0^u \frac{\ln(1+x)}{e^x - 1} dx + \quad (48)$$

$$2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{2^n (-1)^m}{2k+1} e^{2^{n+1}(m+1)} \left(\frac{e^{-2^n(m+1)(2+u)}}{2^n(m+1)} + \sum_{r=0}^{2k} \binom{2k+1}{r} (-2)^{2k+1-r} (2^n(m+1))^{2k-r} \Gamma(r-2k, 2^n(m+1)(2+u)) \right)$$

Remark: $\Gamma(x, y)$ is the incomplete Gamma function.

Entry 12. for $0 \leq u < 2\pi$ we have

$$b = \frac{u}{2} - \frac{(1+u)\ln(1+u)}{2} + 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left(\frac{u}{2+u} \right)^{2n+1} {}_2F_1 \left(1, 2n+1, 2n+2, \frac{u}{2+u} \right) + \sum_{n=1}^{\infty} \frac{1}{2n+1} \left(\frac{u}{2+u} \right)^{2n+1} \sum_{k=1}^n \frac{(-1)^{k-1} B_k (2+u)^{2k}}{(2n-2k+1)(2k)!} {}_2F_1 \left(1, 2n-2k+1, 2n+2, \frac{u}{2+u} \right) + \int_u^{\infty} \frac{\ln(1+x)}{e^x - 1} dx \quad (49)$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function and B_k are the Bernoulli numbers.

Entry 13. for $i = \sqrt{-1}$ we have

$$b = \int_0^{\infty} \frac{\ln(1 + \ln(1 + ix))}{x(1 + ix)} dx \quad (50)$$

$$b = \int_0^{\infty} \frac{1}{x(1+x^2)} \left(\ln \left(\sqrt{(1 + \ln(\sqrt{1+x^2}))^2 + (\tan^{-1}(x))^2} \right) + x \tan^{-1} \left(\frac{\tan^{-1}(x)}{1 + \ln(\sqrt{1+x^2})} \right) \right) dx \quad (51)$$

Entry 14. for $0 \leq u \leq 1, \nu > 1$ we have

$$b = -\text{Li}_2(-u) + \frac{u - (1+u)\ln(1+u)}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n u^{n+3}}{n+3} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k B_{k+1}}{(n-2k+1)(2k+2)!} + \int_u^{\nu} \frac{\ln(1+x)}{e^x - 1} dx + \int_{\nu}^{\infty} \frac{\ln(x)}{e^x - 1} dx + \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{(-1)^{k-1} (n-k+1)^{k-1} \Gamma(1-k, (n-k+1)\nu)}{k} \quad (52)$$

where $\text{Li}_2(x)$ is the polylogarithm function, $\Gamma(x, y)$ is the incomplete Gamma function and B_k are the Bernoulli numbers.

Entry 15.

$$b = \int_0^1 \frac{\ln(1+x)}{e^x - 1} dx + \frac{\ln(2)}{2(e-1)} + \sum_{n=1}^{\infty} \frac{\ln(n+2)}{e^{n+1} - 1} + 2 \int_0^{\infty} \frac{(e \cos(x) - 1) \tan^{-1}(x/2) - e \sin(x) \ln(\sqrt{4+x^2})}{(1+e^2 - 2e \cos(x))(e^{2\pi x} - 1)} dx \quad (53)$$

Entry 16.

$$b = \int_0^1 \frac{\ln(1+x)}{e^x - 1} dx + \sum_{n=1}^{\infty} \ln(n+1) \ln \left(1 + \frac{e-1}{e(e^n - 1)} \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} e^{-nk} (m! - \Gamma(m+1, k))}{m(n+1)^m k^{m+1}} \quad (54)$$

Remark: $\Gamma(x, y)$ is the incomplete Gamma function.

Entry 17.

$$b = -e\gamma + e \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)^2} + \int_0^{\infty} \frac{e^{-x} \ln(1+x)}{e^x - 1} dx \quad (55)$$

$$b = -e\gamma + e \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)^2} + \sum_{n=2}^{\infty} \frac{e^n \Gamma(0, n)}{n} \quad (56)$$

where γ is the Euler-Mascheroni constant and $\Gamma(x, y)$ is the incomplete Gamma function.

Entry 18. for $u > \ln(2)$ we have

$$b = e \Gamma(0, 1+u) + e^{-u} \ln(1+u) + \int_0^u \frac{\ln(1+x)}{e^x - 1} dx - \sum_{n=2}^{\infty} (-1)^{n-1} \int_u^{\infty} \frac{\ln(1+x)}{(e^x - 1)^n} dx \quad (57)$$

Remark: $\Gamma(x, y)$ is the incomplete Gamma function.

Entry 19. for $u \geq 0$ we have

$$b = \int_0^u \frac{\ln(1+x)}{e^x - 1} dx - \ln(1+u) \ln(1 - e^{-u}) - \int_u^{\infty} \frac{\ln(1 - e^{-x})}{1+x} dx \quad (58)$$

Entry 20. for $0 \leq u \leq 1$ we have

$$b = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(n! \zeta(n+1) - \sum_{k=0}^n \binom{n}{k} k! u^{n-k} \text{Li}_{k+1}(e^{-u}) \right) + \int_u^{\infty} \frac{\ln(1+x)}{e^x - 1} dx \quad (59)$$

where $\zeta(x)$ is the Riemann zeta function and $\text{Li}_k(x)$ is the Polylogarithm function.

Entry 21.

$$b = \frac{1}{2} \sum_{n=0}^m \int_0^{\infty} \frac{\ln(1+x)}{\sinh(2^n x)} dx + \int_0^{\infty} \frac{\ln(1+x)}{e^{2^{m+1}x} - 1} dx, \quad m = 0, 1, 2, 3, \dots \quad (60)$$

$$b = \frac{1}{2} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{\ln(1+x)}{\sinh(2^n x)} dx \quad (61)$$

Entry 22. for $a < 1/2$ we have

$$b = \int_0^{\infty} \frac{\cosh(2ax) \ln(1+2x)}{\sinh(x) \cosh((1-2a)x)} dx - \int_0^{\infty} \frac{e^{-(1-2a)x} \ln(1+2x)}{\cosh((1-2a)x)} dx \quad (62)$$

$$b = \int_0^{\infty} \frac{e^{-(1-2a)x} \ln(1+2x)}{\sinh((1-2a)x)} dx - \int_0^{\infty} \frac{\sinh(2ax) \ln(1+2x)}{\sinh(x) \sinh((1-2a)x)} dx \quad (63)$$

III. References

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